Quantum spin Hall effect of light (and Photonic analogs of Topological Insulators)

Konstantin Y. Bliokh, in collaboration with
A. Bekshaev, D. Leykam, D. Smirnova, M. Lein, F Nori

Science **(2015)**

Physics Reports **(2015)**

Nature Photonics **(2016)**

NJP **(2018)**

PRL (2018)

Nature Communications **(2019)**

Topological non–Hermitian origin of surface acoustic waves

Konstantin Y. Bliokh,

In collaboration with

D. Leykam, J. Dressel, F. Nori

•K.Y. Bliokh, F. Nori
Transverse spin and surface waves in acoustic metamaterials
Phys. Rev. B **99**, 020301(R) (2019). [PDF][Link][arXiv]

•K.Y. Bliokh, F. Nori
Spin and orbital angular momenta of acoustic beams
Phys. Rev. B **99**, 174310 (2019). [PDF][Link][arXiv]

•K.Y. Bliokh, F. Nori
Klein-Gordon Representation of Acoustic Waves and Topological Origin of Surface Acoustic Modes
Phys. Rev. Lett. **123**, 054301 (2019). [PDF][Link][arXiv][Suppl. Info.]

•I.D. Toftul, K.Y. Bliokh, M.I. Petrov, F. Nori
Acoustic Radiation Force and Torque on Small Particles as Measures of the Canonical Momentum and Spin Densities
Phys. Rev. Lett. **123**, 183901 (2019). [PDF][Link][arXiv][Suppl. Info.]

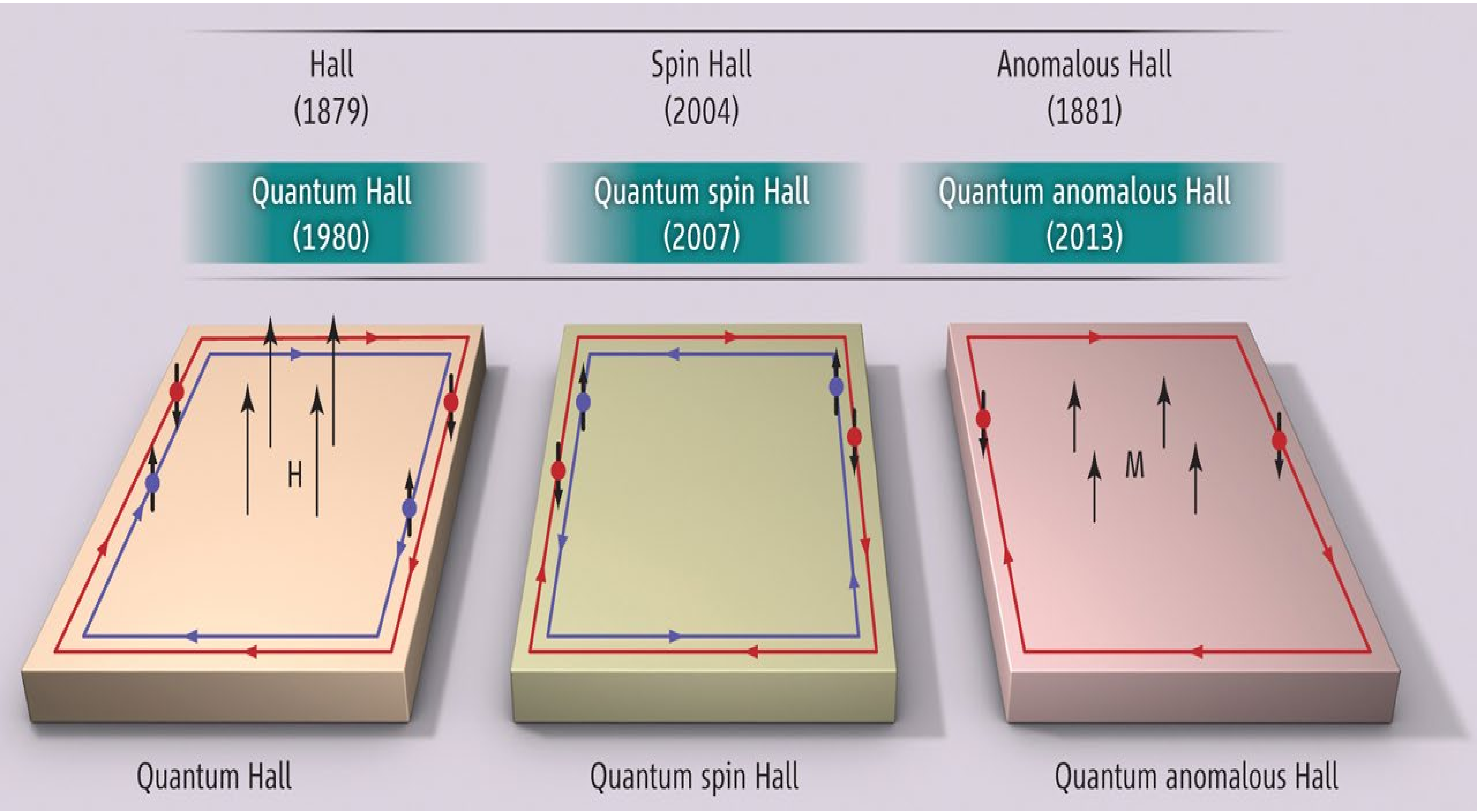
•L. Burns, K.Y. Bliokh, F. Nori, J. Dressel
Acoustic versus electromagnetic field theory: scalar, vector, spinor representations and the emergence of acoustic spin
New Journal of Physics **22**, 053050 (2020). [PDF][Link][arXiv]

•D. Leykam, K.Y. Bliokh, F. Nori
Edge modes in two-dimensional electromagnetic slab waveguides: Analogs of acoustic plasmons
Phys. Rev. B **102**, 045129 (2020). [PDF][Link][arXiv]

•K.Y. Bliokh, Y.P. Bliokh, F. Nori
Ponderomotive forces, Stokes drift, and momentum in acoustic and electromagnetic waves
Phys. Rev. A Letter **106**, L021503 (2022). [PDF][Link][arXiv]

Very quick overview of various types of Quantum Hall Effects (for electrons).

Before considering the Quantum Spin Hall Effect (QSHE) **for light**.



Quantum Hall trio: For all three quantum Hall effects, electrons flow through the lossless edge channels, with the rest of the system insulating.

QSHE (Quantum Spin Hall Effect) and

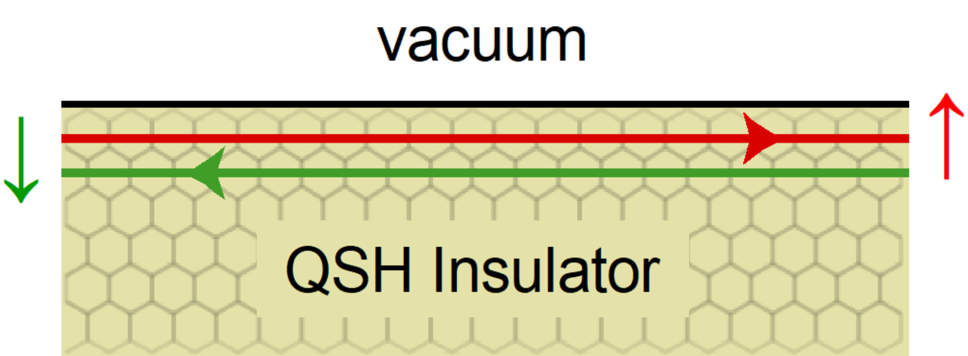
Topological Insulators

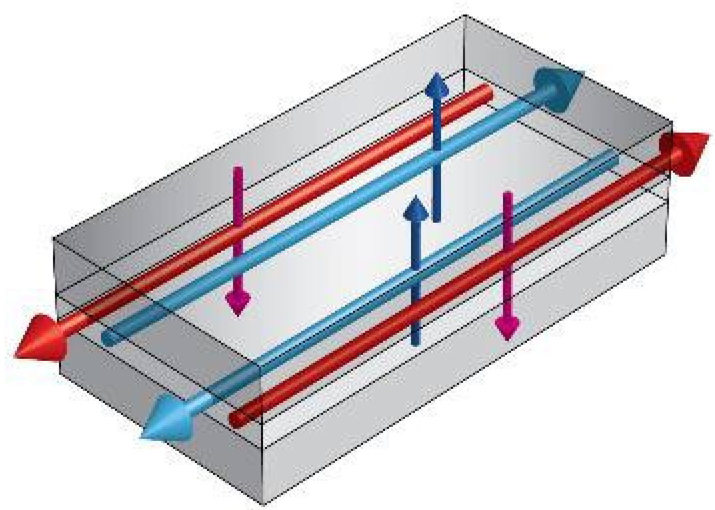
for electrons

QSHE and topological insulators

The quantum spin Hall effect means the presence of topologically–protected **edge modes** at the interface between two 2D insulators. Such modes are characterized by strong **spin–momentum locking**: **opposite spins propagate in opposite directions.**

“Spin filtered” or “helical” edge states



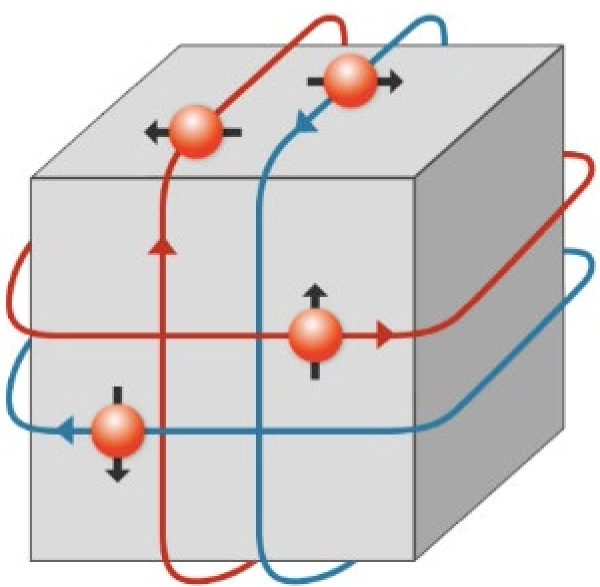


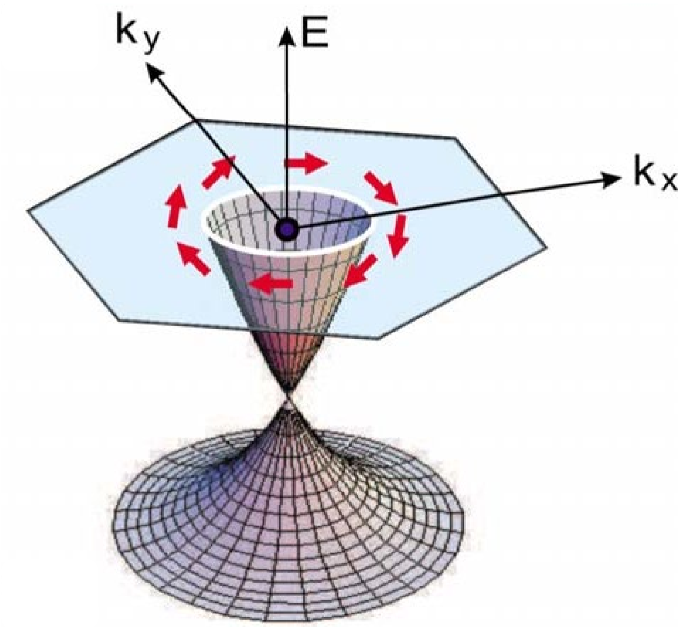
Kane & Mele, *PRL* (2005); Bernevig, Hughes, Zhang, *Science* (2006)

QSHE and topological insulators

A 3D **topological insulator** is a 3D generalization of such states.

Such insulator exhibits 2D surface modes, which are
helical massless fermions with spin–momentum locking
(vortex spin texture):





Hiseh *et al.*, *Nature* (2009); Hasan & Kane, *RMP* (2010)

Now let's look for a
Photonic counterpart
of the electronic
Quantum Spin Hall effect.

QSHE of light:
Surface modes with
spin–momentum locking

It has been suggested that **photonic topological insulators** can be created in ***complex metamaterials*** structures.

We have shown that pure ***free-space light*** already possesses intrinsic QSHE,

and that ***simple natural* materials** (such as ***metals*** supporting surface plasmon-polariton modes) exhibit some features that resemble topological insulators.

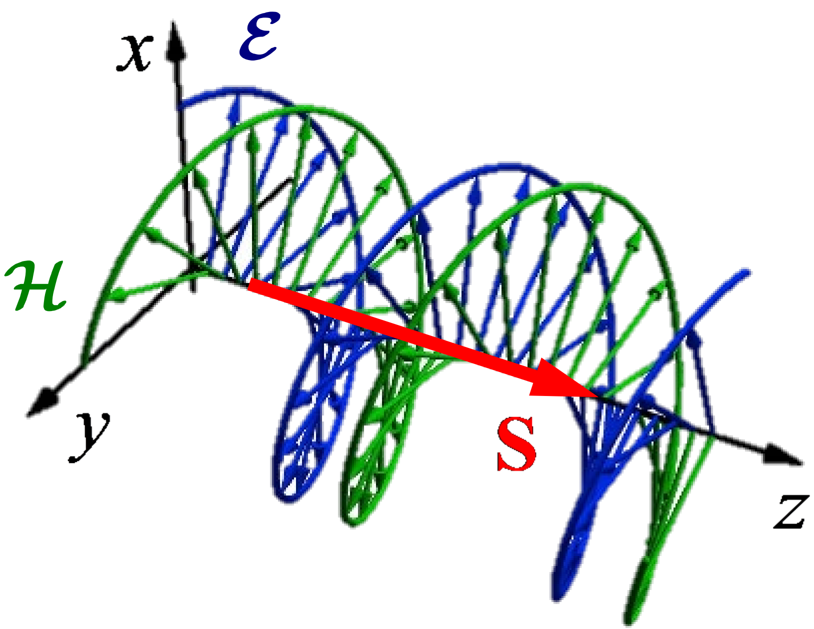
We show that the transverse spin in evanescent waves (our work from 2012-14) and **spin-controlled unidirectional excitation of surface modes or waveguide modes** can be interpreted as manifestations of the QSHE of light.

Basic spin properties of light

The bulk modes for free light are propagating plane waves:

$$\mathbf{E}^{\sigma} \propto \mathbf{e}^{\sigma} \exp(ikz), \quad \mathbf{e}^{\sigma} = \frac{\bar{\mathbf{x}} + i\sigma\bar{\mathbf{y}}}{\sqrt{2}}$$

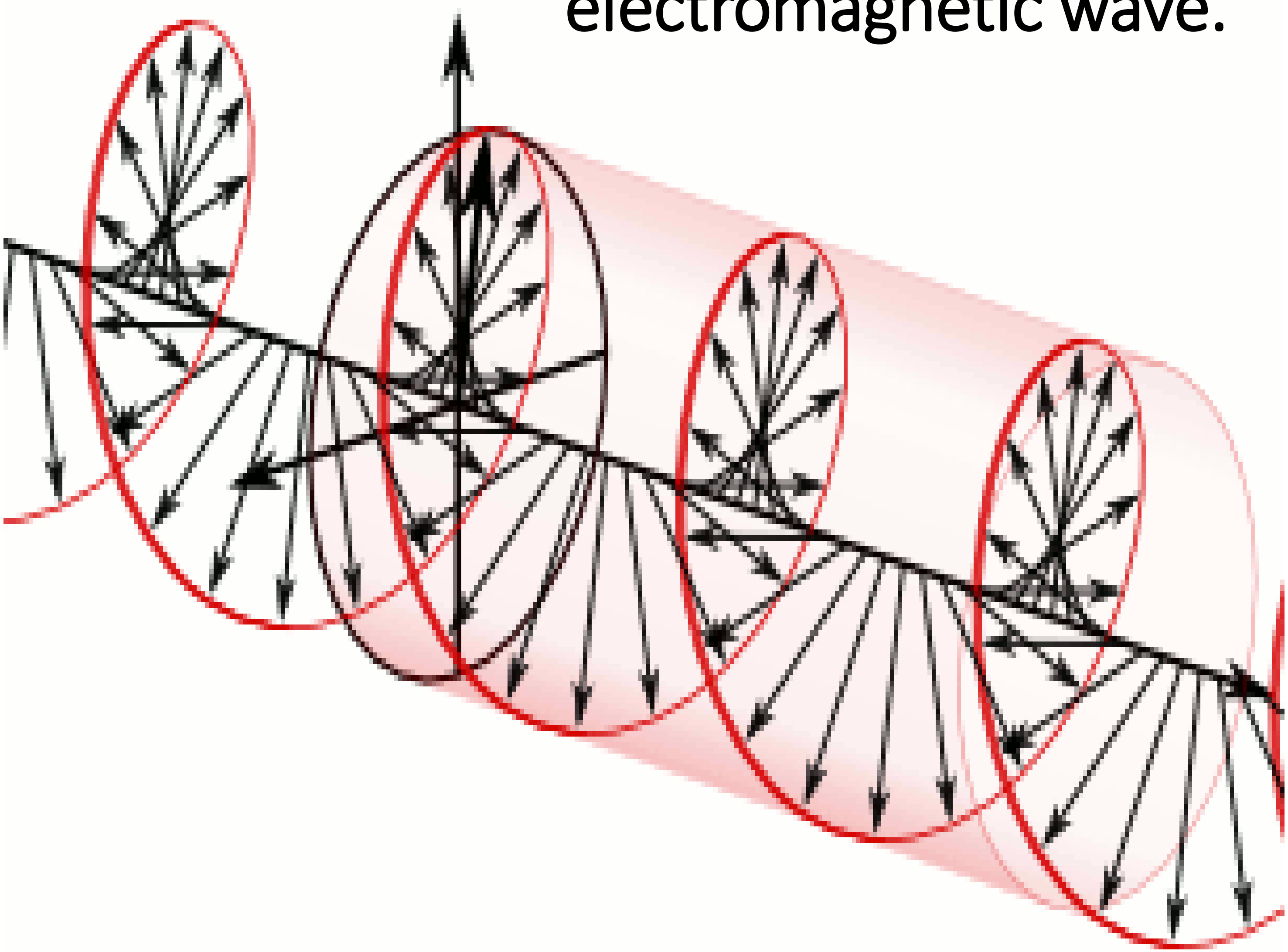
Here $\sigma = \pm 1$ is the helicity, and photons carry spin:



$$\mathbf{S} = \sigma \frac{\mathbf{k}}{k}$$

Longitudinal helicity–dependent spin

The electric field vectors of a traveling circularly-polarized electromagnetic wave.



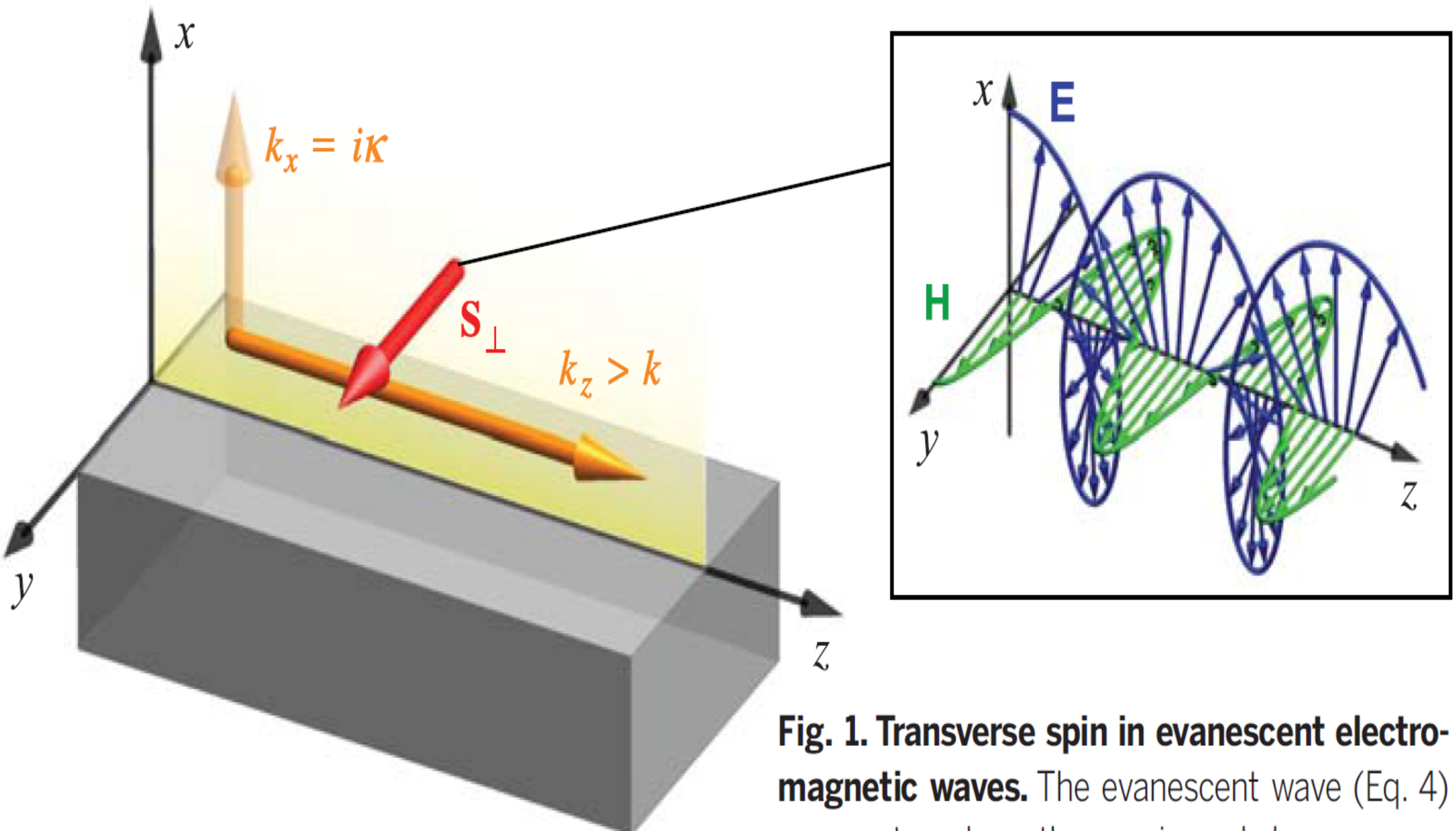
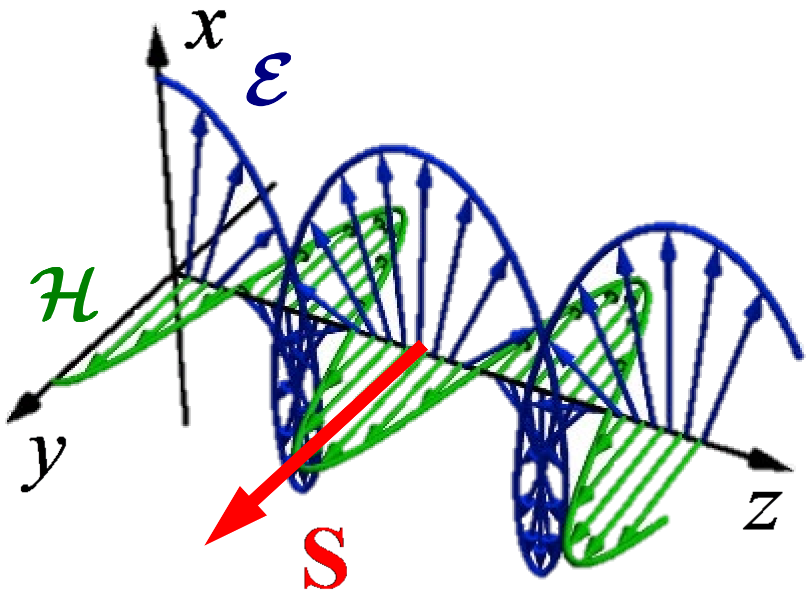


Fig. 1. Transverse spin in evanescent electromagnetic waves. The evanescent wave (Eq. 4) propagates along the z axis and decays exponentially in the $x > 0$ semi-space. (Inset) The instantaneous distributions of the electric and magnetic wave fields for the case of linear transverse-magnetic polarization, $\xi = (1, 0)^T$. The cycloidal (x, z)–plane rotation of the electric field generates the transverse spin $\mathbf{S}_\perp$ (Eq. 5) (*20, 21*). The sign of the transverse spin depends on the direction of propagation of the evanescent wave.

Surface modes and transverse spin

Evanescent modes have spin. Let us consider their evanescent–wave tails: $\mathbf{k} = k_z \bar{\mathbf{z}} + i\kappa \bar{\mathbf{x}}$

$$\mathbf{E}^{\sigma} \propto \mathbf{e}^{\sigma} \exp\left(ik_z z - \kappa x\right), \quad \mathbf{e}^{\sigma} = \left(\bar{\mathbf{x}} - i\frac{\kappa}{k_z}\bar{\mathbf{z}}\right)$$



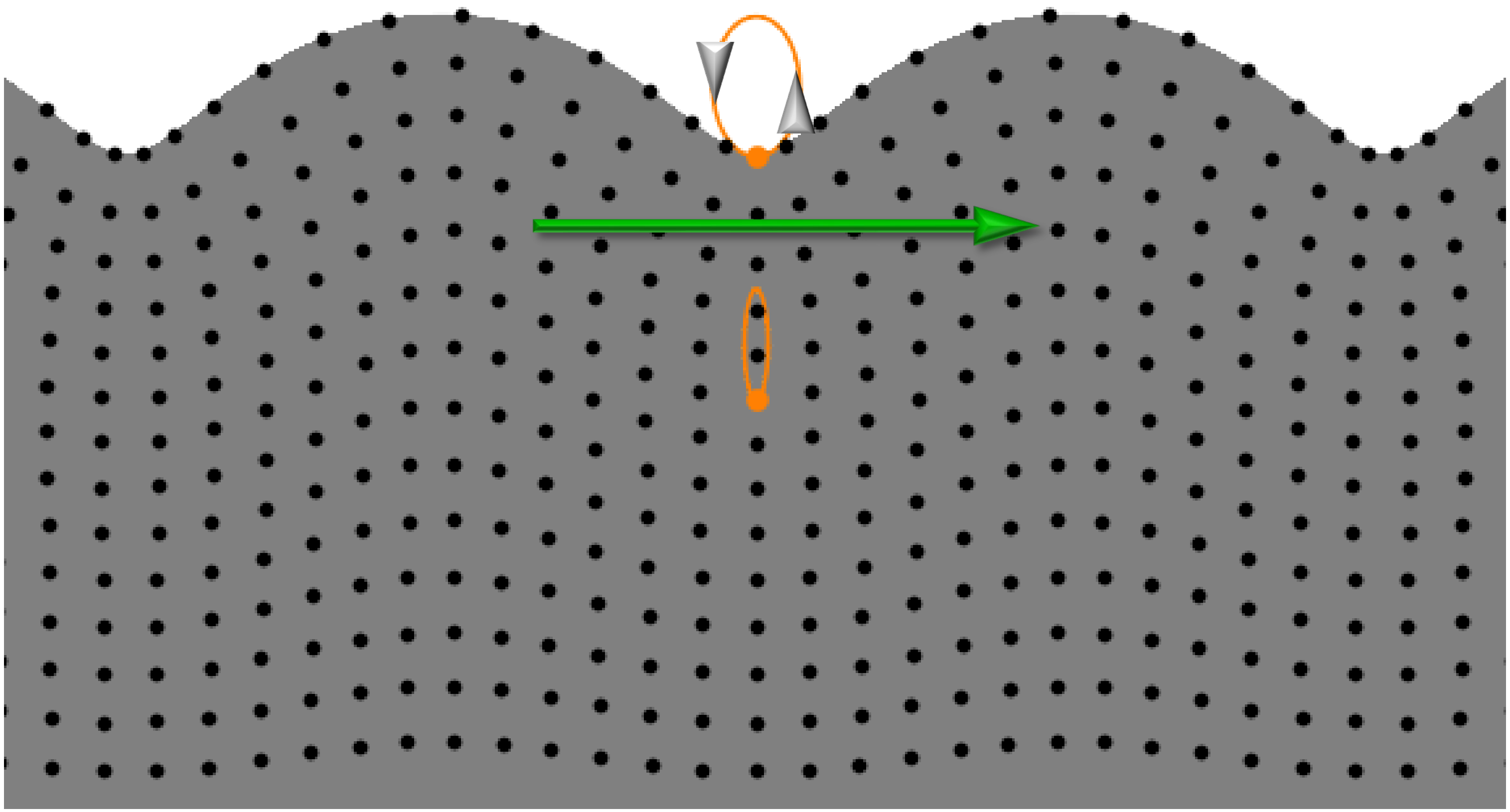
$$\mathbf{S}_{\perp} = \frac{\mathrm{Re}\,\mathbf{k} \times \mathrm{Im}\,\mathbf{k}}{(\mathrm{Re}\,\mathbf{k})^2}$$

Transverse helicity–independent spin!

Bliokh & Nori, *PRA* (2012) **Bliokh, Bekshaev, Nori, *Nature Comm.* (2014)**

Types of waves

Surface waves (e.g., Rayleigh or ocean waves):



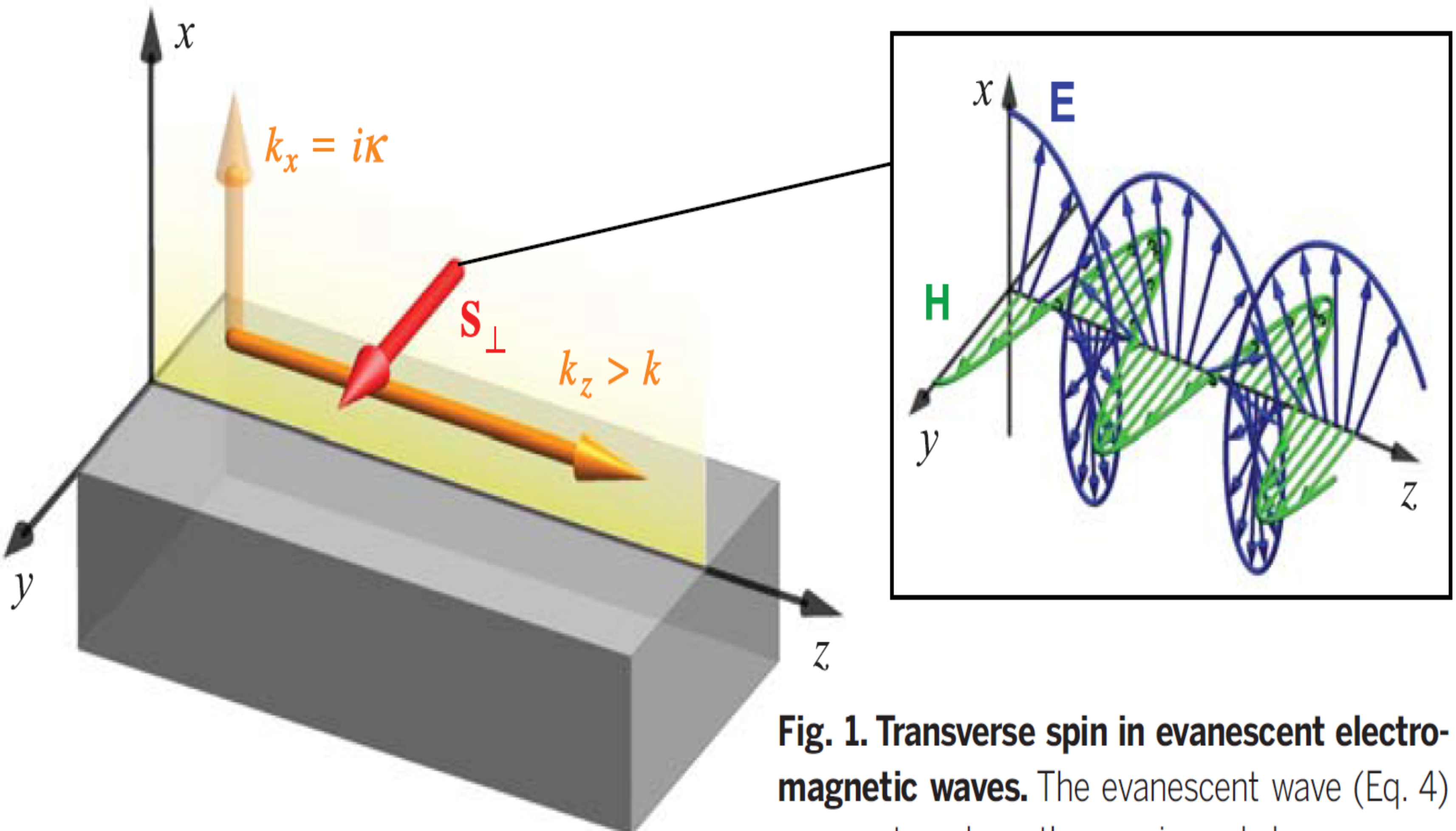


Fig. 1. Transverse spin in evanescent electromagnetic waves. The evanescent wave (Eq. 4) propagates along the z axis and decays exponentially in the $x > 0$ semi-space. (Inset) The instantaneous distributions of the electric and magnetic wave fields for the case of linear transverse-magnetic polarization, $\xi = (1, 0)^T$. The cycloidal (x, z)–plane rotation of the electric field generates the transverse spin $\mathbf{S}_\perp$ (Eq. 5) (*20, 21*). The sign of the transverse spin depends on the direction of propagation of the evanescent wave.

Surface modes and transverse spin

The nature of this transverse spin is similar to the circular motion of water in surface ocean waves:

M. Stone, *Science* (2015) . “Perspective” to our paper in the same issue.

Extraordinary Momentum and Spin in Evanescent Waves

Bliokh, Bekshaev, Nori

Nature Communications (2014).

Additional results in:

Physics Reports (2015)

Nature Photonics (2016)

NJP (2018),

PRL (2018)

Nature Communications (2019)

Early work in: Bliokh & Nori, PRA (2012).

Well-known textbook statements:

- **Momentum of light is determined by the Poynting vector**

- **Momentum of light is directed along the wave vector and is independent of polarization**

- **Spin angular momentum of light is determined by the circular polarization and is directed along the wave vector**

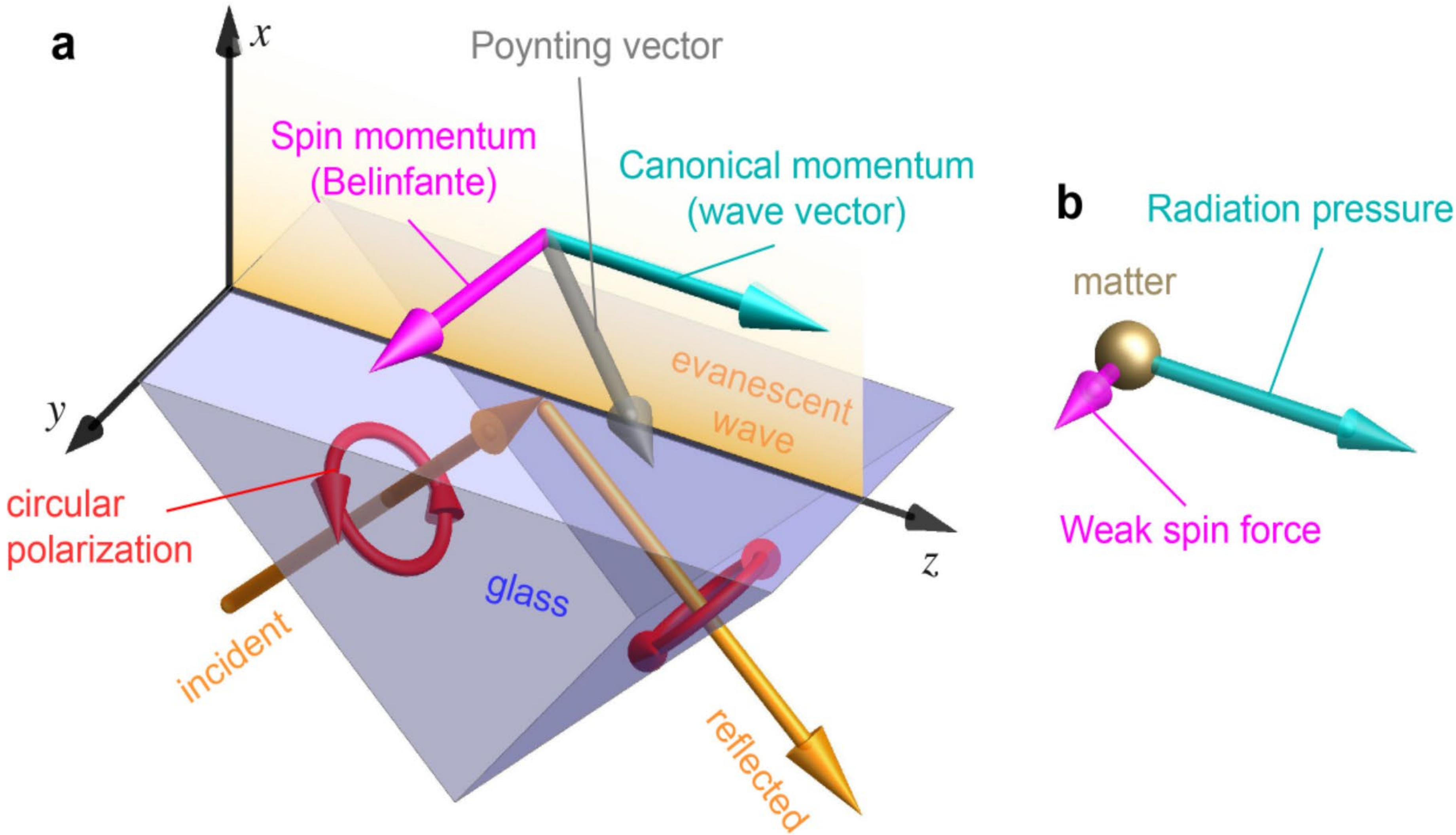
Our results challenge textbook statements:

- ~~Momentum of light is determined by the Poynting vector~~
- ~~Momentum of light is directed along the wave vector and is independent of polarization~~
- ~~Spin angular momentum of light is determined by the circular polarization and is directed along the wave vector~~

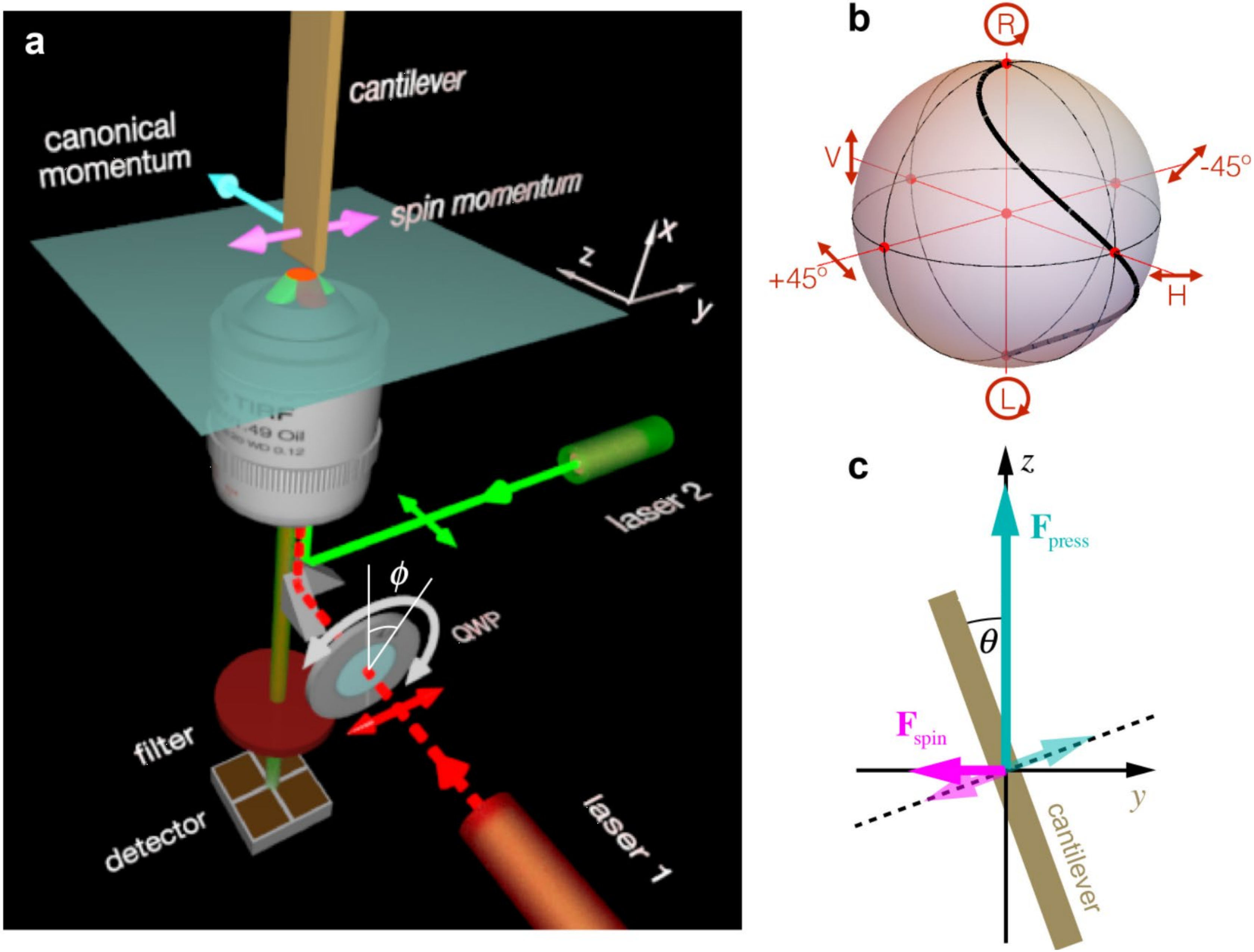
Transverse spin and momentum

Antognozzi, Bermingham, Harniman, Simpson, Senior, Hayward, Hoerber, Dennis, Bekshaev, Bliokh, Nori

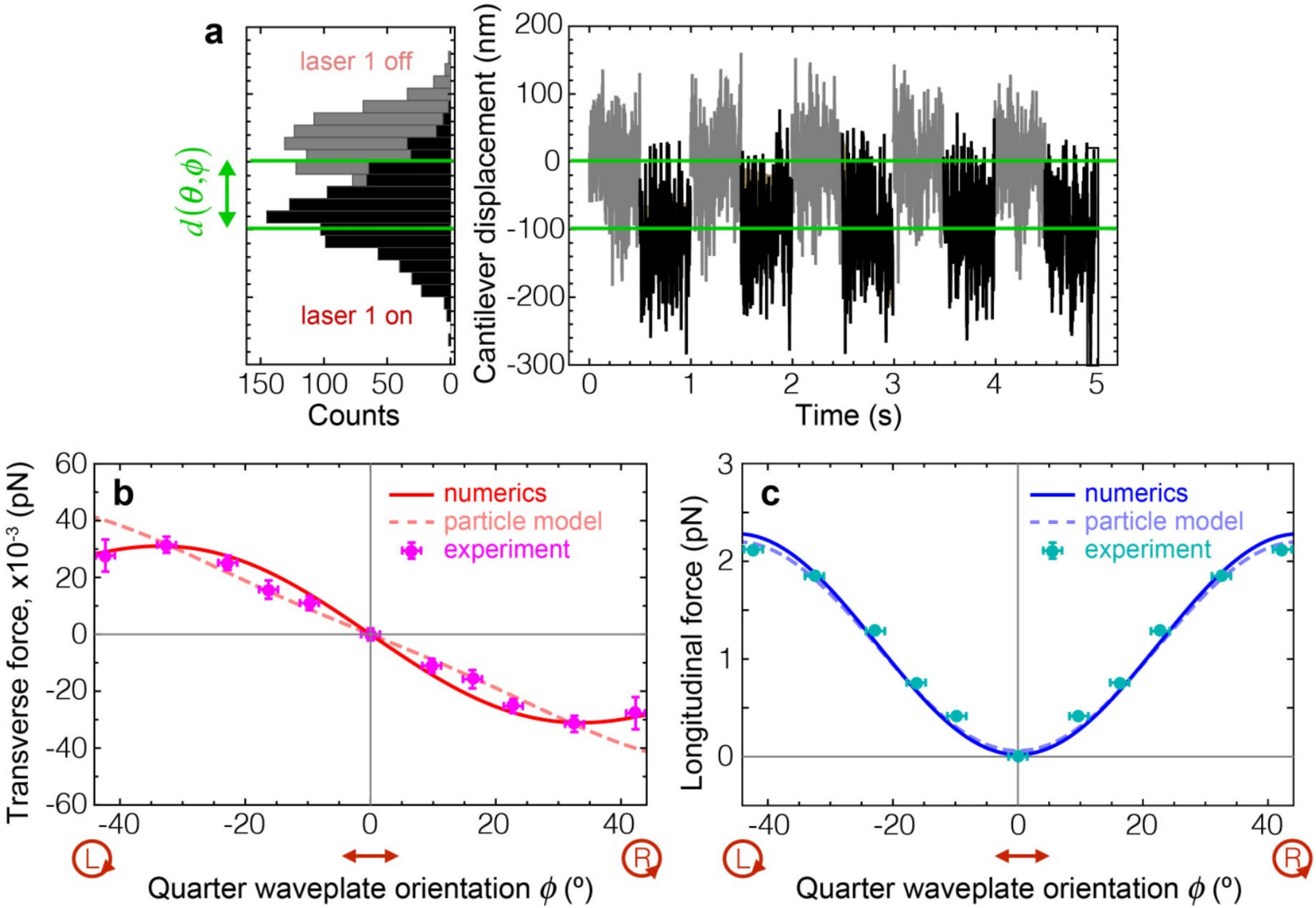
Direct measurements of the extraordinary optical momentum and transverse spin-dependent force using a nano-cantilever. **Nature Physics (2016).**



Transverse spin and momentum



Transverse spin and momentum



Three-Dimensional Measurement of the Helicity-Dependent Forces on a Mie Particle

Liu, et al (the group of Federico Capasso)
Phys. Rev. Lett. **120**, 223901 – Published 31 May 2018

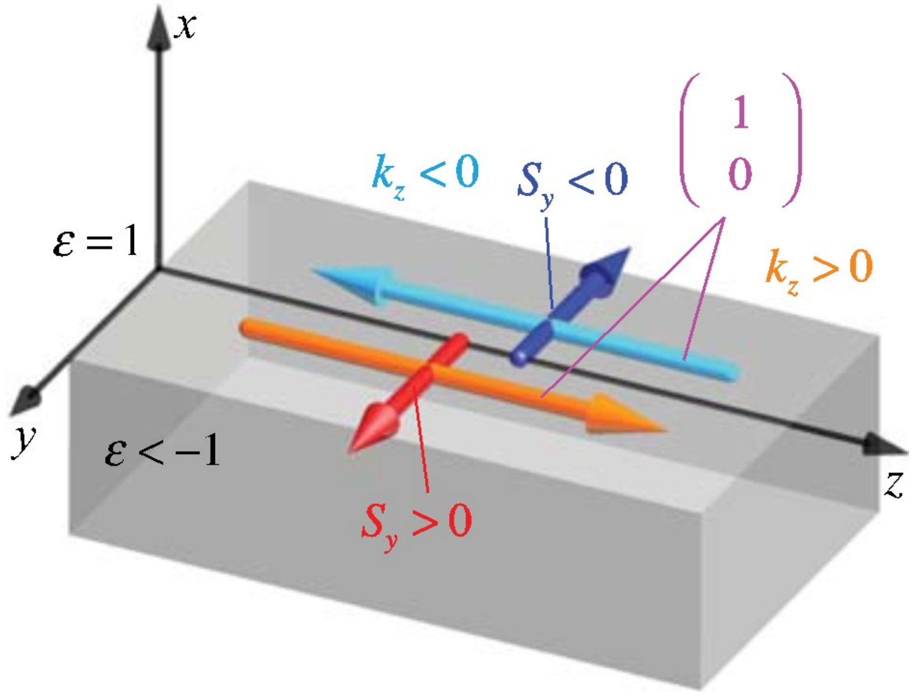
They report the simultaneous measurement of all components of this polarization-dependent **optical force by using a 3D force spectroscopy technique with femtonewton sensitivity.**

The vector force fields are compared quantitatively with theoretical calculations as the polarization state of the incident light is varied and show excellent agreement.

By plotting the 3D motion of the Mie particle in response to the switched force field, they obtained **visual evidence of the effect of spin momentum on the Poynting vector of an evanescent optical field.**

Quantum Spin Hall Effect (= QSHE) of light

This unusual transverse spin (independent of the polarization) survives in the TE or TM surface modes. Most importantly, opposite directions of propagation correspond to opposite transverse spins:



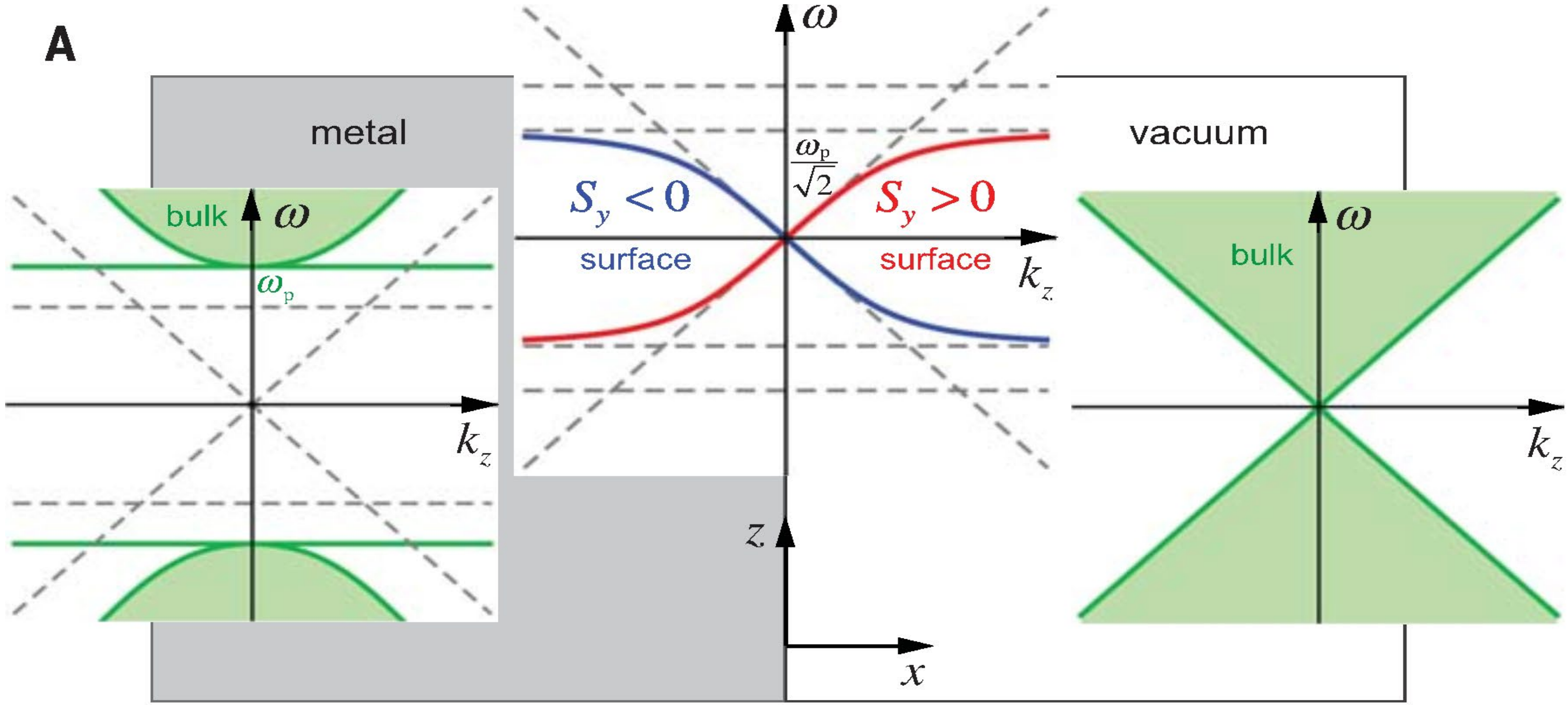
The QSHE of light !

$$\langle \mathbf{S}_{\text{surf}} \rangle = \frac{1}{\sqrt{-\varepsilon}} \bar{\mathbf{k}}_{\text{surf}} \times \bar{\mathbf{n}}$$

Bliokh, Smirnova, Nori, *Science* (2015)

Quantum Spin Hall Effect (= QSHE) of light

The metal-vacuum interface resembles (using a CM analogy) the interface between a semi-metal and an insulator.



Metal = Insulator (for photons! Below ω_p)

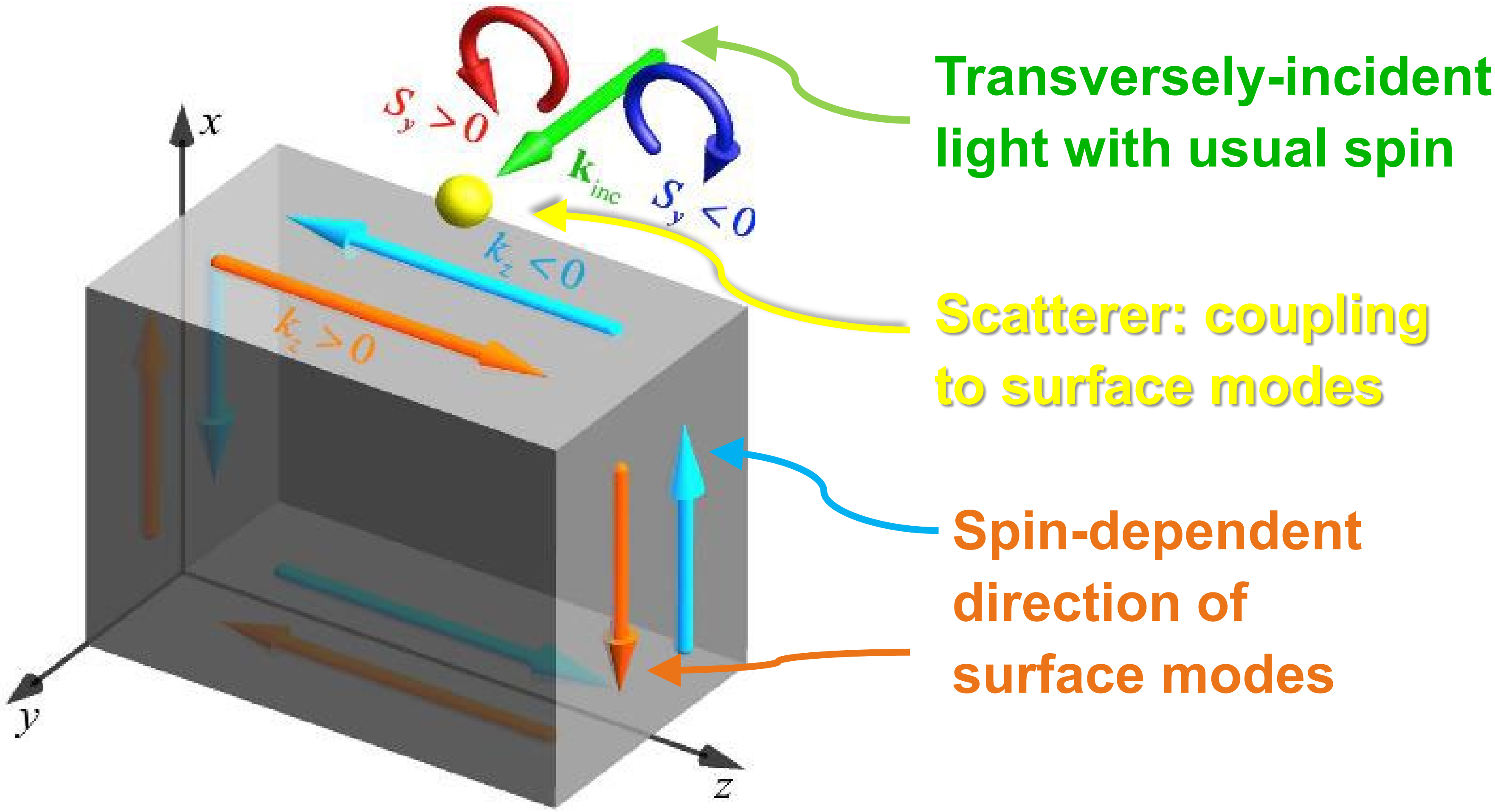
Optical spin-momentum locking has been observed in many separate experiments.

Some involved **Surface Plasmon Polaritons** (SPPs) at metal-vacuum interfaces (which resemble the interface between a semi-metal and an insulator).

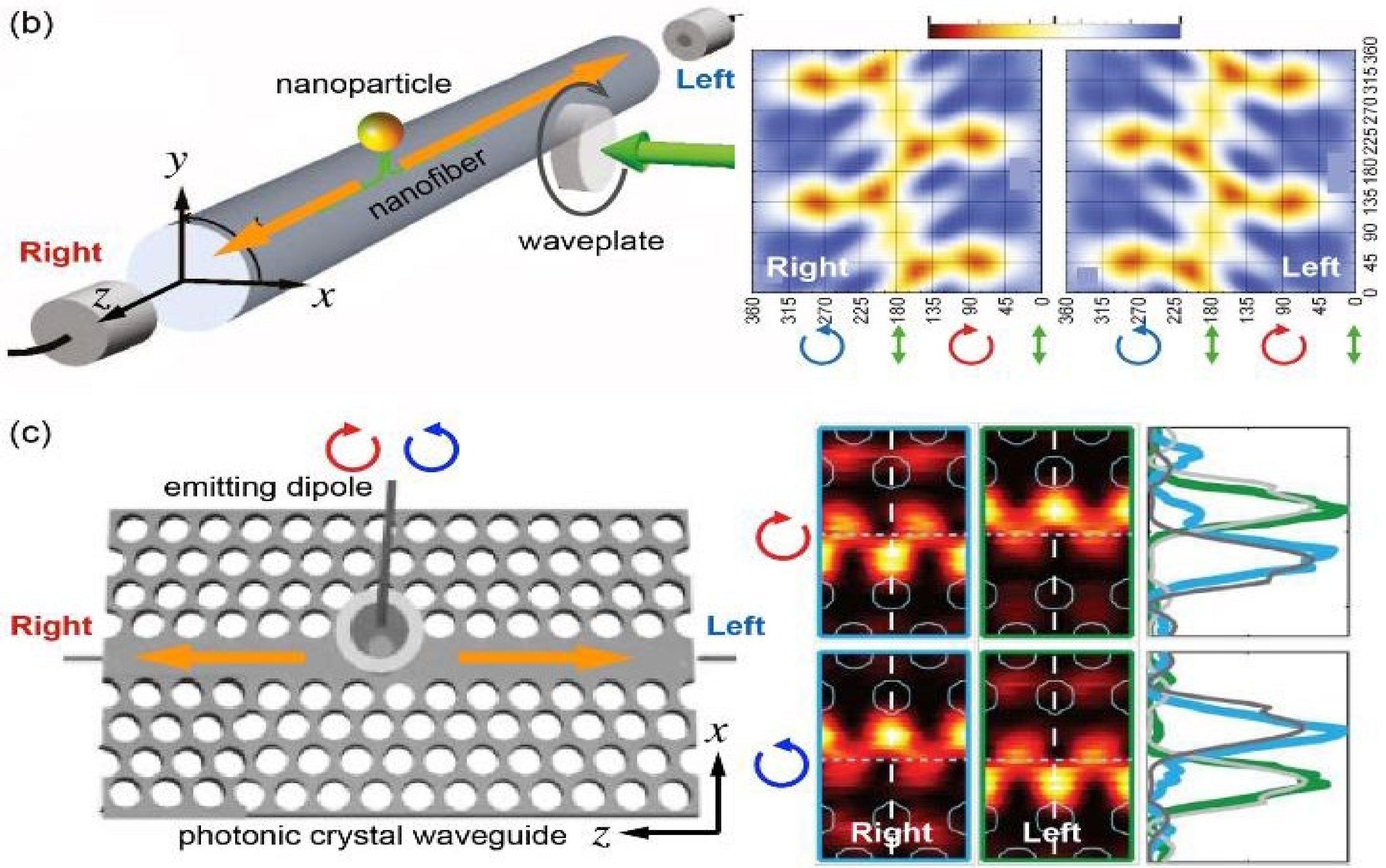
The SSP modes exhibit spin-momentum locking, which are typical for electron QSHE states.

Quantum Spin Hall Effect (= QSHE) of light: experiments

Since 2013 several groups have independently reported experiments on spin–dependent unidirectional excitation of surface or waveguided Maxwell waves:

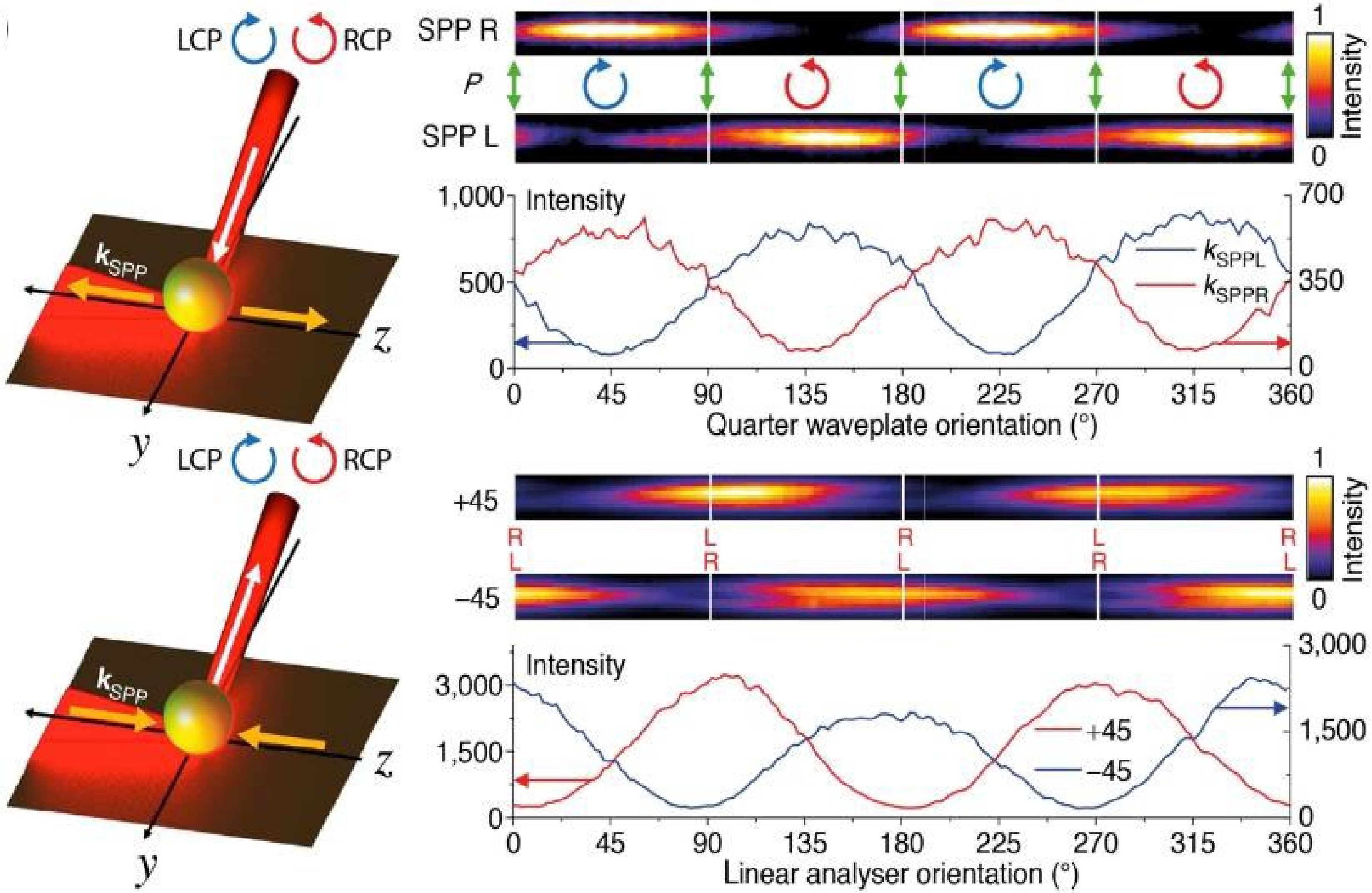


Quantum Spin Hall Effect (= QSHE) of light: experiments



Petersen *et al.*, *Science* (2014); Mitsch *et al.*, *Nature Commun.* (2014)
le Feber *et al.* *Nature Commun.* (2015)

Quantum Spin Hall Effect (= QSHE) of light: experiments



O'Connor *et al.*, *Nature Commun.* (2014)

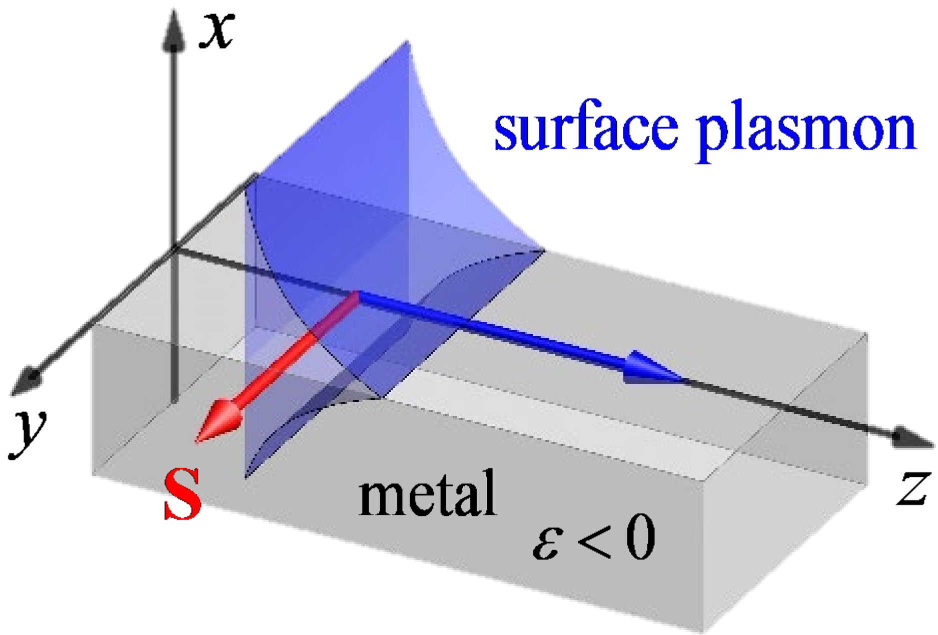
- ✓ **We have shown that pure free-space light already possesses intrinsic QSHE,**
- ✓ **and that simple natural materials (such as metals supporting surface plasmon-polariton modes) exhibit features that resemble topological insulators.**
- ✓ **We show that the transverse spin in evanescent waves (our work from 2012-14) and spin-controlled unidirectional excitation of surface or waveguide modes can be interpreted as manifestations of the QSHE of light.**
- ✓ **Light possesses intrinsic QSHE, i.e., strong spin–momentum locking in surface Maxwell modes.**
- ✓ **The transverse polarization–independent spin in evanescent waves (stemming from the transversality and SOI of light) is responsible for it.**
- ✓ **It differs in its origin from the QSHE of electrons (fermions). Spin–momentum coupling rather than spinor–momentum coupling.**
- ✓ **It seems that the dual symmetry between magnetic and electric properties plays an important role in the QSHE of light, but this is not fully clarified yet.**

The question

Since 2014, we were considering this question:

Can the very basic surface waves in Maxwell equations be described as topological surface modes?

Do surface plasmons at metal–dielectric interfaces have a topological origin?



Topological non-Hermitian origin of surface Maxwell waves

Konstantin Y. Bliokh, D. Leykam, M. Lein, F. Nori

Nature Communications, **10**, 580 (2019)

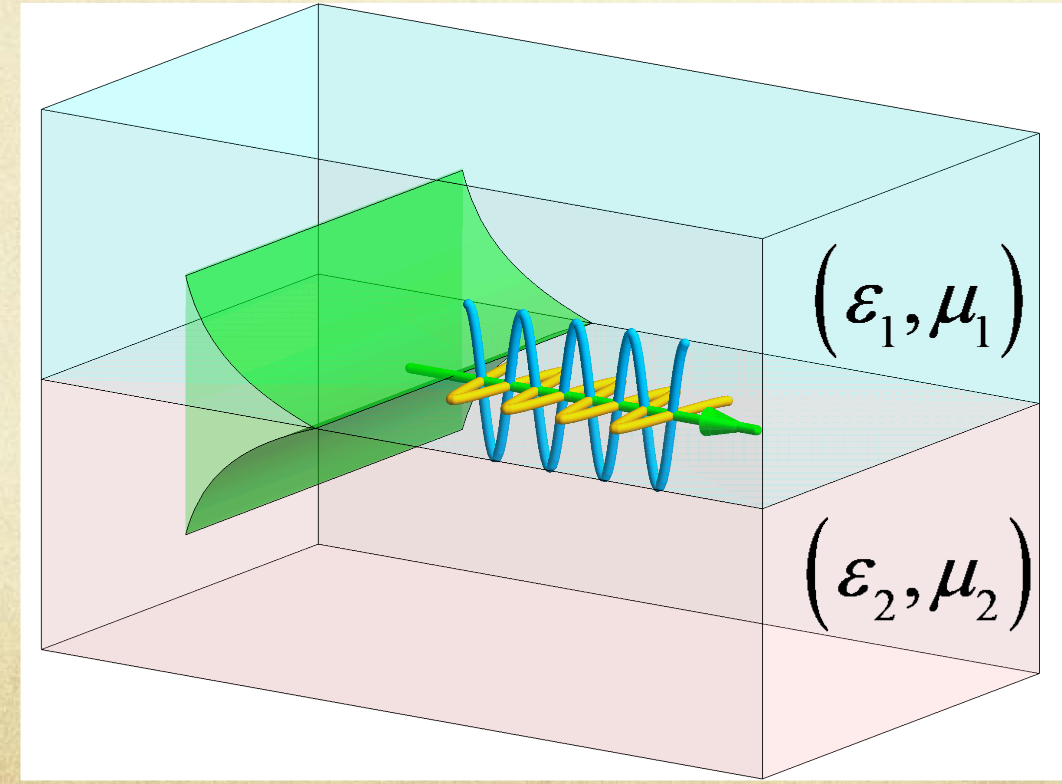
Surface modes of Maxwell equations

Surface Maxwell modes

Maxwell equations (isotropic, homogeneous, lossless):

$$\nabla \times \mathbf{E} = -\mu \partial_t \mathbf{H}, \quad \nabla \times \mathbf{H} = \varepsilon \partial_t \mathbf{E}, \quad \nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{E} = 0$$

Surface modes at an interface between two media:

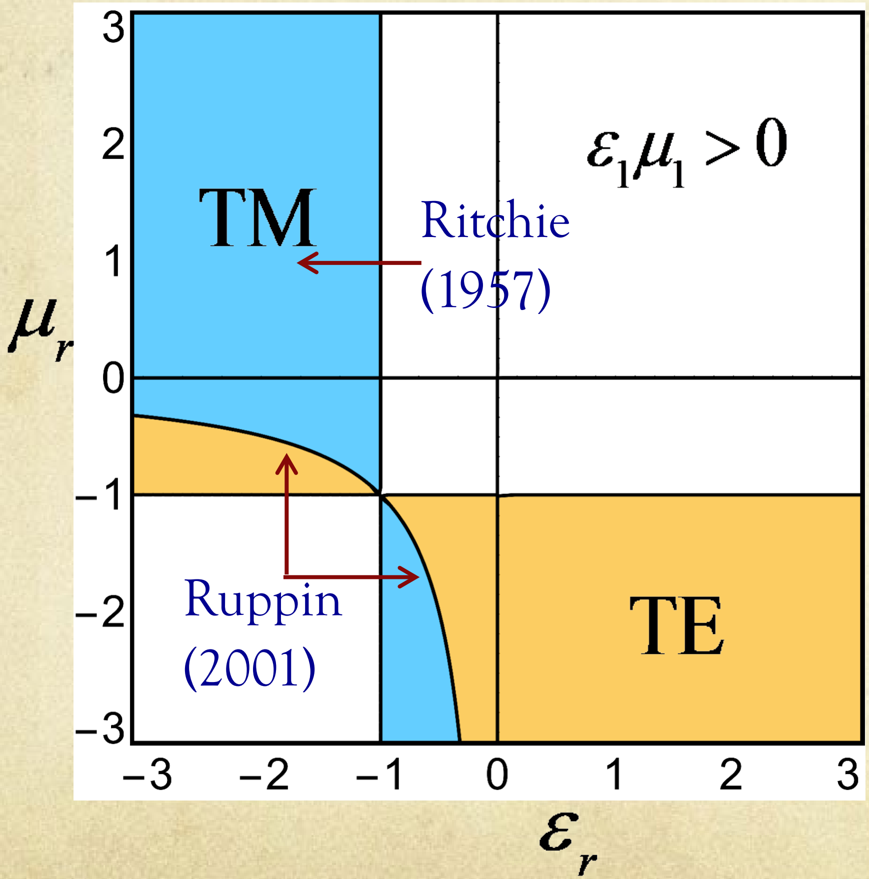


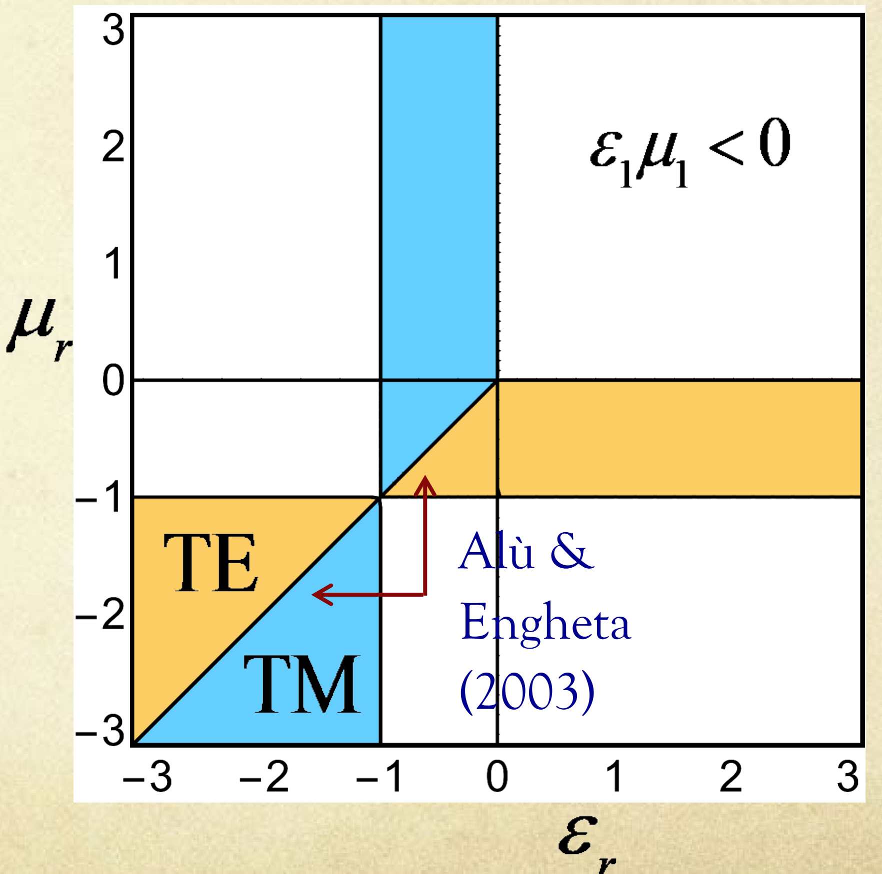
TM polarization

TE polarization

Surface Maxwell modes

"Phase diagram" of surface TE/TM Maxwell modes in the $(\varepsilon_r, \mu_r) = (\varepsilon_2/\varepsilon_1, \mu_2/\mu_1)$ plane:





Shadrivov *et al.* *PRE* (2004); Kats *et al.* *PRL* (2007); Bliokh *et al.* *Nat. Comm.* (2019)

Surface Maxwell modes

These diagrams are deformed with deformations of the interface (see, e.g., surface plasmons on a sphere), but there are some general robust features:

- The TM-mode can exist only at interfaces where the permittivity changes sign: $\mathrm{sgn}\left(\varepsilon_r\right) = -1$.
- The TE-mode can exist only at interfaces where the permeability changes sign: $\mathrm{sgn}\left(\mu_r\right) = -1$.

Why is this so? Is there a fundamental reason for this?

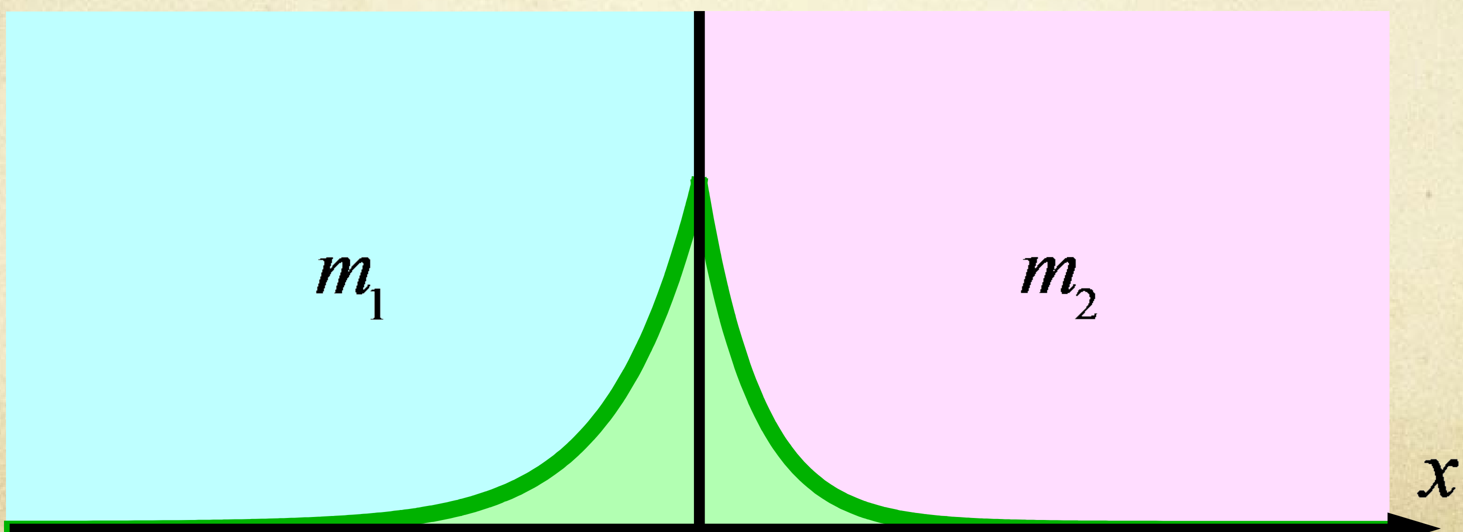
Topological surface modes of the Dirac equation

Topological Dirac modes

The Dirac equation (1D, for simplicity):

$$\hat{\mathcal{H}}\psi \equiv \hat{p}\hat{\sigma}_z + m\hat{\sigma}_x \equiv \begin{pmatrix} -i\partial_x & m \\ m & i\partial_x \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \omega\psi$$

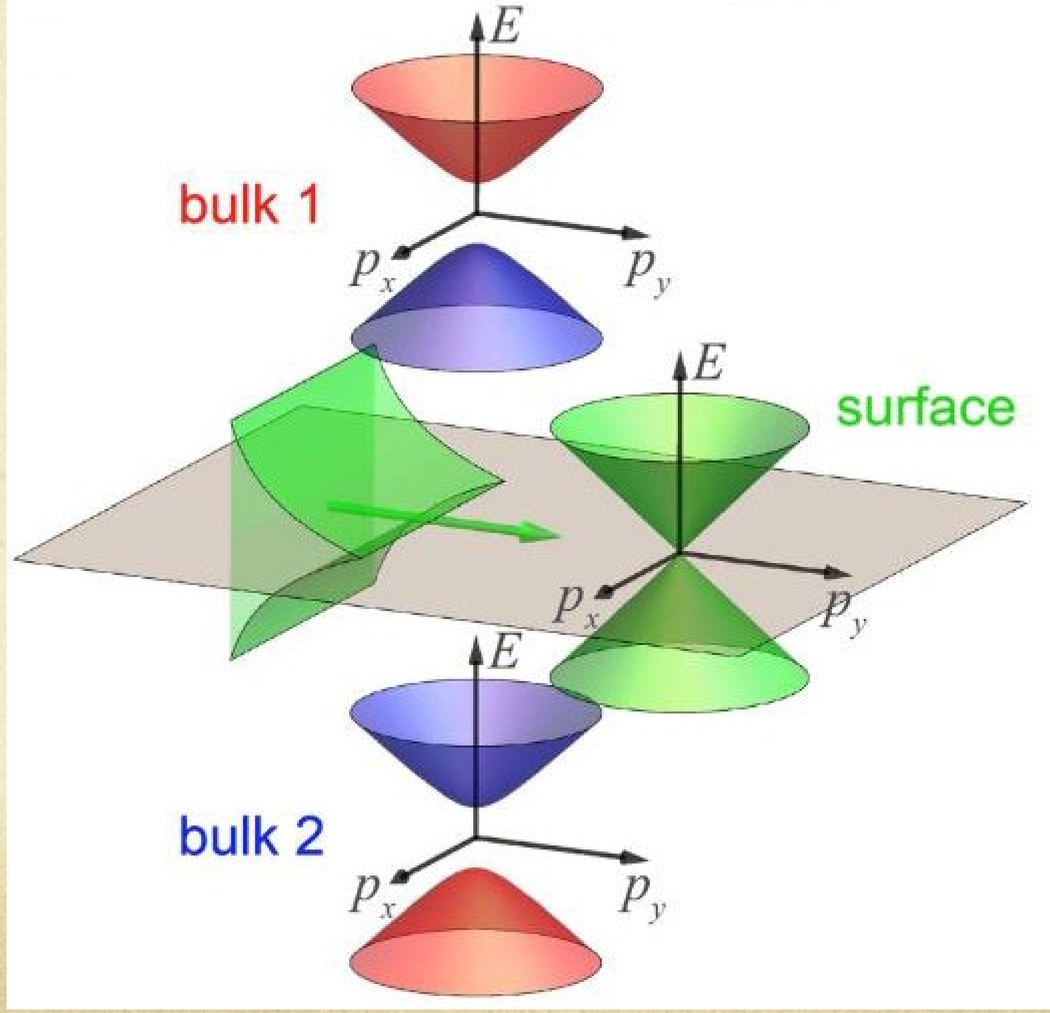
The surface mode can appear at an interface between two media with different masses:



Jackiw & Rebbi *PRD* (1976)

Topological Dirac modes

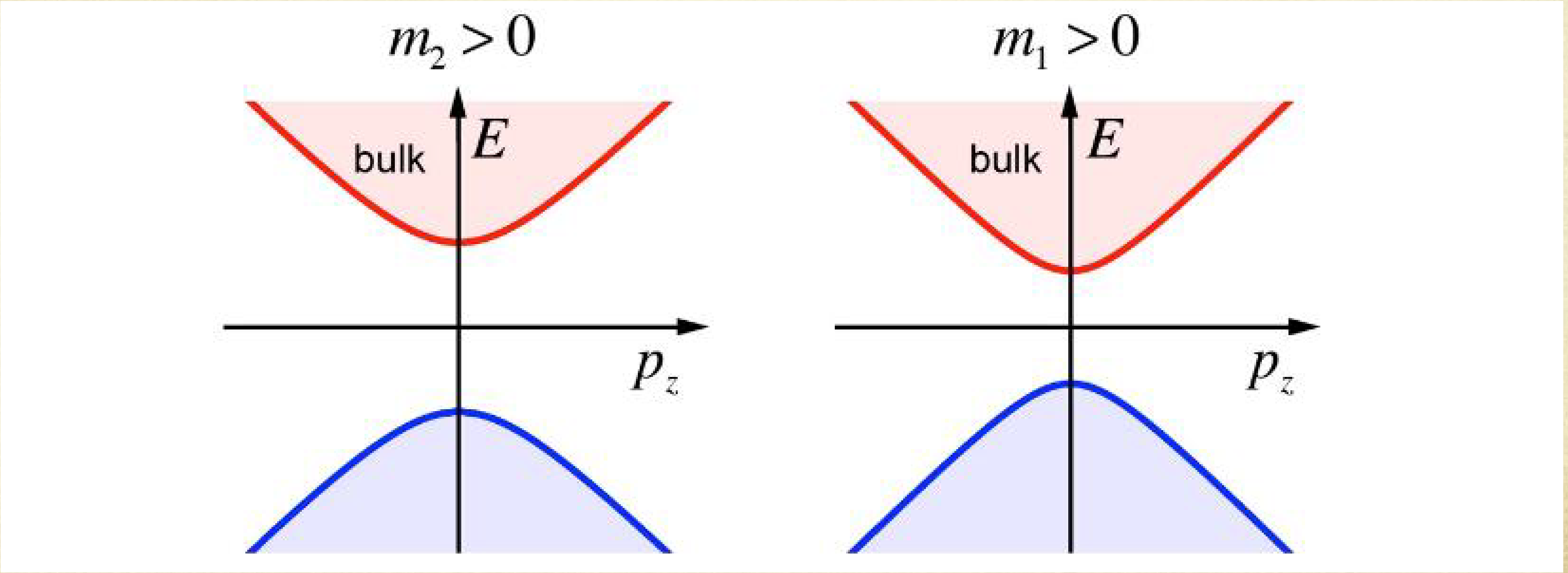
The same mode occurs in 2D and 3D and it has a zero-mass spectrum:



Topological Dirac modes

Most importantly, the Dirac surface mode appears only at interfaces where the mass changes its sign: $\mathrm{sgn}(m_r) = -1$.

This is related to the nontrivial Möbius-strip-like (Z_2) topology of the Dirac Hamiltonian and bulk eigenmodes:



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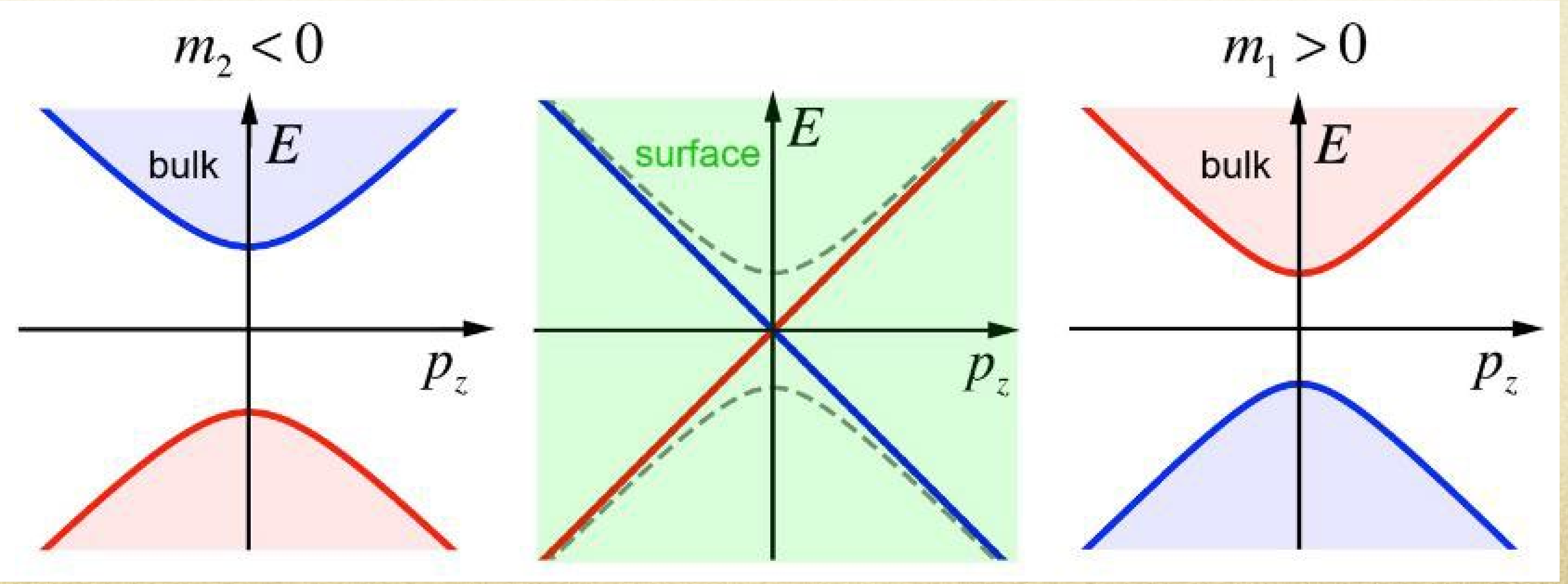
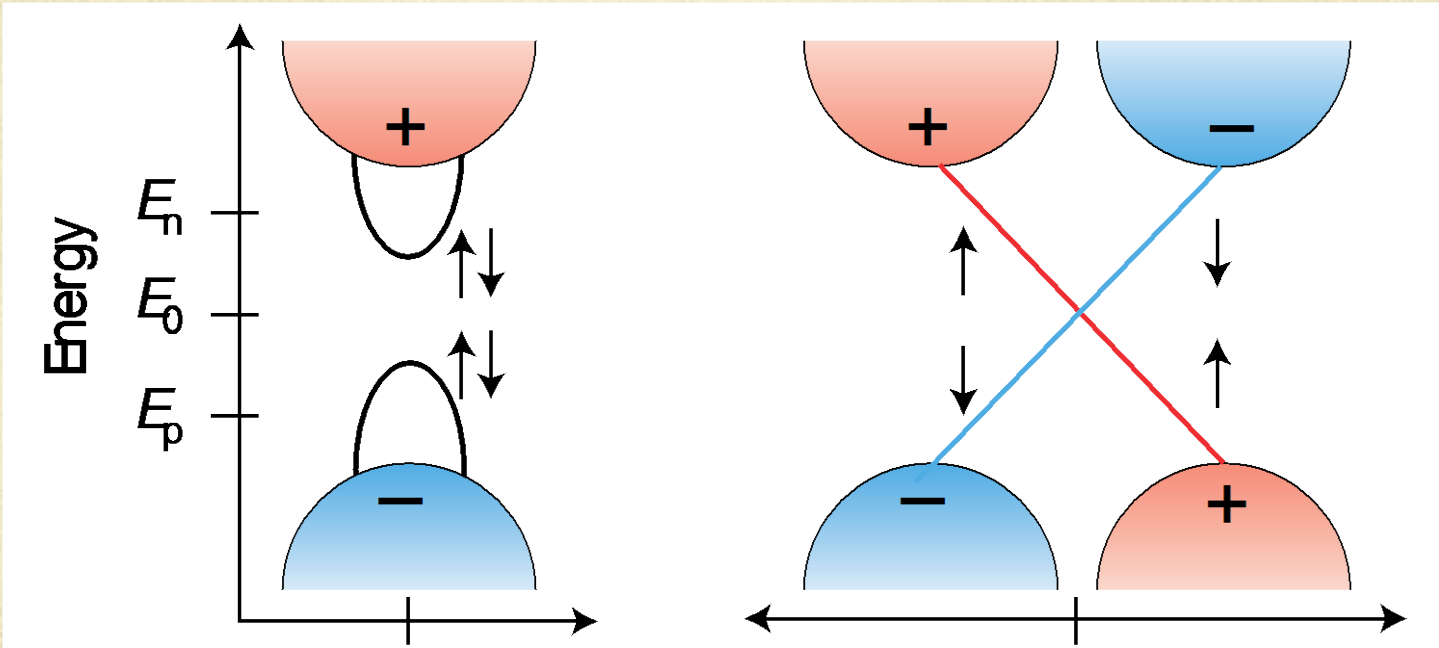
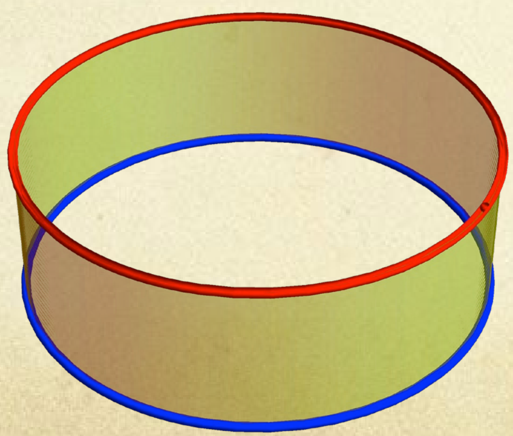
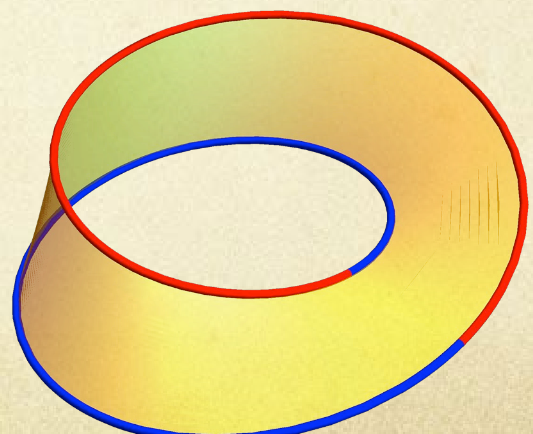


Topological Dirac modes







Hasan & Kane *RMP* (2010), Qi & Zhang (2010), ...

Topological Dirac modes

One can introduce a topological Z_2 invariant:

$$w(m) = \frac{1}{2}\left[1 - \operatorname{sgn}(m)\right] = (0,1)$$

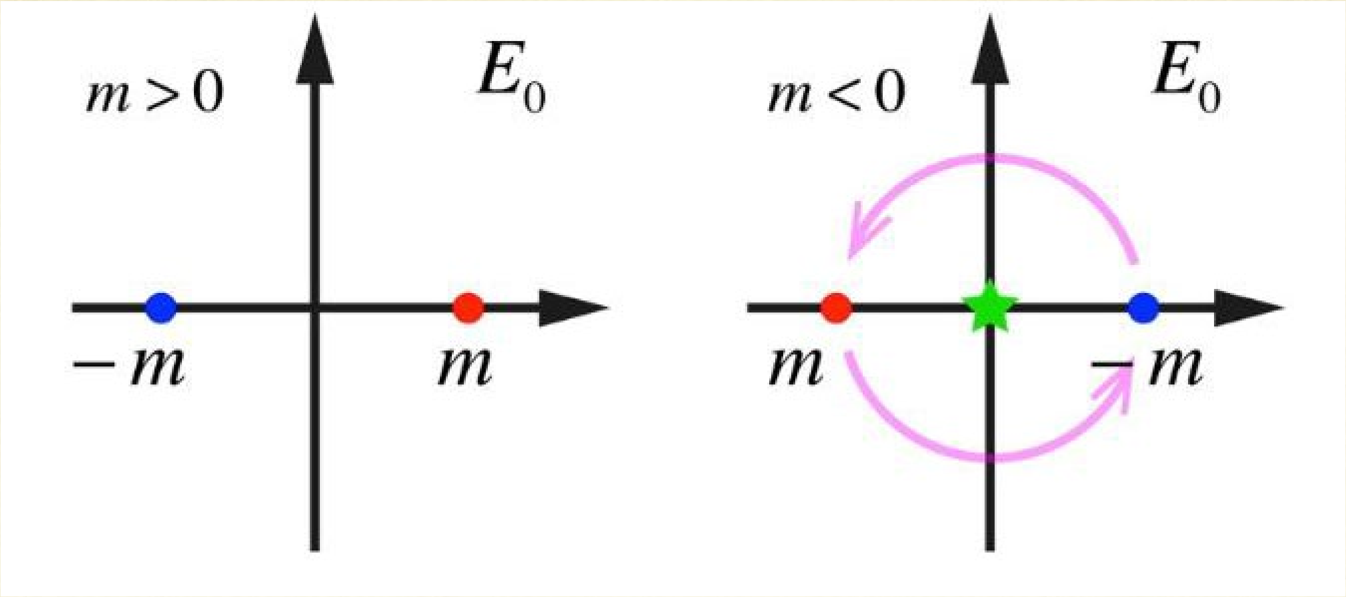
There is a bulk-boundary correspondence, which determines the number of surface modes:

$$N = \left| w(m_2) - w(m_1) \right| = w(m_r) = (0,1)$$

Hasan & Kane *RMP* (2010), Qi & Zhang (2010), ...

Topological Dirac modes

Notably, one can present the flip in the mass sign as a π rotation in the complex-mass plane:



$$w(m)=\frac{1}{\pi}\mathrm{Arg}(m)=(0,1)$$

Bliokh *et al. Nat. Commun.* (2019)

Weyl form of Maxwell equations and topological properties of the helicity

Maxwell equations and helicity

Relativistic Dirac/Weyl-like form of Maxwell equations:

$$\left(\hat{\mathbf{S}}\cdot\hat{\mathbf{p}}\right)\Psi = i\hat{\sigma}^{(\mathrm{m})}\,\partial_t\Psi$$

$\hat{\mathbf{p}} = -i\nabla,\quad \hat{\mathbf{S}} = \text{spin-1 matrices}$

$$\begin{pmatrix} \nabla\times & 0 \\ 0 & \nabla\times \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \omega\begin{pmatrix} 0 & i\mu \\ -i\varepsilon & 0 \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

Here the key operator is **helicity**:

$$\hat{\mathfrak{S}} = \frac{\hat{\mathbf{S}}\cdot\hat{\mathbf{p}}}{|\mathbf{p}|} = \frac{1}{\left|\sqrt{\varepsilon\mu}\right|}\begin{pmatrix} 0 & i\mu \\ -i\varepsilon & 0 \end{pmatrix}$$

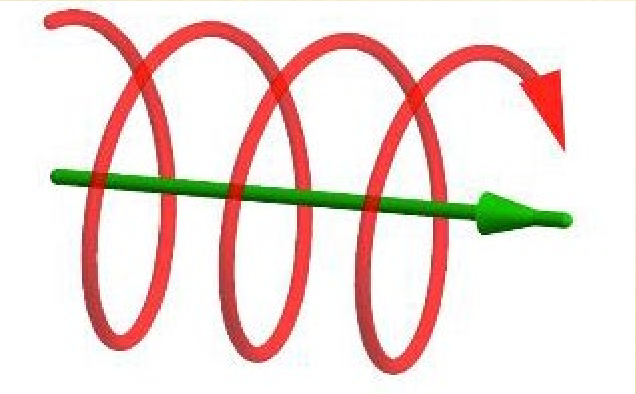
Majorana, Oppenheimer, Berry, Bialynicki-Birula, ..., Alpeggiani *et al. PRL* (2018)

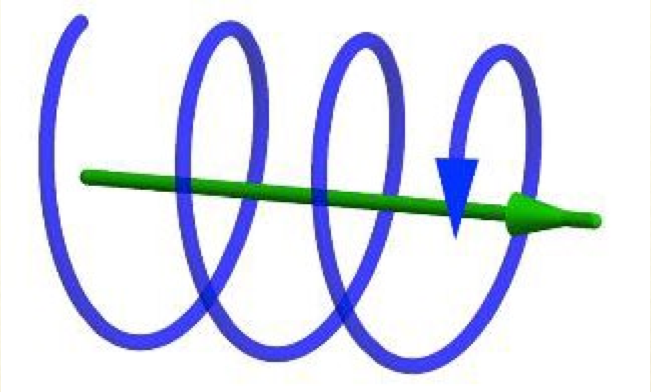
Maxwell equations and helicity

Helicity is a very important property of light, which is determined by the projection of spin $\mathbf{S}$ onto the momentum direction $\mathbf{p}/|\mathbf{p}|$.

It is also related to the dual symmetry between $\mathbf{E}$ and $\mathbf{H}$.

Free-space light has two helicity eigenstates with $\mathfrak{S} = \pm 1$ (e.g., circularly polarized plane waves):



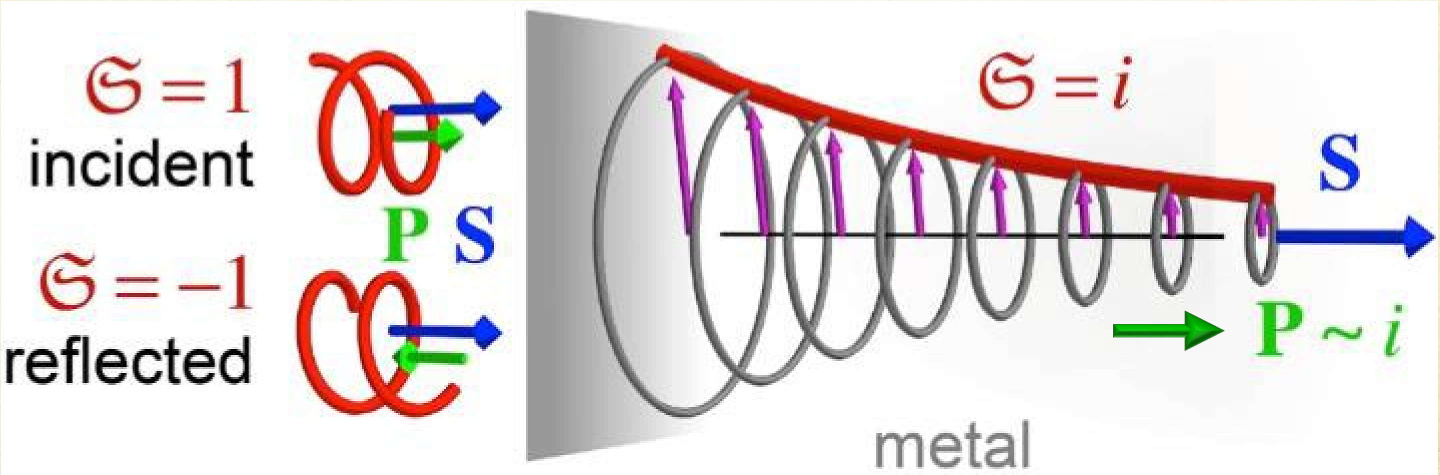


Calkin (1965); QED;...; G. Molina-Terriza *et al.*; Bliokh *et al.*

Maxwell equations and helicity

The helicity operator in a lossless medium is generally **non-Hermitian**, and its eigenvalues are complex: $|\mathfrak{S}| = 1$.

This makes physical sense: e.g., in metals ($\varepsilon\mu < 0$) the momentum (wavevector) becomes imaginary, while the spin (polarization rotation) remains real:

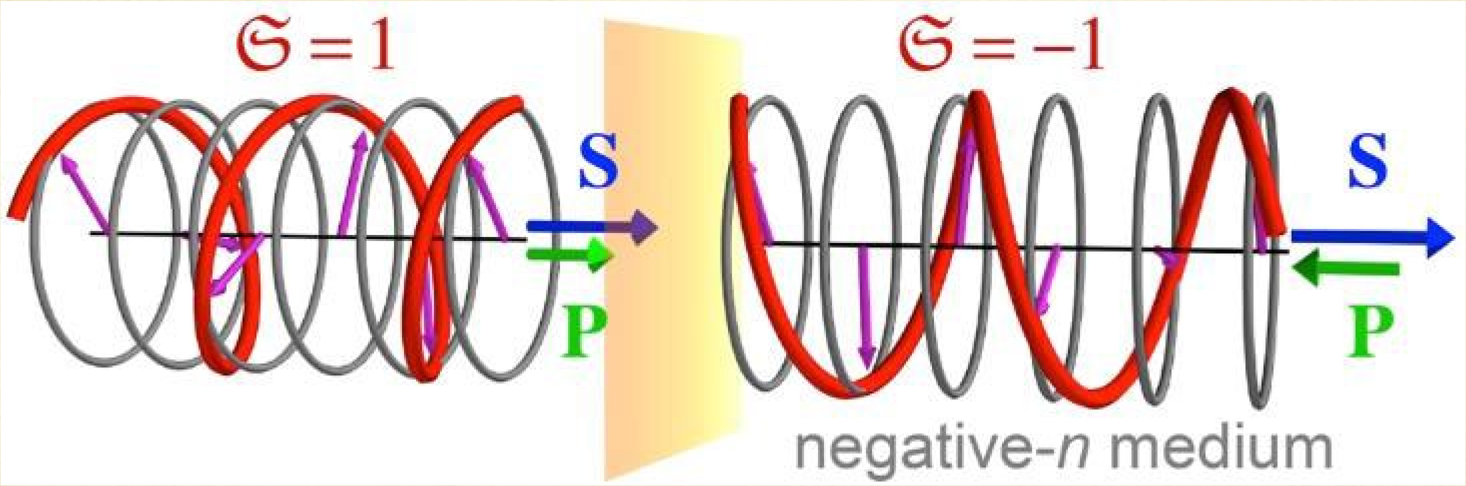


Alpeggiani *et al. PRL* (2018); Bliokh *et al. NC* (2019)

Maxwell equations and helicity

The helicity operator in a lossless medium is generally **non-Hermitian**, and its eigenvalues are complex: $|\mathfrak{S}| = 1$.

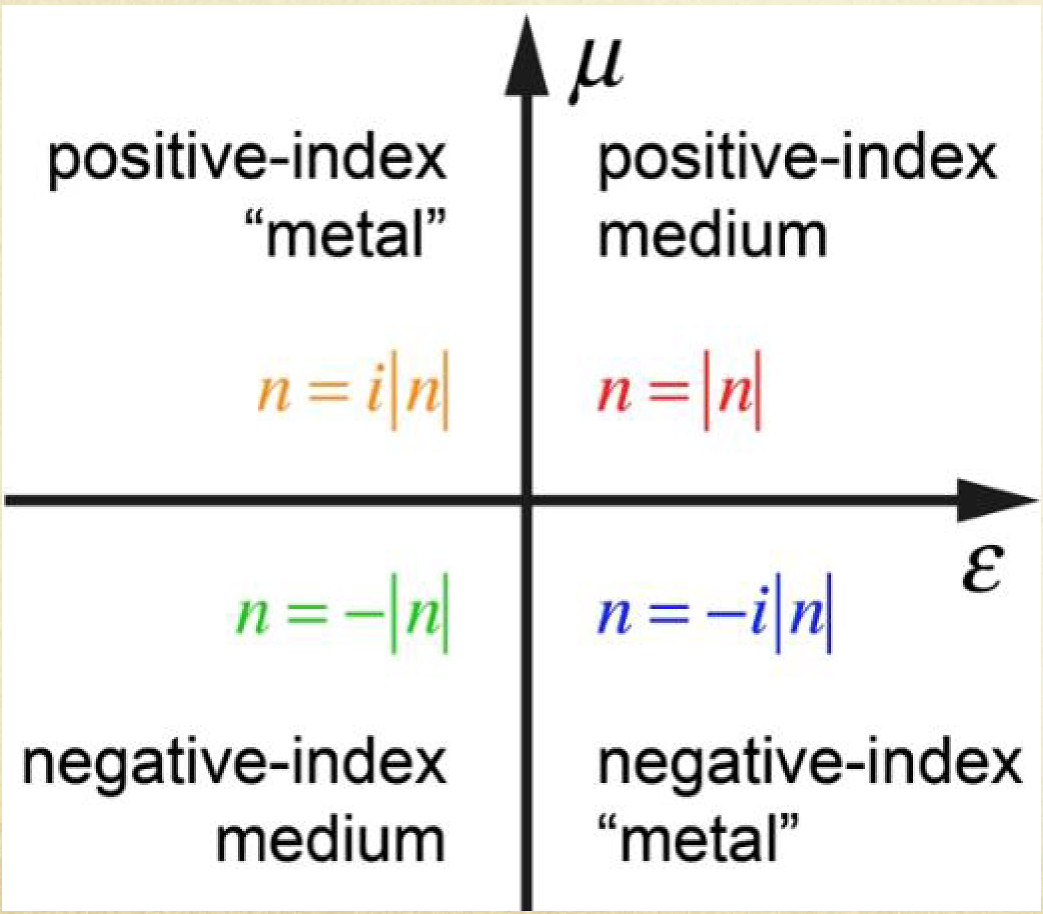
Furthermore, helicity changes its sign in negative-index media, $(\varepsilon, \mu) \to (-\varepsilon, -\mu)$: $\mathfrak{S} \to -\mathfrak{S}$ (because the momentum flips):



Alpeggiani *et al. PRL* (2018); Bliokh *et al. NC* (2019)

Maxwell equations and helicity

Summarizing this, we obtain properties of the photon helicity in the four types of optical isotropic media:



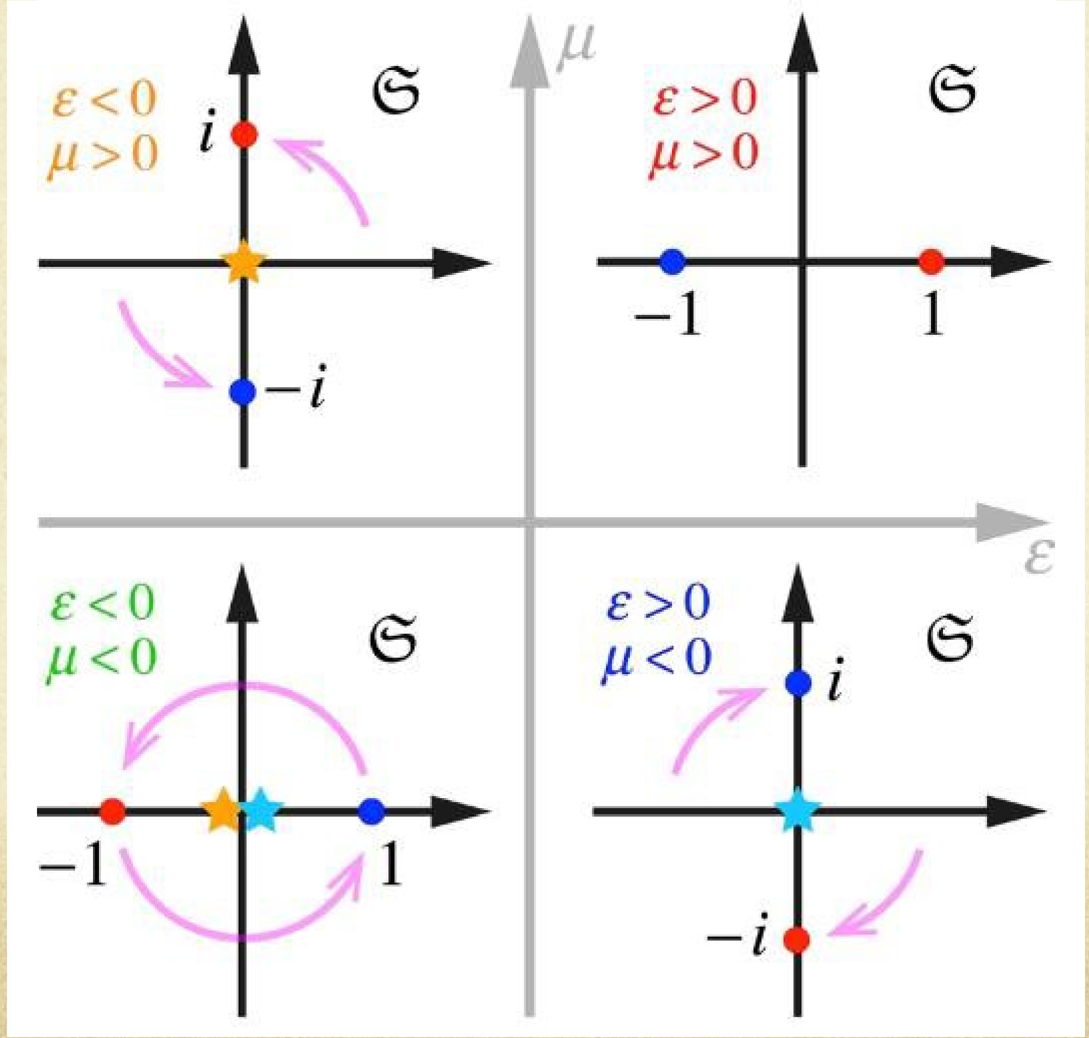
$$\hat{\mathfrak{S}} = \frac{1}{|n|}\begin{pmatrix} 0 & i\mu \\ -i\varepsilon & 0 \end{pmatrix}$$

$$\mathfrak{S} = \pm\frac{n}{|n|}$$

Bliokh *et al. Nat. Commun.* (2019)

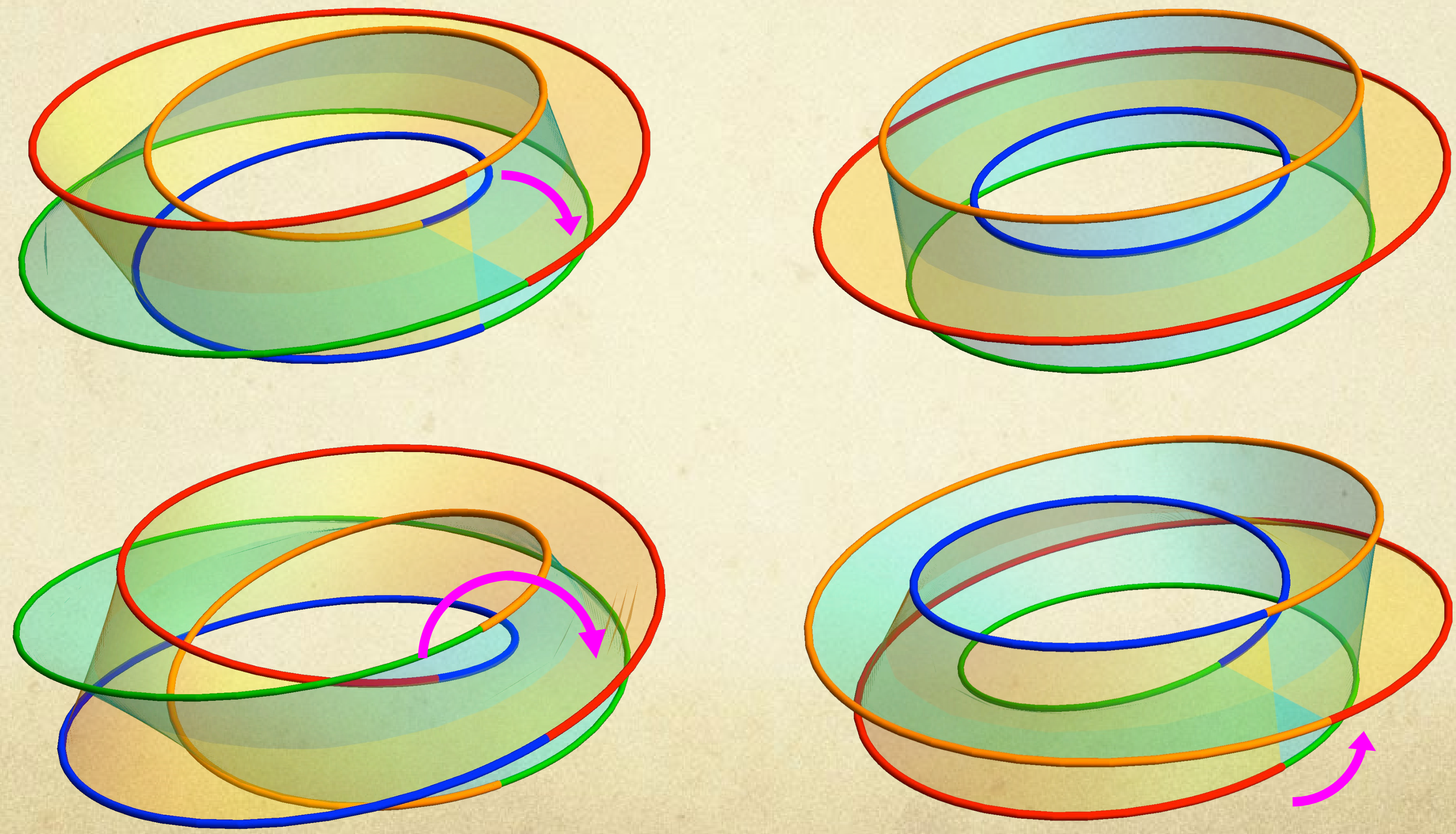
Maxwell equations and helicity

Transitions between these four types of media can be regarded as discrete rotations in the complex helicity plane:



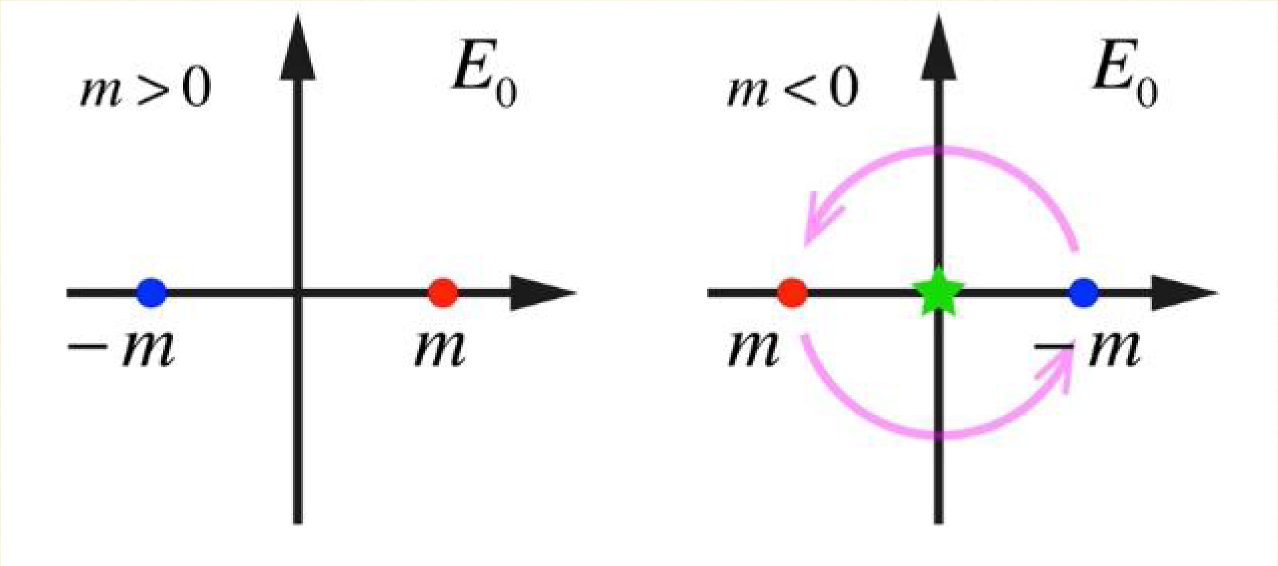
Maxwell equations and helicity

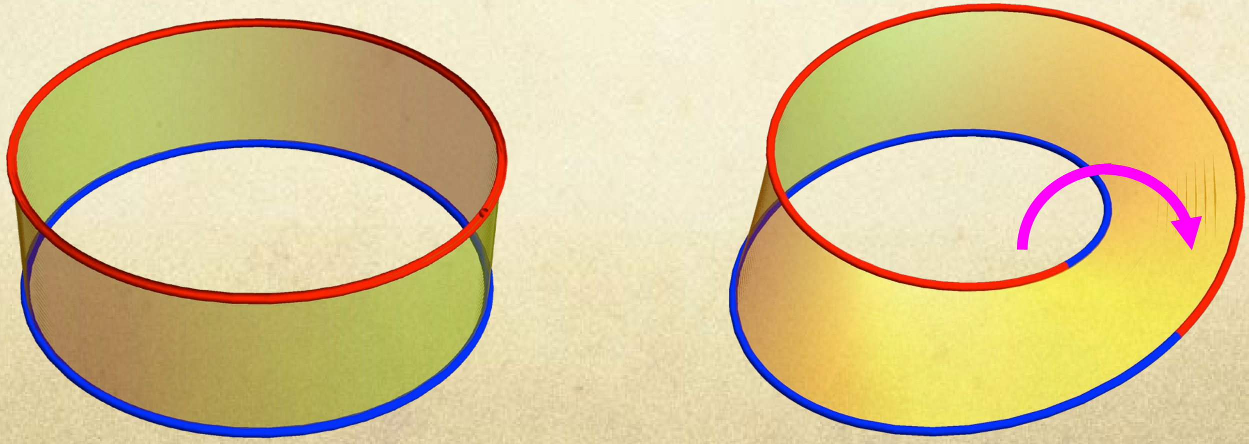
This maps the interface helicity properties to the $Z_4 = Z_2 \times Z_2$ (double-Möbius) group:



Maxwell equations and helicity

Comparing this with the Z_2 features of the Dirac Hamiltonian:





Maxwell equations and helicity

We introduce the topological Z_4 "helicity winding" number:

$$w(\varepsilon,\mu)=\frac{2}{\pi}\mathrm{Arg}(n)=(0,1,2,-1)$$

or, equivalently, two topological Z_2 numbers:

$$w^{\mathrm{TM}}(\varepsilon)=\frac{1}{2}\left[1-\mathrm{sgn}(\varepsilon)\right]=(0,1)$$

$$w^{\mathrm{TE}}(\mu)=\frac{1}{2}\left[1-\mathrm{sgn}(\mu)\right]=(0,1)$$

Bliokh *et al. Nat. Commun.* (2019)

Maxwell equations and helicity

One fundamental novelty should be emphasized:

All previous works on topological insulators considered topological properties of the Hamiltonian operator.

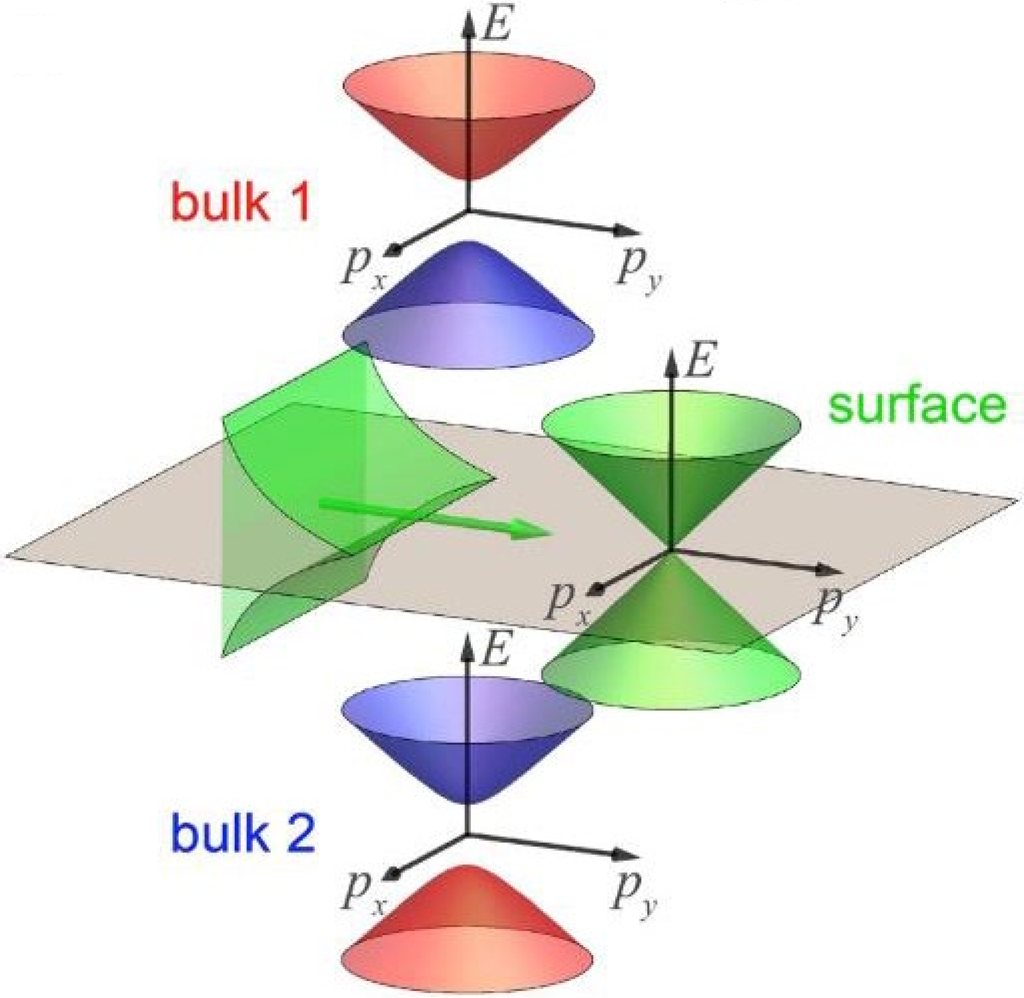
In contrast, we consider the topology of the helicity operator in Maxwell equations.

$$\text{topology of } \hat{\mathcal{H}} \;\rightarrow\; \text{topology of } \hat{\mathfrak{S}}$$

This considerably extends the topological approach: any conserved-quantity operator can be considered.

Topological Dirac modes

The 1D mode of Rebbi-Jackiw also occurs in 2D and 3D and it has a zero–mass spectrum:

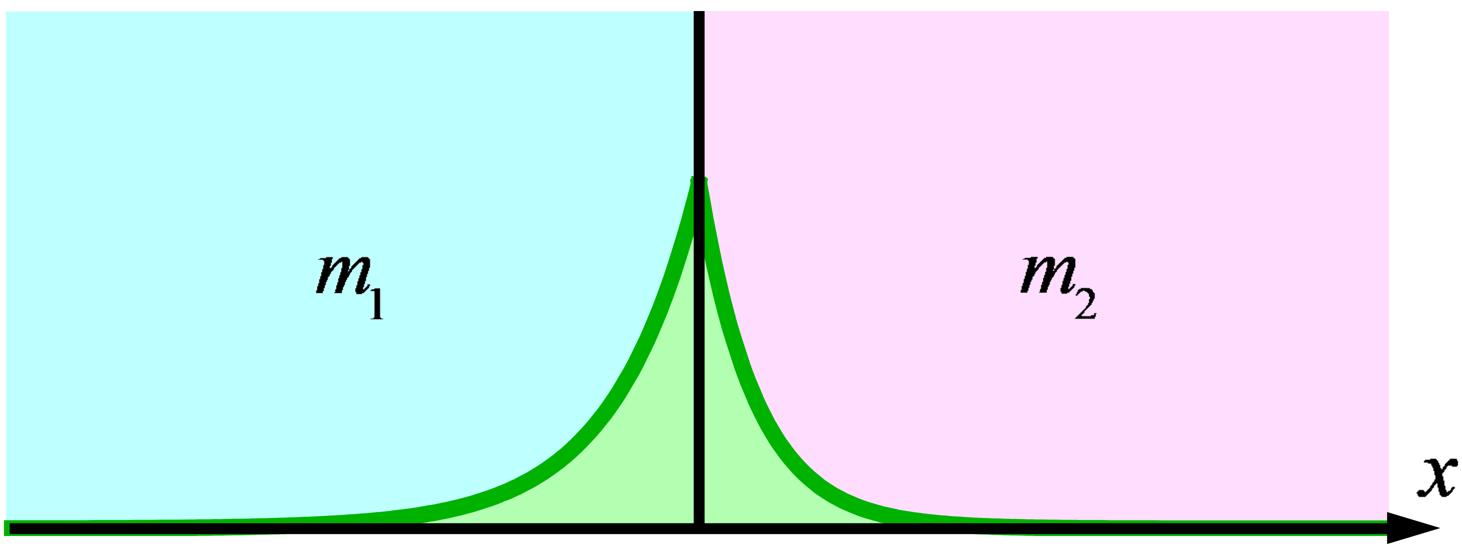


Topological Dirac modes

The Dirac equation (1D, for simplicity):

$$\hat{\mathcal{H}}\psi \equiv \hat{p}\hat{\sigma}_z + m\hat{\sigma}_x \equiv \begin{pmatrix} -i\partial_x & m \\ m & i\partial_x \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \omega\psi$$

The surface mode can appear at an interface between two media with different masses:

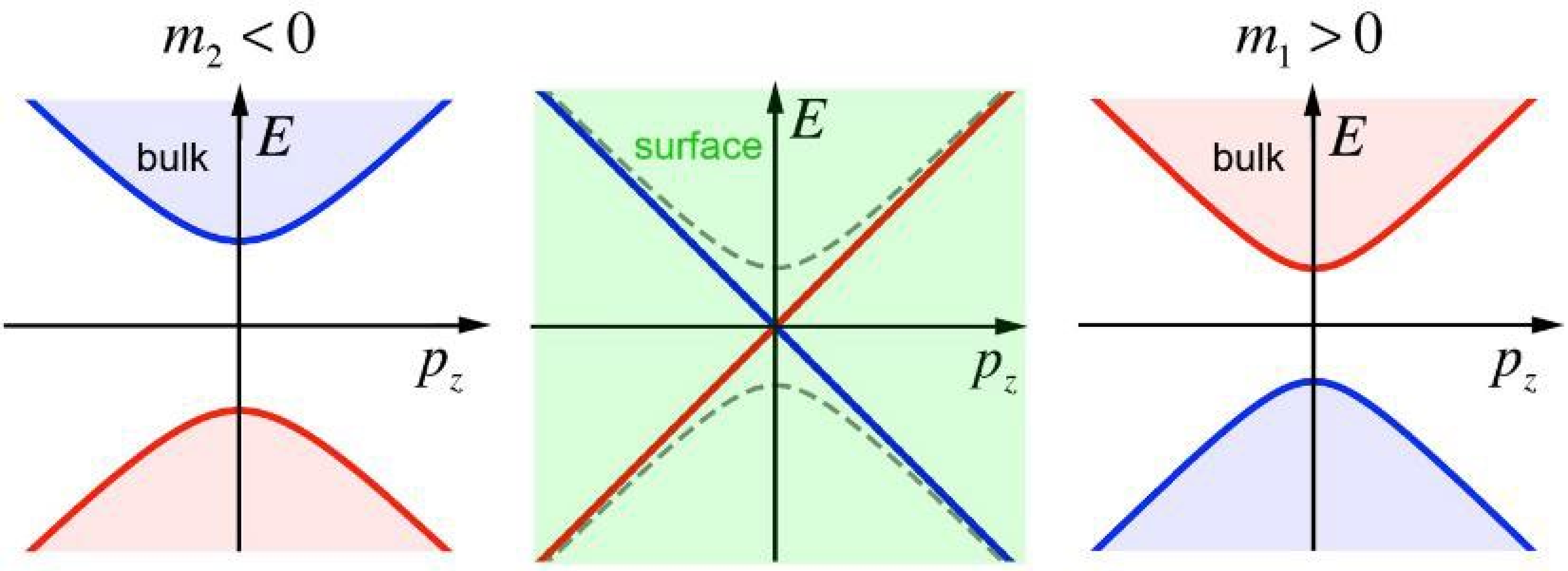


Jackiw & Rebbi *PRD* (1976)

Topological Dirac modes

Most importantly, the Dirac surface mode appears only at **interfaces where the mass changes its sign**: $\mathrm{sgn}(m_r) = -1$.

This is related to the nontrivial Möbius–strip–like (Z_2) topology of the Dirac Hamiltonian and bulk eigenmodes:



Topological Dirac modes

One can introduce a topological Z_2 invariant:

$$w(m) = \frac{1}{2}\left[1 - \operatorname{sgn}(m)\right] = (0,1)$$

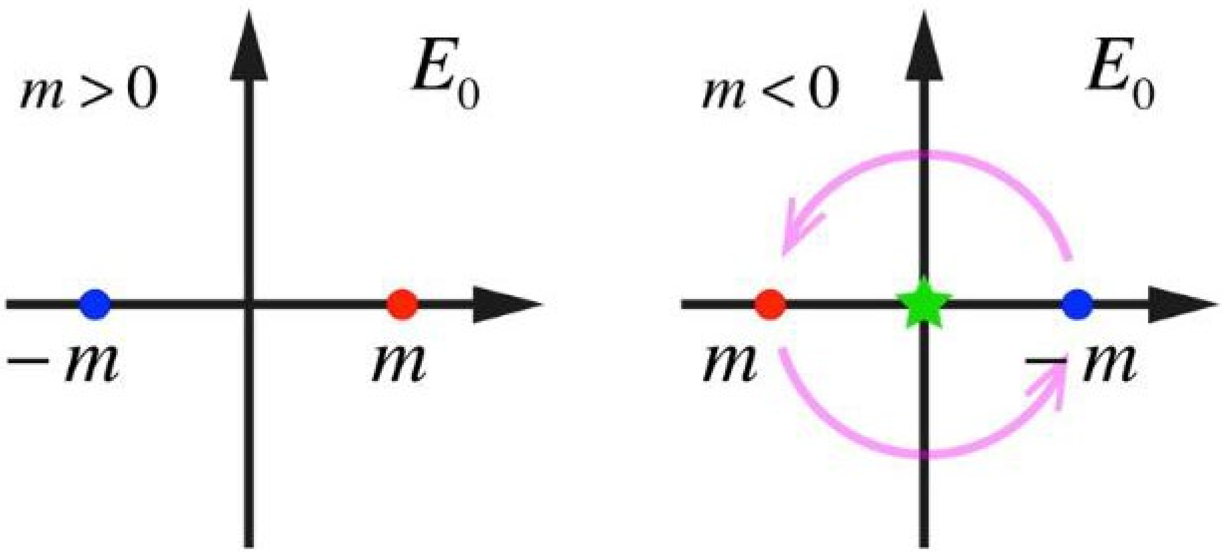
There is a bulk–boundary correspondence, which determines the number of surface modes:

$$N = \left| w(m_2) - w(m_1) \right| = w(m_r) = (0,1)$$

Hasan & Kane *RMP* (2010), Qi & Zhang (2010), …

Topological Dirac modes

Notably, one can represent the flip in the mass sign as a π rotation in the complex–mass plane:



$$w(m) = \frac{1}{\pi}\mathrm{Arg}(m) = (0,1)$$

Bliokh *et al. Nat. Commun.* (2019)

Maxwell equations and helicity

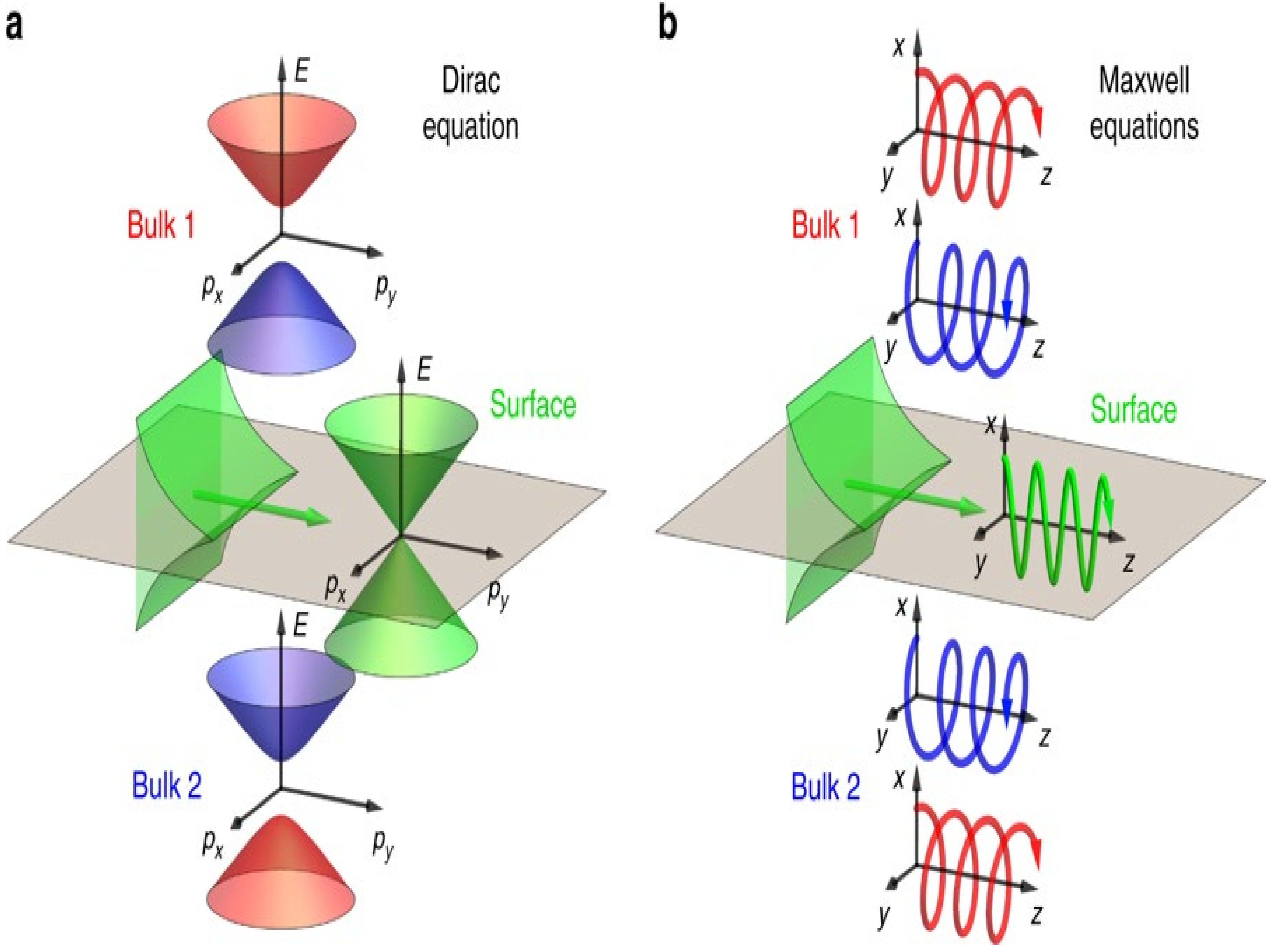
One fundamental novelty should be emphasized:

All previous works on topological insulators considered topological properties of the Hamiltonian operator.

In contrast, we consider the topology of the helicity operator in Maxwell equations.

$$\text{topology of } \hat{\mathcal{H}} \;\rightarrow\; \text{topology of } \hat{\mathfrak{S}}$$

This considerably extends the topological approach: any conserved–quantity operator can be considered.



a
E
Dirac
equation
Bulk 1
p_x
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E
Surface
p_x
p_y
E
Bulk 2
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Maxwell
equations
Bulk 1
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Surface
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Bulk 2
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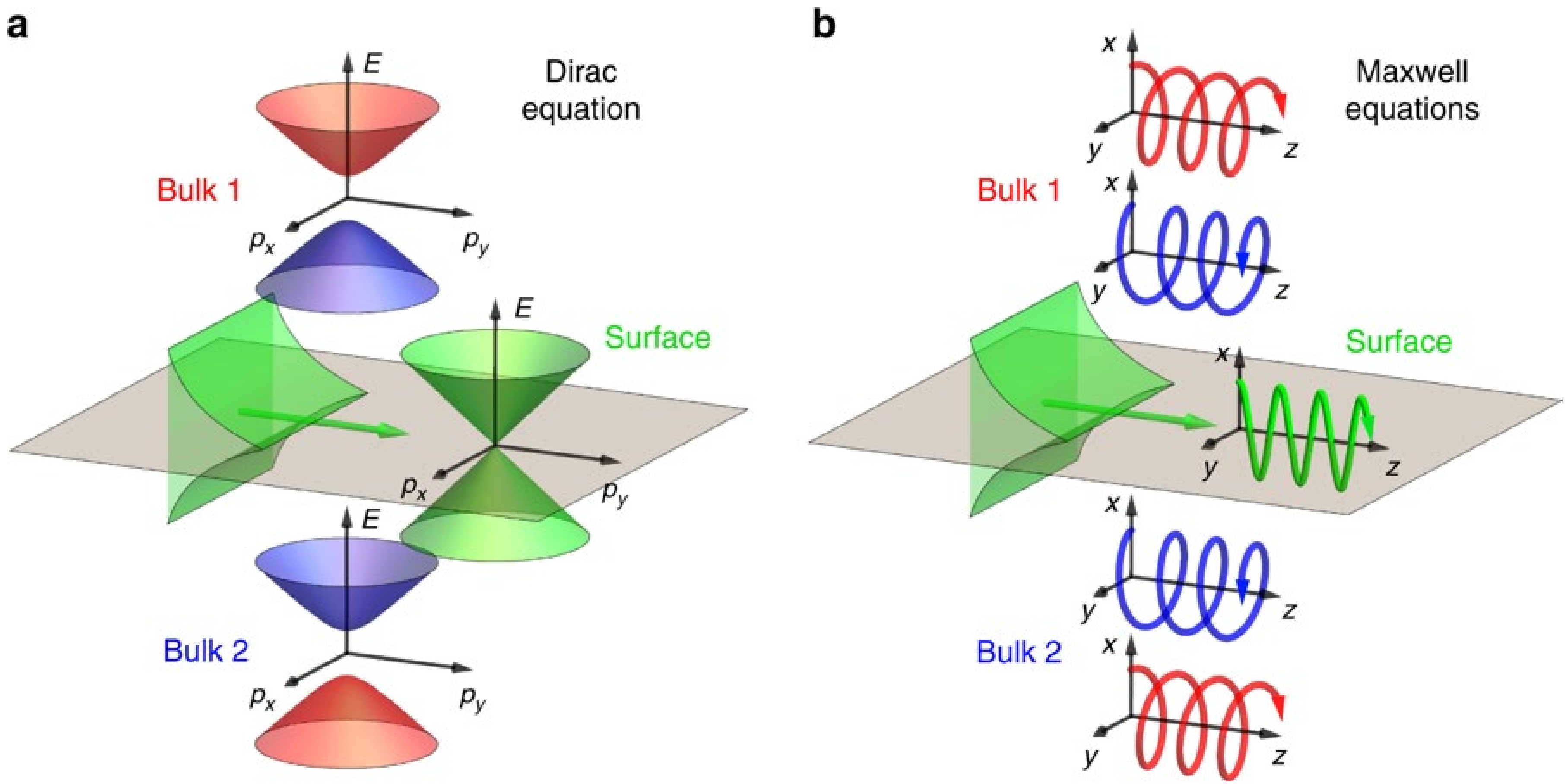
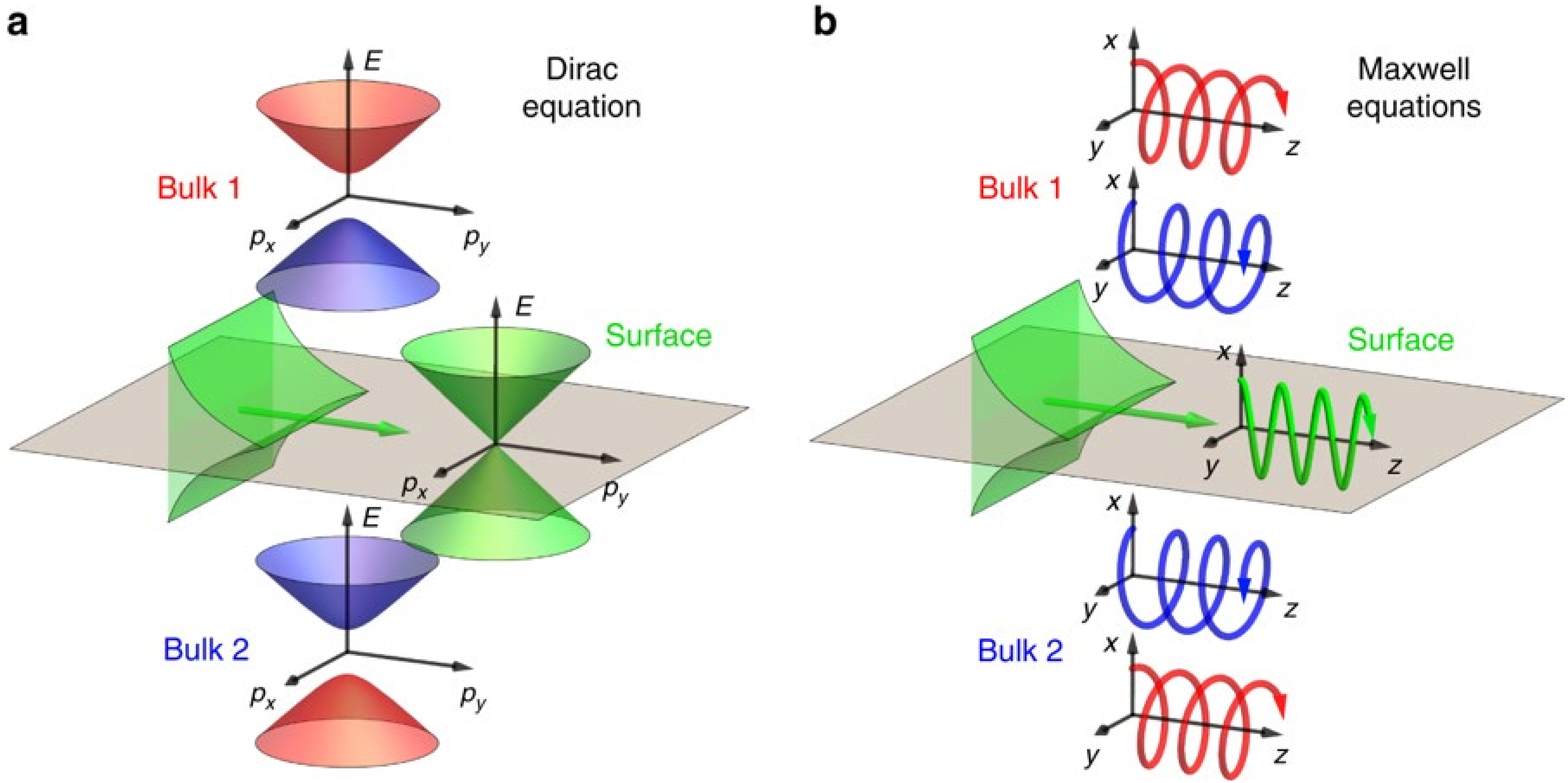
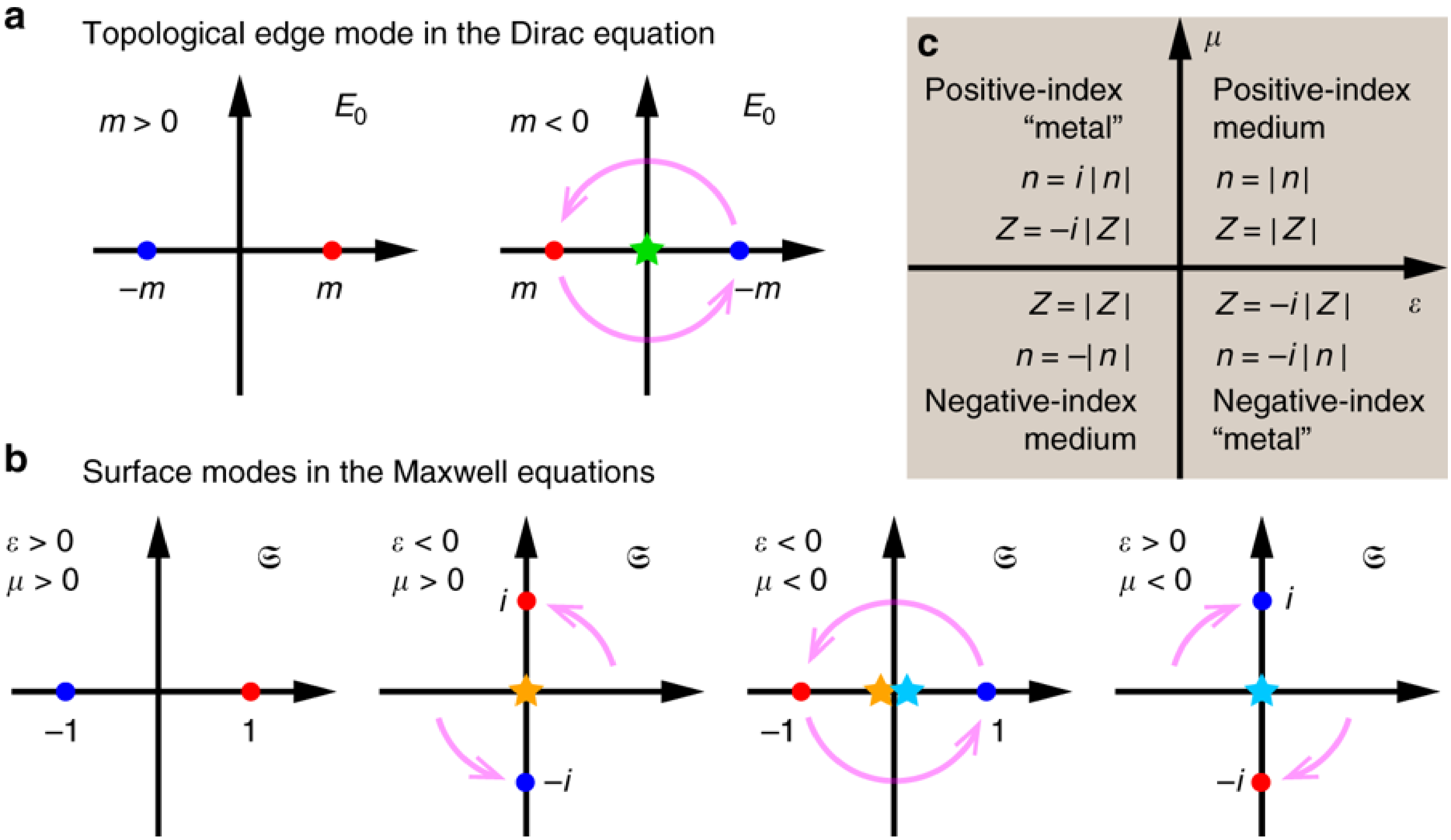


Schematics of topological surface modes in the Dirac and Maxwell equations. **a** The Dirac equation with a finite mass *m* is characterized by the gapped bulk spectrum $E(\mathbf{p})$. An interface between "media" with opposite-sign masses $\pm m$, and bulk spectra (schematically shown in red and blue), supports **topological surface modes with massless spectrum** (shown in green).

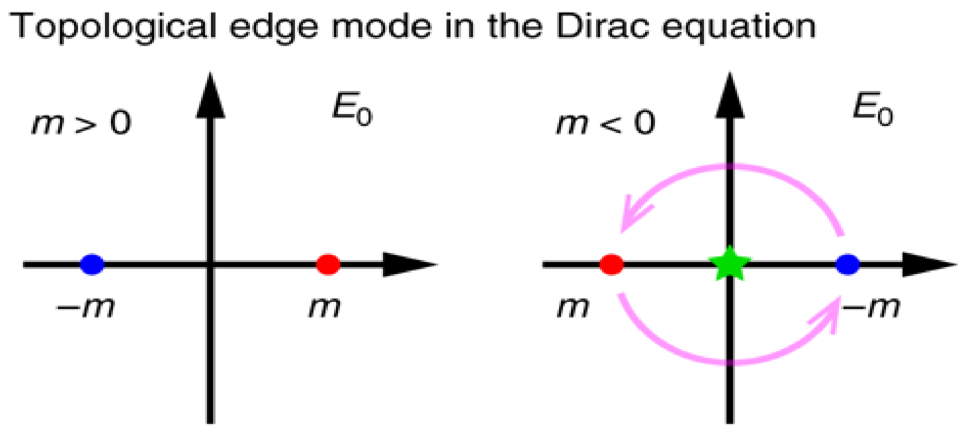


b Maxwell equations possess massless bulk spectra (not shown here), which are double-degenerate with respect to opposite helicity states. These bulk helicity eigenmodes have opposite circular polarizations, i.e., chiral spatial distributions of the electric or magnetic field (shown in red and blue here). An interface between two media with different helicity properties (controlled by the signs of the permittivity ε and permeability μ of the medium) supports zero-helicity surface waves with transverse-electric or transverse-magnetic linear polarizations (shown in green).



Winding of the energy and helicity spectra in the Dirac and Maxwell equations.

a

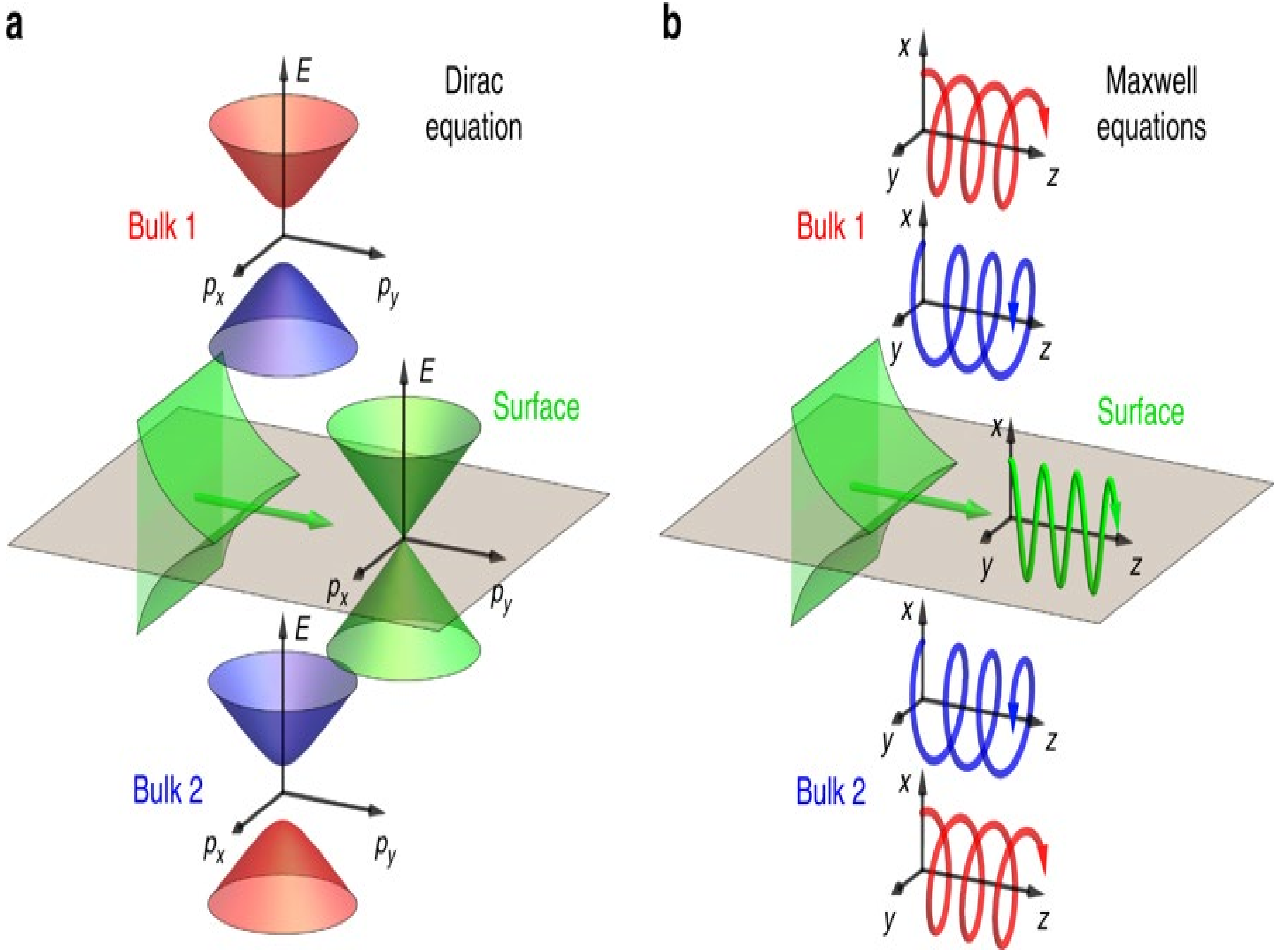


Winding of the energy spectra in the Dirac equation.

a Changing the **sign of the mass *m*** in the Dirac equation is equivalent to a **π rotation** (shown by the arrows) of the rest-energy spectrum $E_0 \equiv E(\mathbf{0}) = m$ in the complex-mass plane.

This results in a **single zero-mass surface mode** (shown by the star symbol) protected by the topological Z_2 winding number.

The dot and star symbols with their colors correspond to the rest-energy spectra of the bulk and surface modes shown in Fig. 1a.


a
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Bulk 1
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p_y
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Surface
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Bulk 2
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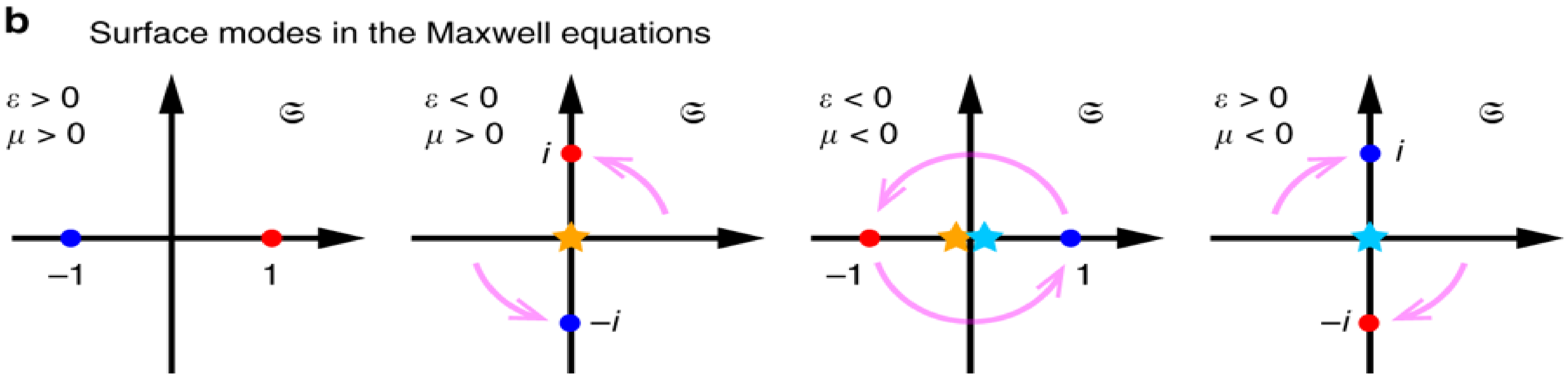
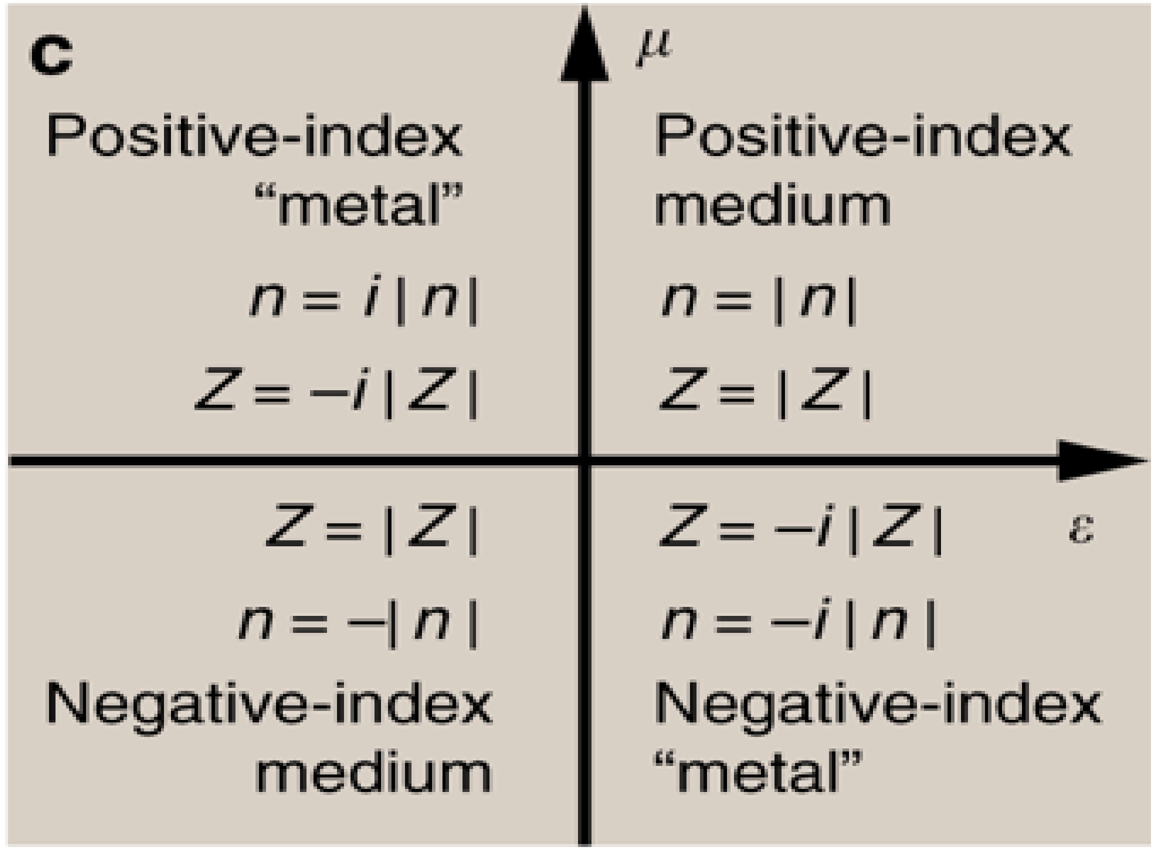


Winding of the helicity spectra in Maxwell equations.

b Changing the signs of the permittivity ε and permeability μ in Maxwell equations **produces $\pm\pi/2$ and π rotations of the helicity spectrum in the complex helicity plane.**

This results in **the appearance of one or two zero-helicity** (transverse-electric and transverse-magnetic) **surface modes** (the star symbols) described by the topological Z_4 number.



c The **medium-index diagram** showing the **signs of the refractive index *n* and impedance *Z*** in **four possible types of media.**

Topological non–Hermitian origin of surface acoustic waves

Konstantin Y. Bliokh,

In collaboration with

D. Leykam, J. Dressel, F. Nori

•K.Y. Bliokh, F. Nori
Transverse spin and surface waves in acoustic metamaterials
Phys. Rev. B **99**, 020301(R) (2019). [PDF][Link][arXiv]

•K.Y. Bliokh, F. Nori
Spin and orbital angular momenta of acoustic beams
Phys. Rev. B **99**, 174310 (2019). [PDF][Link][arXiv][Erratum]

•K.Y. Bliokh, F. Nori
Klein-Gordon Representation of Acoustic Waves and Topological Origin of Surface Acoustic Modes
Phys. Rev. Lett. **123**, 054301 (2019). [PDF][Link][arXiv][Suppl. Info.]

•I.D. Toftul, K.Y. Bliokh, M.I. Petrov, F. Nori
Acoustic Radiation Force and Torque on Small Particles as Measures of the Canonical Momentum and Spin Densities
Phys. Rev. Lett. **123**, 183901 (2019). [PDF][Link][arXiv][Suppl. Info.]

•L. Burns, K.Y. Bliokh, F. Nori, J. Dressel
Acoustic versus electromagnetic field theory: scalar, vector, spinor representations and the emergence of acoustic spin
New Journal of Physics **22**, 053050 (2020). [PDF][Link][arXiv]

•D. Leykam, K.Y. Bliokh, F. Nori
Edge modes in two-dimensional electromagnetic slab waveguides: Analogs of acoustic plasmons
Phys. Rev. B **102**, 045129 (2020). [PDF][Link][arXiv]

•K.Y. Bliokh, Y.P. Bliokh, F. Nori
Ponderomotive forces, Stokes drift, and momentum in acoustic and electromagnetic waves
Phys. Rev. A Letter **106**, L021503 (2022). [PDF][Link][arXiv]

TABLE I. Comparison of electromagnetic and acoustic quantities and properties.

	Electromagnetism	Acoustics
Fields	Electric $\mathbf{E}$, magnetic $\mathbf{H}$	Velocity $\mathbf{v}$, pressure P
Constraints	$\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{H} = 0$	$\nabla \times \mathbf{v} = 0$
Medium parameters	Permittivity ε, permeability μ	Density ρ, compressibility β
Energy density	$\frac{1}{4}\left(\varepsilon\|\mathbf{E}\|^2 + \mu\|\mathbf{H}\|^2\right)$	$\frac{1}{4}\left(\rho\|\mathbf{v}\|^2 + \beta\|P\|^2\right)$
Spin AM density	$\frac{1}{4}[\varepsilon \mathrm{Im}(\mathbf{E}^* \times \mathbf{E}) + \mu \mathrm{Im}(\mathbf{H}^* \times \mathbf{H})]$	$\frac{1}{2}\rho \mathrm{Im}(\mathbf{v}^* \times \mathbf{v})$
Transverse spin density in an evanescent wave	$\frac{\omega \mathbf{S}}{W} = \frac{\mathrm{Re}\mathbf{k} \times \mathrm{Im}\mathbf{k}}{(\mathrm{Re}\mathbf{k})^2}$	$\frac{\omega \mathbf{S}}{W} = 2\frac{\mathrm{Re}\mathbf{k} \times \mathrm{Im}\mathbf{k}}{(\mathrm{Re}\mathbf{k})^2}$
Surface waves	$\frac{\varepsilon_2}{\varepsilon_1} < 0$ (TM), $\frac{\mu_2}{\mu_1} < 0$ (TE)	$\frac{\rho_2}{\rho_1} < 0$ (TM-like)
Dispersive corrections	$(\varepsilon, \mu) \to (\tilde{\varepsilon}, \tilde{\mu})$	$(\rho, \beta) \to (\tilde{\rho}, \tilde{\beta})$

	Acoustics	Electromagnetism
Real fields	$\Psi^{\mu} = (P, \mathbf{v})$	$\boldsymbol{\Psi} = (\mathbf{E}, \mathbf{H})$
Energy density and flux	$W = \frac{1}{2}\Psi^{\mu} \bullet \Psi^{\mu} = \frac{1}{2}\left(\beta P^2 + \rho \mathbf{v}^2\right)$ $\boldsymbol{\Pi} = \frac{1}{2}\Psi^{\mu} \otimes \Psi^{\mu} = P\mathbf{v}$	$W = \frac{1}{2}\boldsymbol{\Psi} \bullet \boldsymbol{\Psi} = \frac{1}{2}\left(\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2\right)$ $\boldsymbol{\Pi} = \frac{1}{2}\boldsymbol{\Psi} \otimes \boldsymbol{\Psi} = \mathbf{E} \times \mathbf{H}$
Connection with "relativistic wavefunctions"	$\Psi^{\mu} = i\hat{p}^{\mu}\psi$ $= \left(-\sqrt{\rho}\,\partial_t \psi, \frac{\nabla \psi}{\sqrt{\rho}}\right)$	$\boldsymbol{\Psi} = \left(\frac{\mathrm{Re}\,\boldsymbol{\psi}}{\sqrt{\varepsilon}}, \frac{\mathrm{Im}\,\boldsymbol{\psi}}{\sqrt{\mu}}\right)$
Relativistic wave equations	$\left(\hat{p}^{\mu} \bullet \hat{p}_{\mu}\right)\psi = 0$	$\left(\hat{S}^{\mu}\hat{p}_{\mu}\right)\boldsymbol{\psi} = 0$
Four-momentum operator	$\hat{p}^{\mu} = \left(i\sqrt{\rho}\,\partial_t, \frac{-i\nabla}{\sqrt{\rho}}\right)$	$\hat{p}^{\mu} = \left(i\sqrt{\varepsilon\mu}\,\partial_t, -i\nabla\right)$
Topological indices	$w(\rho) = \frac{1}{2}\left[1 - \mathrm{sgn}(\rho)\right]$	$w(\varepsilon, \mu) = \frac{1}{2}\left[1 - \mathrm{sgn}(\varepsilon), 1 - \mathrm{sgn}(\mu)\right]$

Relativistic formalisms for acoustic and Maxwell equations:

	Acoustics	**Electromagnetism**
Real fields	$\boldsymbol{\Psi}^{\mu} = (P, \mathbf{v})$	$\boldsymbol{\Psi} = (\mathbf{E}, \mathbf{H})$
Energy density and flux	$W = \frac{1}{2}\boldsymbol{\Psi}^{\mu} \bullet \boldsymbol{\Psi}^{\mu} = \frac{1}{2}(\beta P^2 + \rho \mathbf{v}^2)$ $\boldsymbol{\Pi} = \frac{1}{2}\boldsymbol{\Psi}^{\mu} \otimes \boldsymbol{\Psi}^{\mu} = P\mathbf{v}$	$W = \frac{1}{2}\boldsymbol{\Psi} \bullet \boldsymbol{\Psi} = \frac{1}{2}(\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2)$ $\boldsymbol{\Pi} = \frac{1}{2}\boldsymbol{\Psi} \otimes \boldsymbol{\Psi} = \mathbf{E} \times \mathbf{H}$
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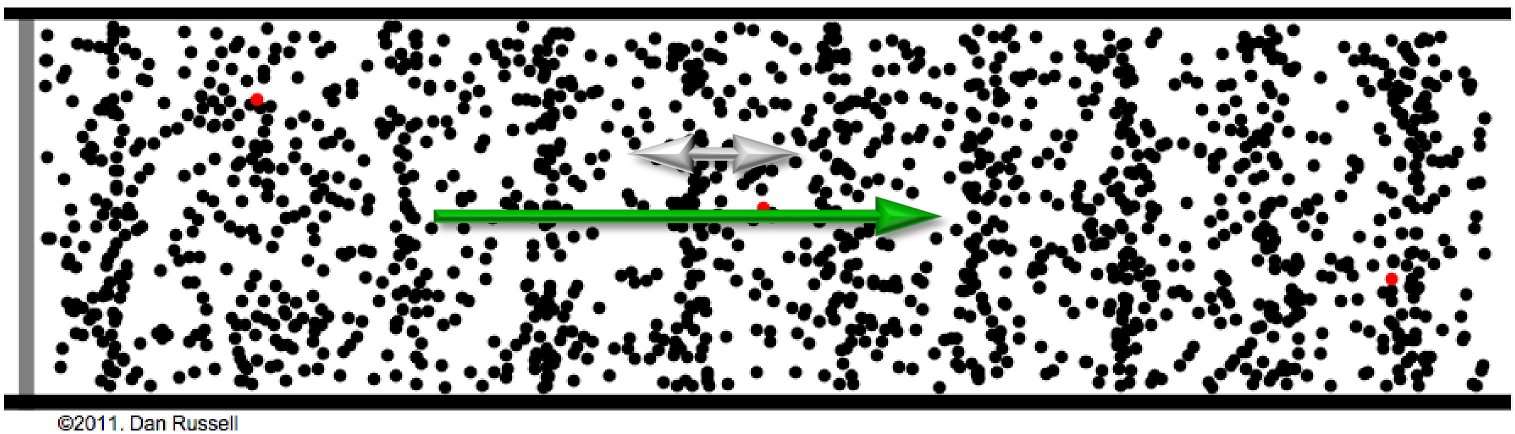
Relativistic wave equations	$(\hat{p}^{\mu} \bullet \hat{p}_{\mu})\psi = 0$	$(\hat{S}^{\mu}\hat{p}_{\mu})\boldsymbol{\psi} = 0$
Four-momentum operator	$\hat{p}^{\mu} = \left(i\sqrt{\rho}\,\partial_t, \frac{-i\nabla}{\sqrt{\rho}}\right)$	$\hat{p}^{\mu} = \left(i\sqrt{\varepsilon\mu}\,\partial_t, -i\nabla\right)$
Topological indices	$w(\rho) = \frac{1}{2}[1 - \mathrm{sgn}(\rho)]$	$w(\varepsilon, \mu) = \frac{1}{2}[1 - \mathrm{sgn}(\varepsilon), 1 - \mathrm{sgn}(\mu)]$

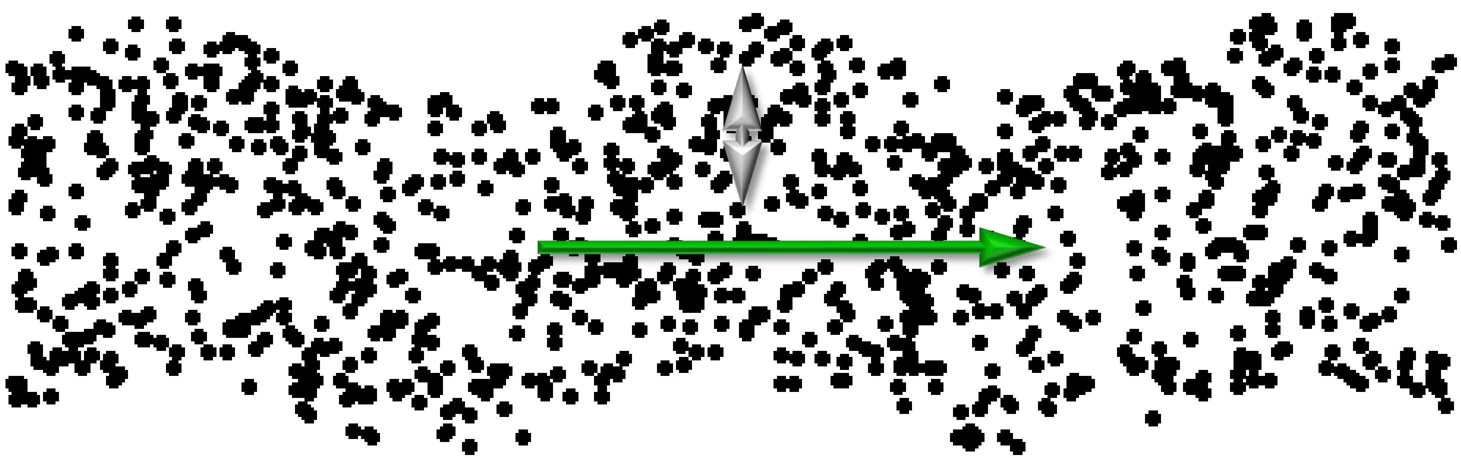
Introduction

Types of waves

Longitudinal (e.g., sound or P–elastic):

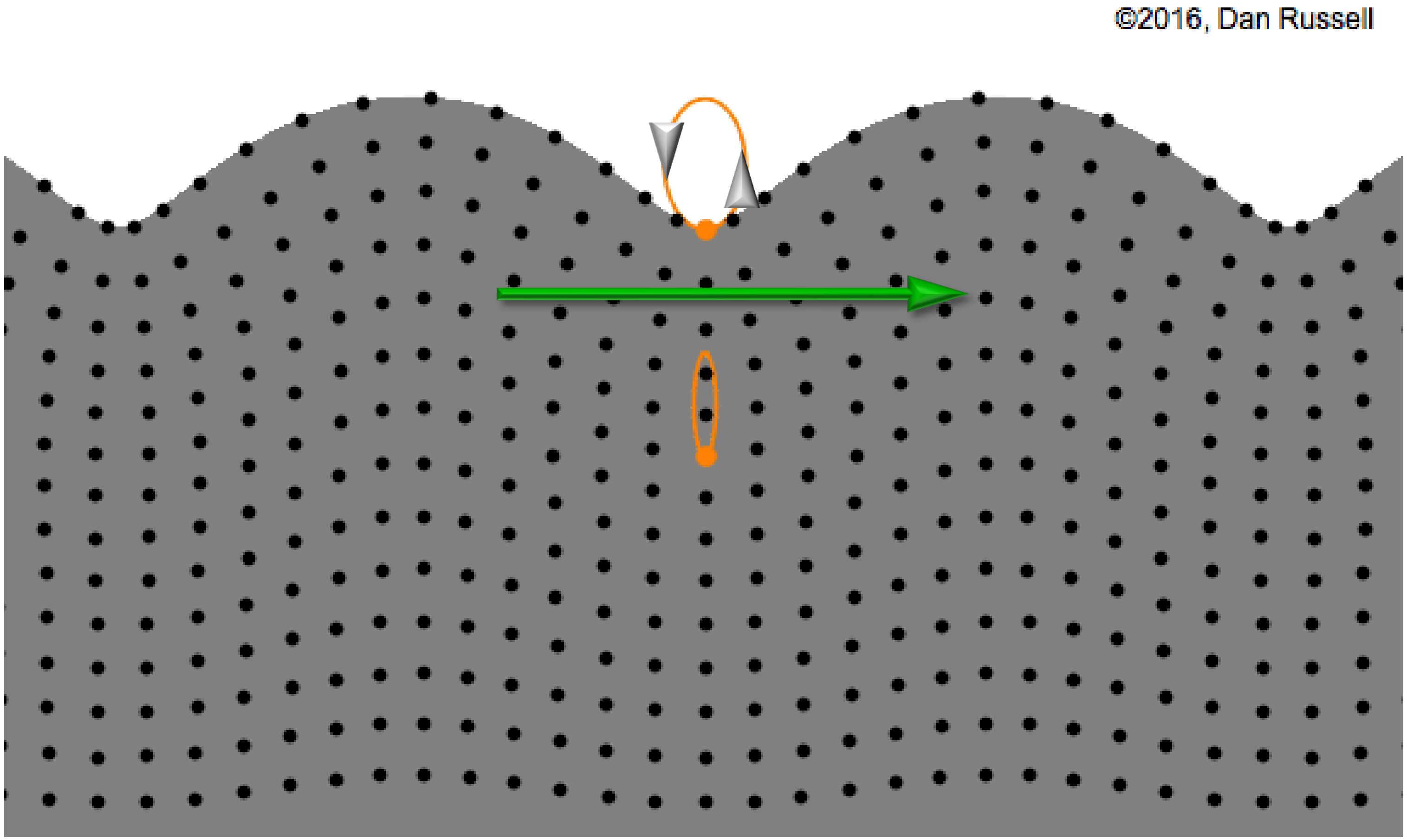


Transverse (e.g., S–elastic):



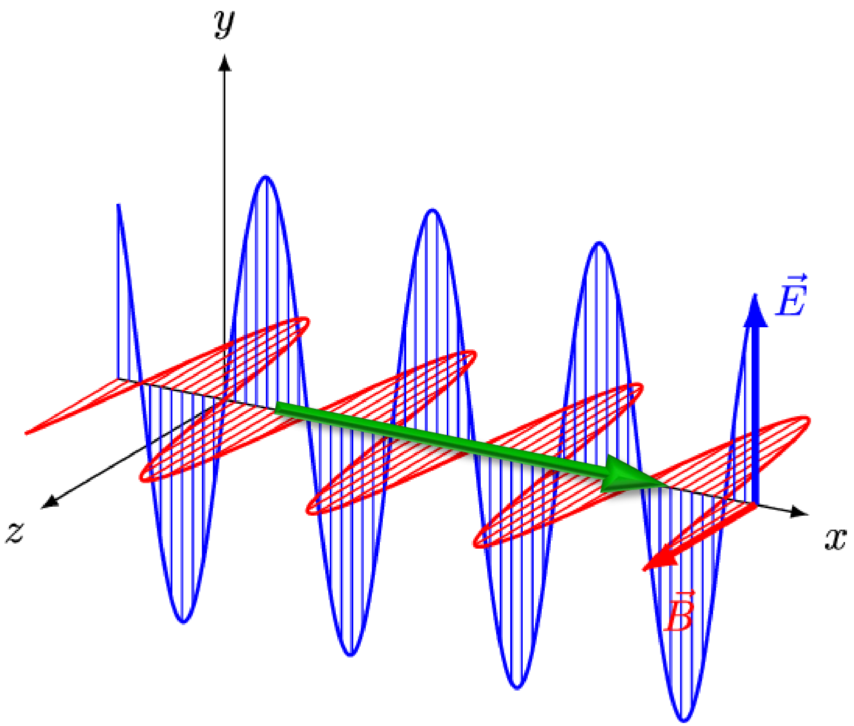
Types of waves

Surface waves (e.g., Rayleigh or ocean waves):

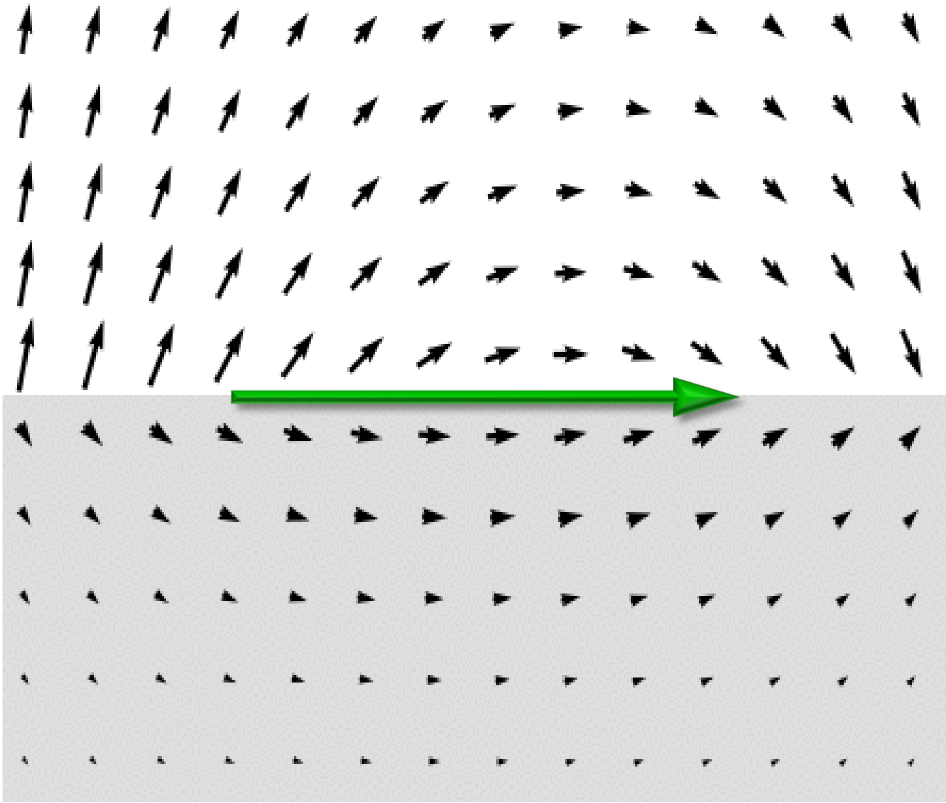


Electromagnetic waves

Bulk electromagnetic:

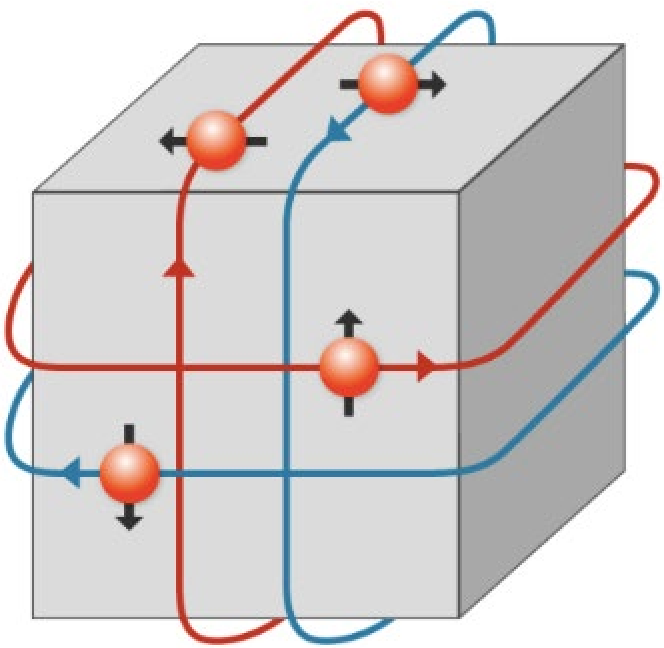


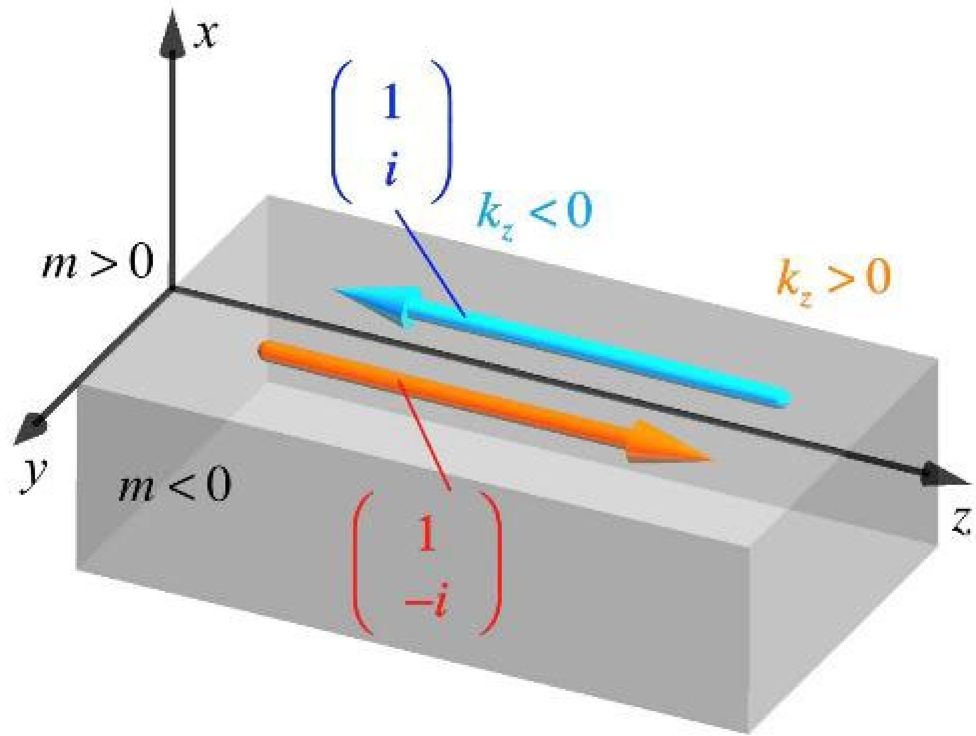
Surface–plasmon:



Analogy with topological insulators

This is analogous to the topological edge/surface states in electron topological insulators:





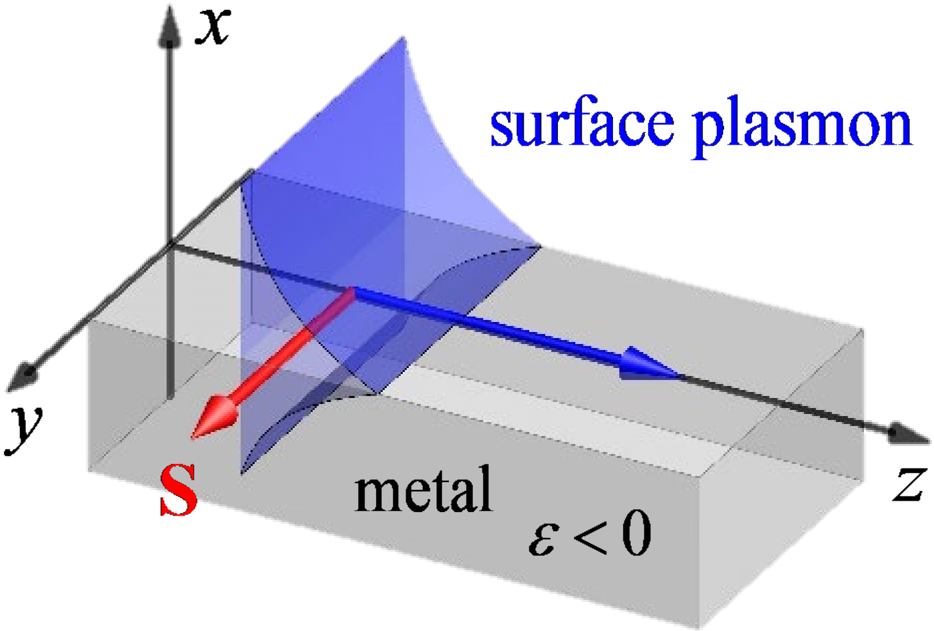
Kane & Mele *PRL* (2005); Bernevig, Hughes, Zhang *Science* (2006)

The question

Since 2014, we were considering this idea:

Can the very basic surface waves in Maxwell equations be described as topological surface modes?

Do surface plasmons at metal-dielectric interfaces have a topological origin (*Nobel Prize* 2016)?



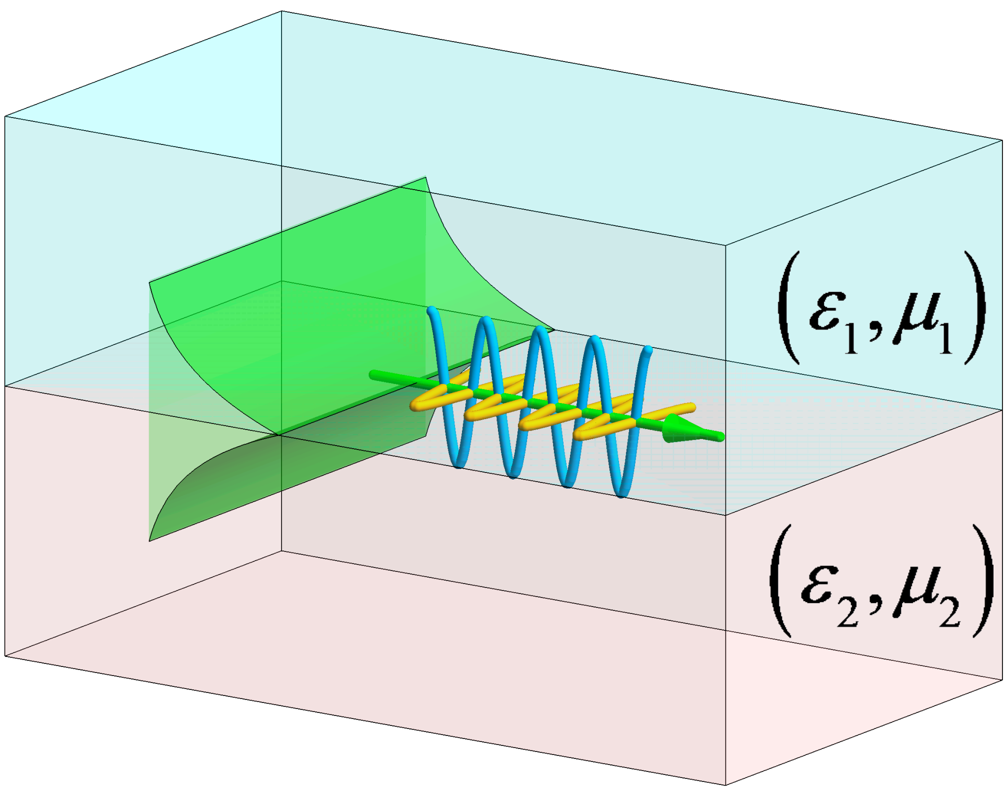
Surface modes of Maxwell equations

Surface Maxwell modes

Maxwell equations (isotropic, homogeneous, lossless):

$$\nabla \times \mathbf{E} = -\mu \partial_t \mathbf{H}, \quad \nabla \times \mathbf{H} = \varepsilon \partial_t \mathbf{E}, \quad \nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{E} = 0$$

Surface modes at an interface between two media:

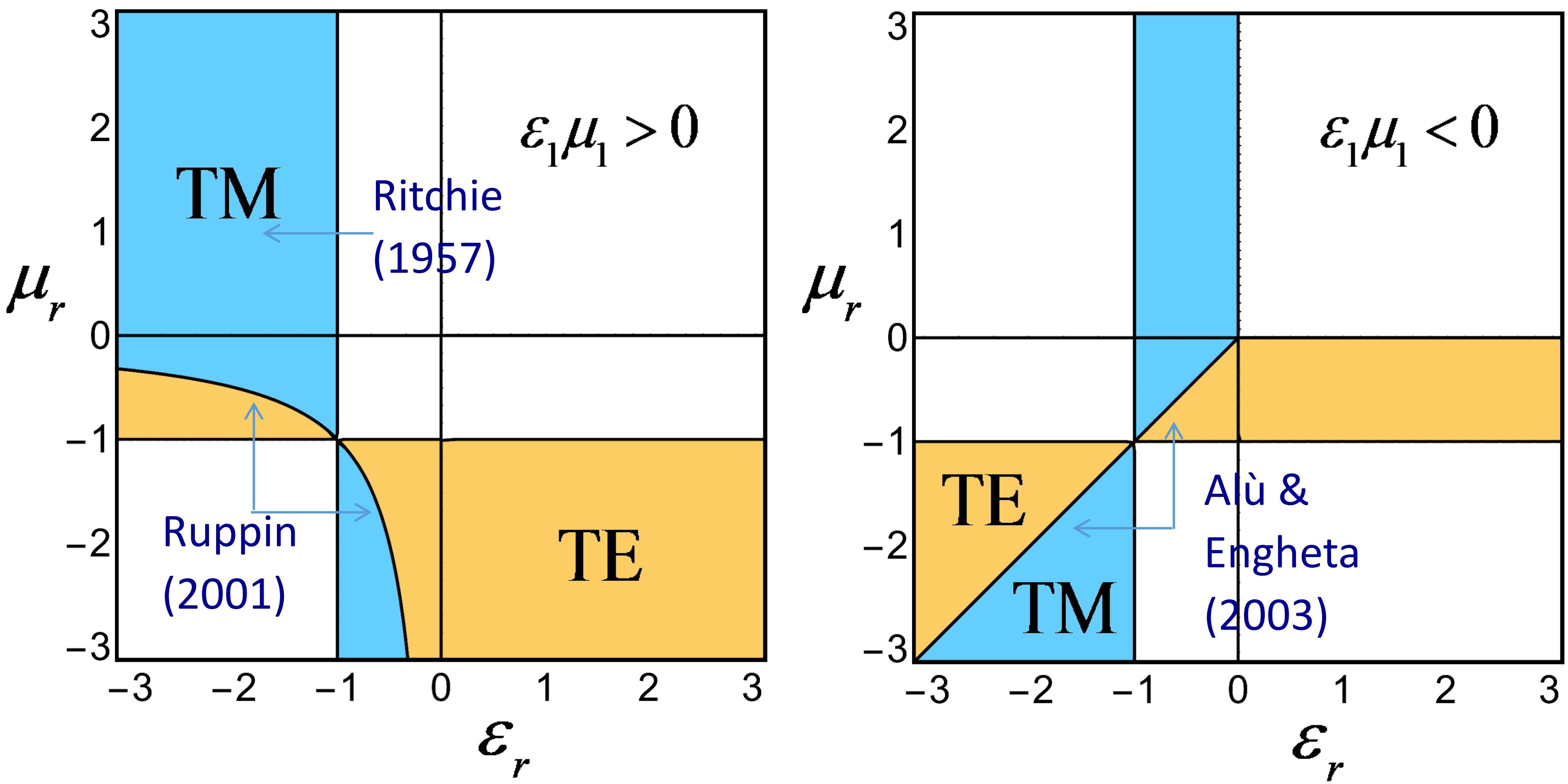


TM polarization

TE polarization

Surface Maxwell modes

"Phase diagram" of surface TE/TM Maxwell modes in the $(\varepsilon_r, \mu_r) = (\varepsilon_2/\varepsilon_1, \mu_2/\mu_1)$ plane:



Shadrivov *et al. PRE* (2004); Kats *et al. PRL* (2007); Bliokh *et al. Nat. Comm.* (2019)

Surface Maxwell modes

These diagrams are deformed with deformations of the interface (see, e.g., surface plasmons on a sphere), but there are some general robust features:

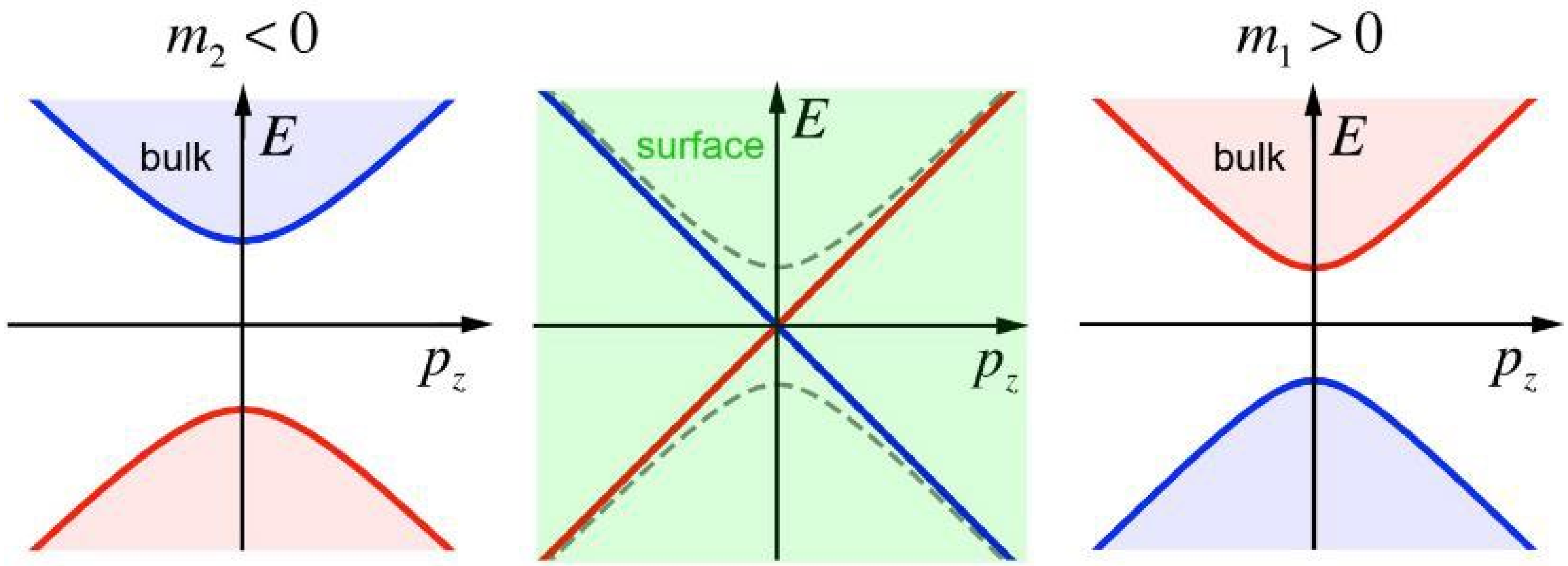
- The TM–mode can exist only at interfaces where the permittivity changes sign: $\mathrm{sgn}(\varepsilon_r) = -1$

- The TE–mode can exist only at interfaces where the permeability changes sign: $\mathrm{sgn}(\mu_r) = -1$

Topological Dirac modes

Most importantly, the Dirac surface mode appears only at interfaces where the mass changes its sign: $\mathrm{sgn}\left(m_r\right) = -1$.

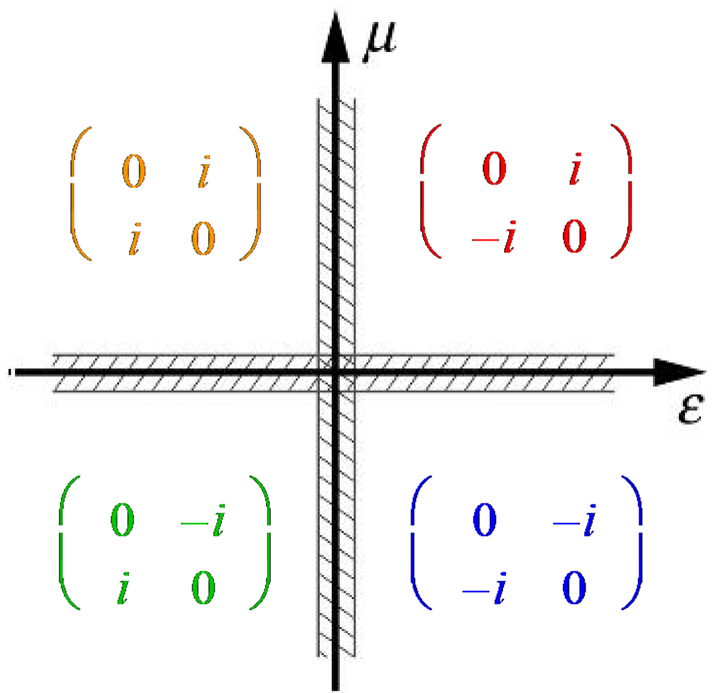
This is related to the nontrivial Möbius–strip–like (Z_2) topology of the Dirac Hamiltonian and bulk eigenmodes:



Topological Maxwell modes

Thus, surface electromagnetic waves (e.g., surface plasmons) have a topological origin related to the non–Hermitian helicity operator.

Helicity is ill–defined at $\varepsilon = 0$ and $\mu = 0$, which split the (ε, μ) plane into the four topologically–different quadrants:

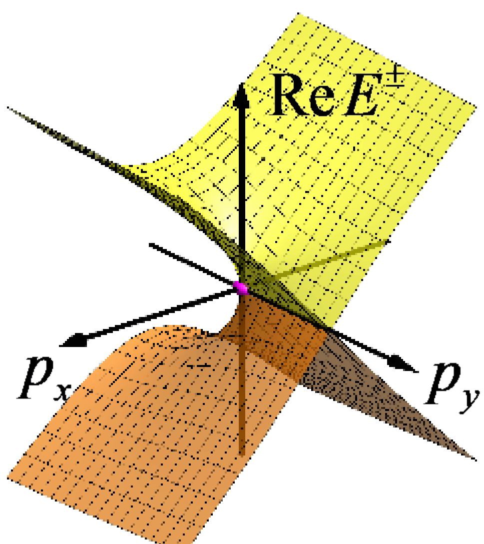


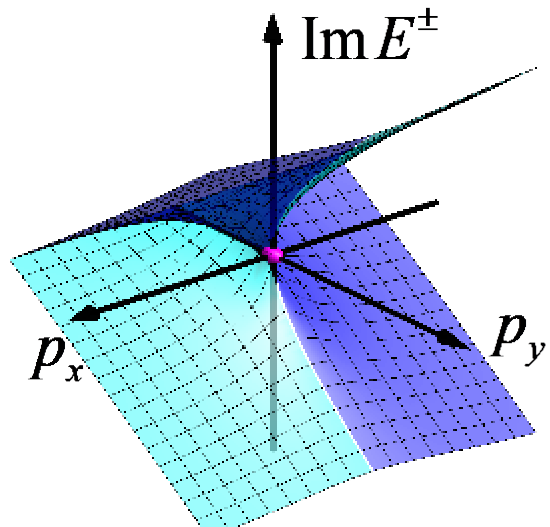
Topological Maxwell modes

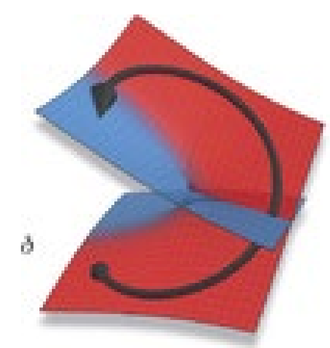
These special values $\varepsilon = 0$ and $\mu = 0$ correspond to the exceptional points (EPs) of the operator

$$\hat{\sigma}^{(\mathrm{m})} = |n|\hat{\mathfrak{S}} = \begin{pmatrix} 0 & i\mu \\ -i\varepsilon & 0 \end{pmatrix}$$

Modern theory of non–Hermitian topological systems: the topological transitions happen at EPs.







Lee (2016); Leykam *et al*. (2017); Gong *et al*. (2018); Kunst *et al*. (2018); Yao, Wang (2018)

Topological Maxwell modes

In our case, we have exceptional points of the helicity operator in a lossless system!

Two eigenstates coalesce in the EP and form a chiral state:

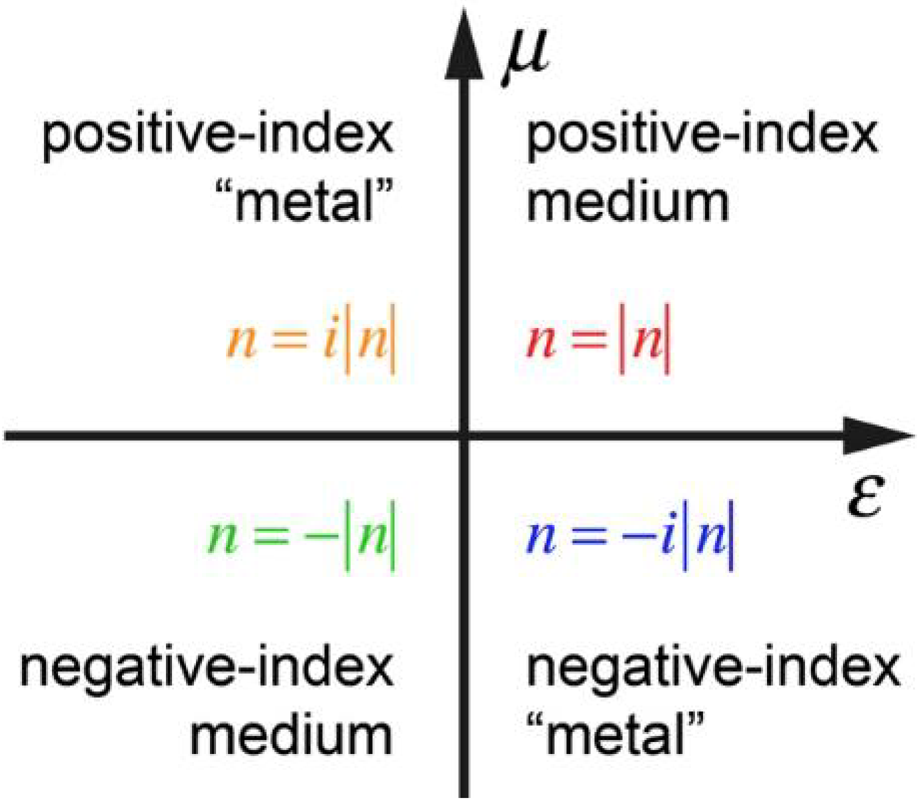
$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \propto \begin{pmatrix} 1 \\ -/+\ i\varepsilon/n \end{pmatrix} \xrightarrow{\varepsilon=0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \xrightarrow{\mu=0} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

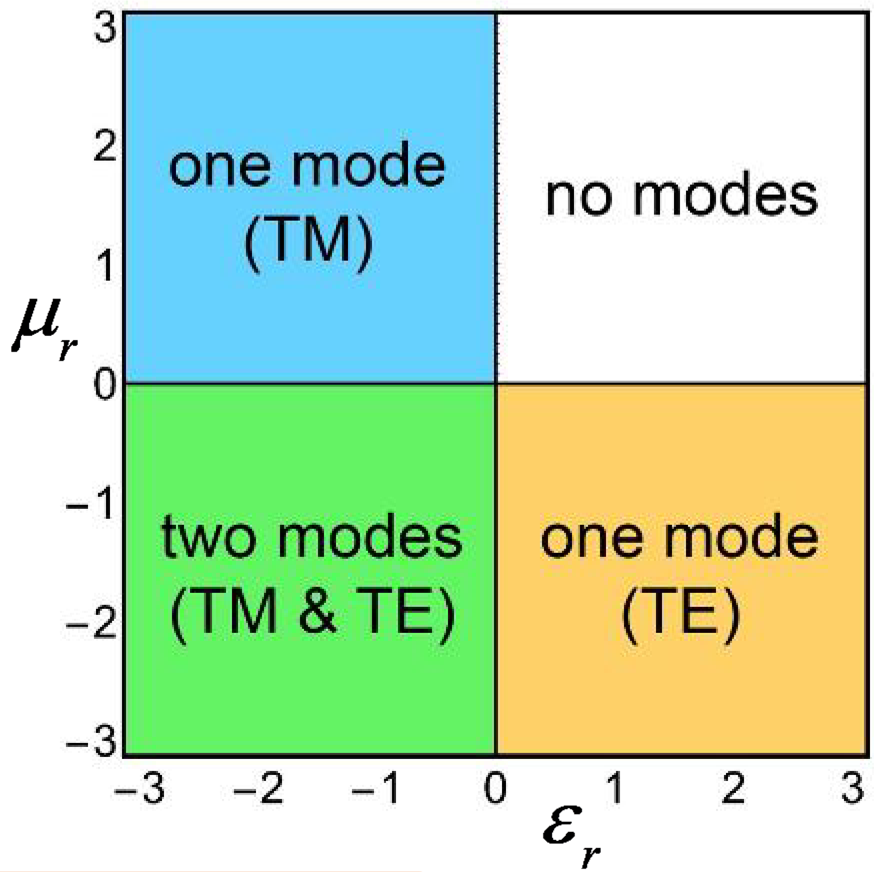
These are electric/magnetic–field states in the ε–near–zero and μ–near–zero materials. A completely new non–Hermitian twist for this field of research!

Silveirinha & Engheta *PRL* (2006); A. Alù *et al*. *IEEE* (2007); Liu *et al*., *PRL* (2008); …

- ✓ Surface electromagnetic modes between isotropic lossless media (e.g., SPPs) have a topological origin, similar to the topological surface modes in the Dirac equation.
- ✓ The topology of the helicity rather than Hamiltonian.
- ✓ Winding of the complex helicity spectrum.
  $Z_4 = Z_2 \times Z_2$ and (ε, μ) instead of Z_2 and m.
- ✓ The bulk–boundary correspondence yields the conditions $\mathrm{sgn}(\varepsilon_r) = -1$ and $\mathrm{sgn}(\mu_r) = -1$ for the TM/TE modes.
- ✓ Helicity is non–Hermitian. So are the surface modes.
  Can be "dark" (imaginary frequency/wavevector).
- ✓ Exceptional points of the helicity at $\varepsilon = 0$ and $\mu = 0$.
  Chiral bulk modes in "index–near–zero" materials.

A new twist for (i) topological systems, (ii) non–Hermitian physics, (iii) metamaterials & plasmonics:



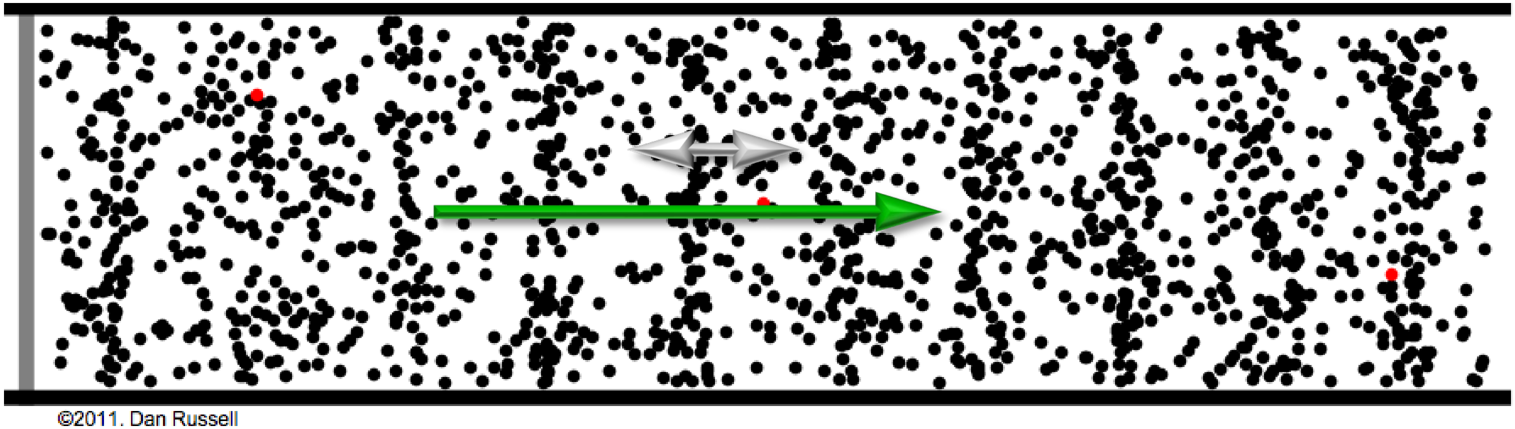


$$\hat{\mathfrak{S}} = \frac{1}{|n|}\begin{pmatrix} 0 & i\mu \\ -i\varepsilon & 0 \end{pmatrix}, \quad \mathfrak{S} = \pm\frac{n}{|n|}$$

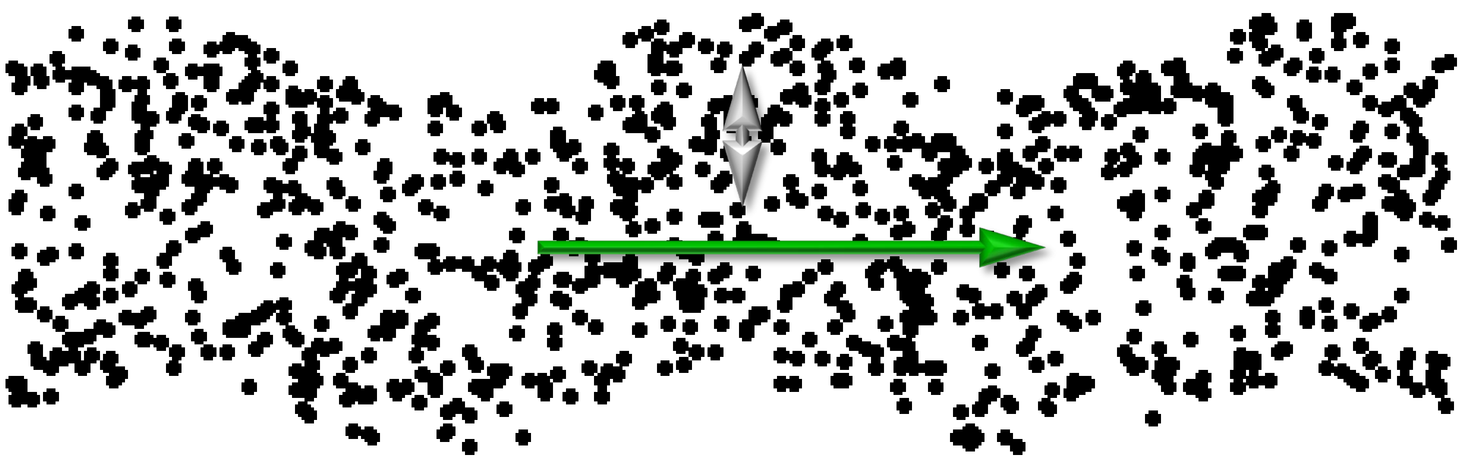
Acoustic (sound) waves

Types of waves

Longitudinal (e.g., sound or P–elastic):

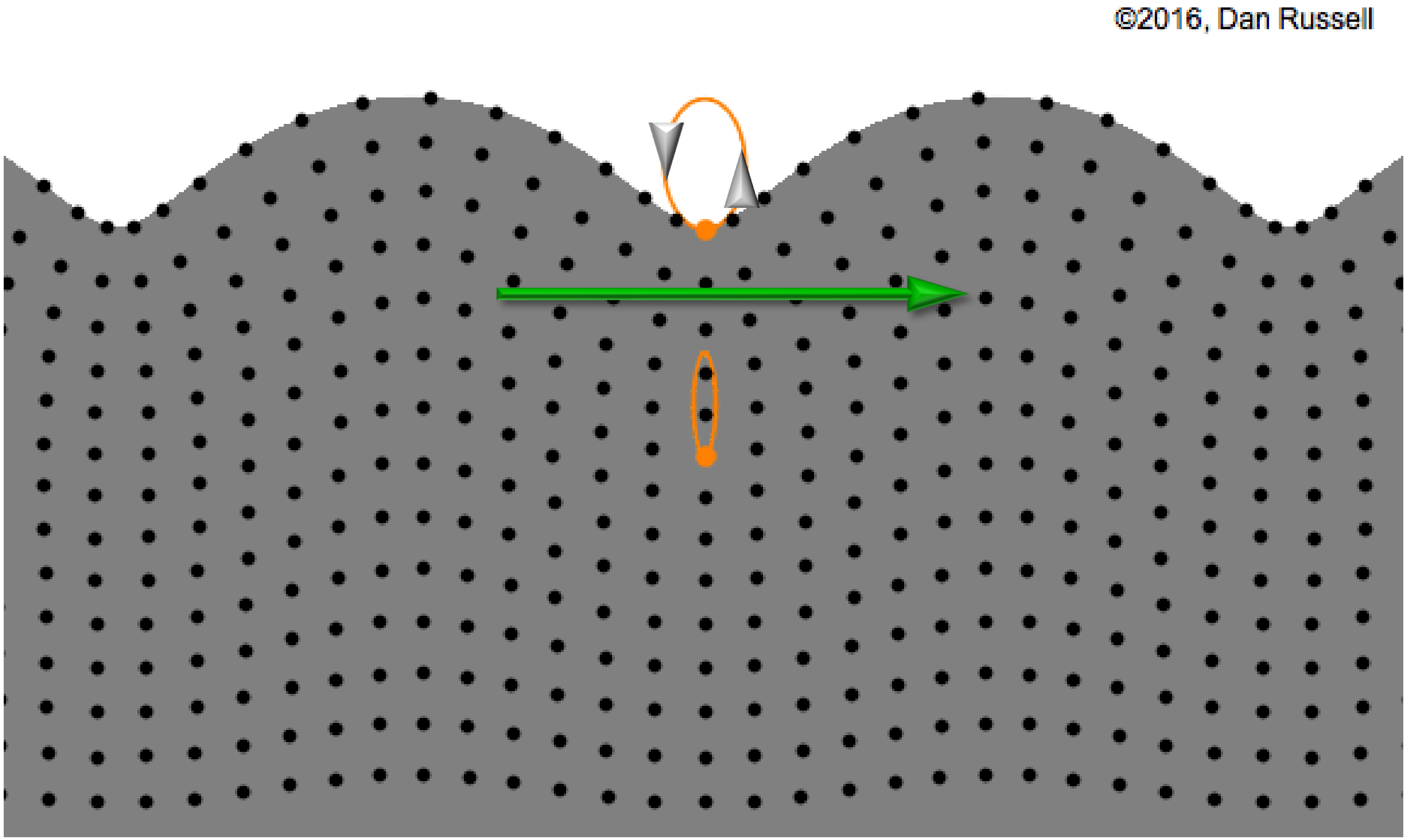


Transverse (e.g., S–elastic):



Types of waves

Surface waves (e.g., Rayleigh or ocean waves):



$$\nabla \cdot \mathbf{v} = i\beta\omega P, \quad \nabla P = i\rho\omega\mathbf{v}.$$

ρ = mass density

β = compressibility = 1 / B (B = bulk modulus)

II. TRANSVERSE SPIN IN EVANESCENT ACOUSTIC WAVES

We start with the equations for monochromatic acoustic waves of frequency ω in a dense medium:

$$\nabla \cdot \mathbf{v} = i\beta\omega P, \quad \nabla P = i\rho\omega\mathbf{v}. \tag{1}$$

Here the variables are the complex velocity $\mathbf{v}(\mathbf{r})$ and the pressure $P(\mathbf{r})$ fields, while the real-valued medium parameters are the mass density ρ and the compressibility $\beta = 1/B$ (B is the bulk modulus).

ρ = mass density

β = compressibility = 1 / *B* (*B* = bulk modulus)

$$\nabla \cdot \mathbf{v} = i\beta\omega P, \quad \nabla P = i\rho\omega\mathbf{v}. \tag{1}$$

Equations (1) support only longitudinal (i.e., curl-free) waves, which follows from the second expression in Eqs. (1): $\nabla \times \mathbf{v} = 0$. Importantly, these are not scalar waves. Indeed, the velocity $\mathbf{v}$ determines vector properties of acoustic waves, even though these are longitudinal. For plane waves with the wave vector $\mathbf{k}$, $\nabla \to i\mathbf{k}$, the dispersion relation and the "longitudinality" condition follow from Eqs. (1):

$$\omega^2 = k^2 c^2 \equiv \frac{k^2}{\rho\beta}, \quad \mathbf{k} \times \mathbf{v} = 0. \tag{2}$$

Evanescent waves can be presented as plane waves with a complex wave vector $\mathbf{k} = \mathrm{Re}\mathbf{k} + i\,\mathrm{Im}\mathbf{k}$ [4,6,16].

Acoustic (longitudinal) waves

Acoustic wave equations (isotropic, lossless):

$$\nabla \cdot \mathbf{v} = -\beta \partial_t P, \quad \nabla P = -\rho \partial_t \mathbf{v}, \quad \nabla \times \mathbf{v} = 0$$

Cf. Maxwell equations:

$$\nabla \times \mathbf{E} = -\mu \partial_t \mathbf{H}, \quad \nabla \times \mathbf{H} = \varepsilon \partial_t \mathbf{E}, \quad \nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{E} = 0$$

There is an electromagnetic–acoustic analogy:

$$(\mathbf{E}, \mathbf{H}) \leftrightarrow (\mathbf{v}, P), \quad (\varepsilon, \mu) \leftrightarrow (\rho, \beta)$$

For example, energy density and flux:

$$\frac{1}{2}\left(\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2\right) \leftrightarrow \frac{1}{2}\left(\rho \mathbf{v}^2 + \beta P^2\right), \qquad \mathbf{E} \times \mathbf{H} \leftrightarrow P\mathbf{v}$$

TABLE I. Comparison of electromagnetic and acoustic quantities and properties.

	Electromagnetism	Acoustics
Fields	Electric $\mathbf{E}$, magnetic $\mathbf{H}$	Velocity $\mathbf{v}$, pressure P
Constraints	$\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{H} = 0$	$\nabla \times \mathbf{v} = 0$
Medium parameters	Permittivity ε, permeability μ	Density ρ, compressibility β
Energy density	$\frac{1}{4}\left(\varepsilon \lvert\mathbf{E}\rvert^2 + \mu \lvert\mathbf{H}\rvert^2\right)$	$\frac{1}{4}\left(\rho \lvert\mathbf{v}\rvert^2 + \beta \lvert P\rvert^2\right)$
Spin AM density	$\frac{1}{4}[\varepsilon \mathrm{Im}(\mathbf{E}^* \times \mathbf{E}) + \mu \mathrm{Im}(\mathbf{H}^* \times \mathbf{H})]$	$\frac{1}{2}\rho \mathrm{Im}(\mathbf{v}^* \times \mathbf{v})$
Transverse spin density in an evanescent wave	$\frac{\omega \mathbf{S}}{W} = \frac{\mathrm{Re}\mathbf{k} \times \mathrm{Im}\mathbf{k}}{(\mathrm{Re}\mathbf{k})^2}$	$\frac{\omega \mathbf{S}}{W} = 2\frac{\mathrm{Re}\mathbf{k} \times \mathrm{Im}\mathbf{k}}{(\mathrm{Re}\mathbf{k})^2}$
Surface waves	$\frac{\varepsilon_2}{\varepsilon_1} < 0$ (TM), $\frac{\mu_2}{\mu_1} < 0$ (TE)	$\frac{\rho_2}{\rho_1} < 0$ (TM-like)
Dispersive corrections	$(\varepsilon, \mu) \to (\tilde{\varepsilon}, \tilde{\mu})$	$(\rho, \beta) \to (\tilde{\rho}, \tilde{\beta})$

The general expression for the time-averaged spin AM density $\mathbf{S}$ can be written by noticing that the medium particles with mass density ρ experience complex displacements $\mathbf{a}$, $\mathbf{v} = -i\omega\,\mathbf{a}$:

$$\mathbf{S} = \frac{\rho}{2}\mathrm{Re}(\mathbf{a}^{*}\times\mathbf{v}) = \frac{\rho}{2\omega}\mathrm{Im}(\mathbf{v}^{*}\times\mathbf{v}). \tag{4}$$

Notably, the entirely similar contribution of oscillating electrons in optical media provides the material contribution to the electromagnetic spin AM [31].

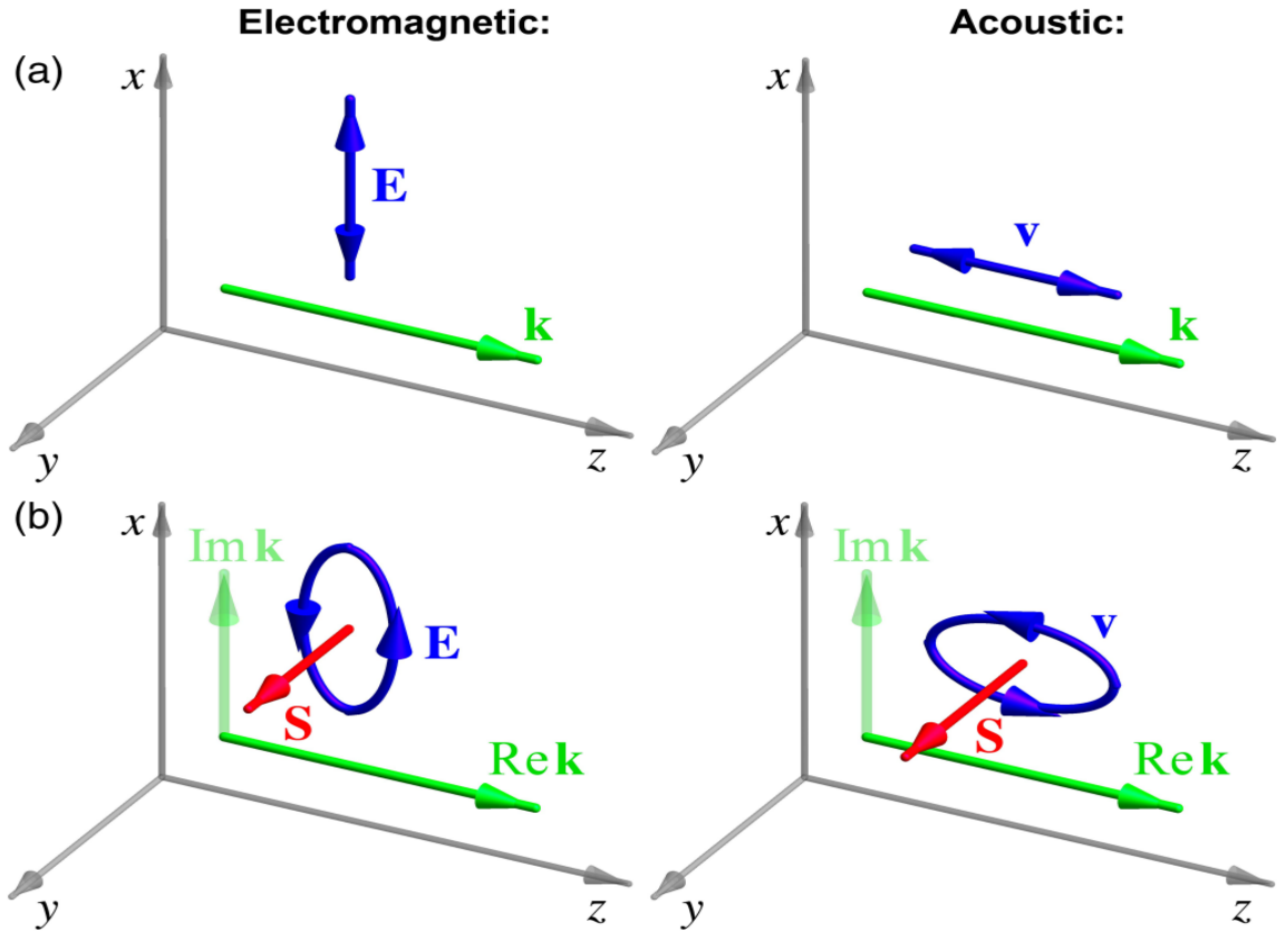
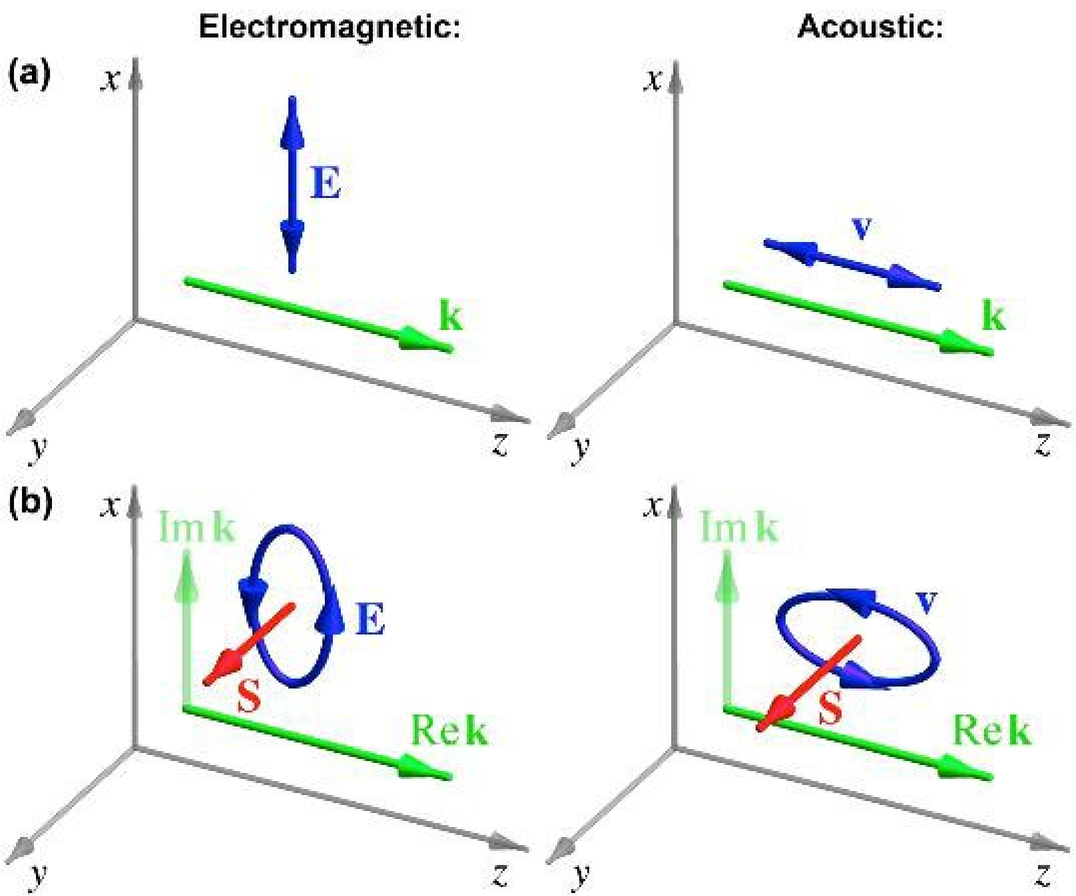


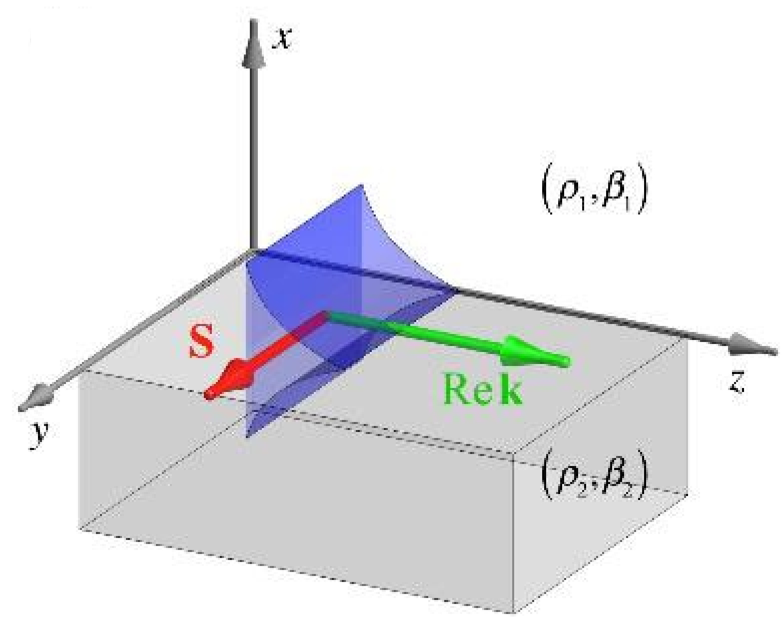
FIG. 1. Appearance of the transverse spin AM in transversal (electromagnetic) and longitudinal (acoustic) waves. (a) Plane waves of these types have linear polarizations orthogonal and parallel to the wave vector, respectively. (b) Evanescent waves with orthogonal real and imaginary parts of the wave vector inevitably have elliptical in-plane polarization because of the transversality ($\mathbf{k} \cdot \mathbf{E} = 0$) and longitudinality ($\mathbf{k} \times \mathbf{v} = 0$) conditions. These elliptical polarizations generate the transverse spin $\mathbf{S}$.

Acoustic (longitudinal) waves

Longitudinal acoustic waves have inherent vector properties and spin AM density described by $\mathbf{v}$:



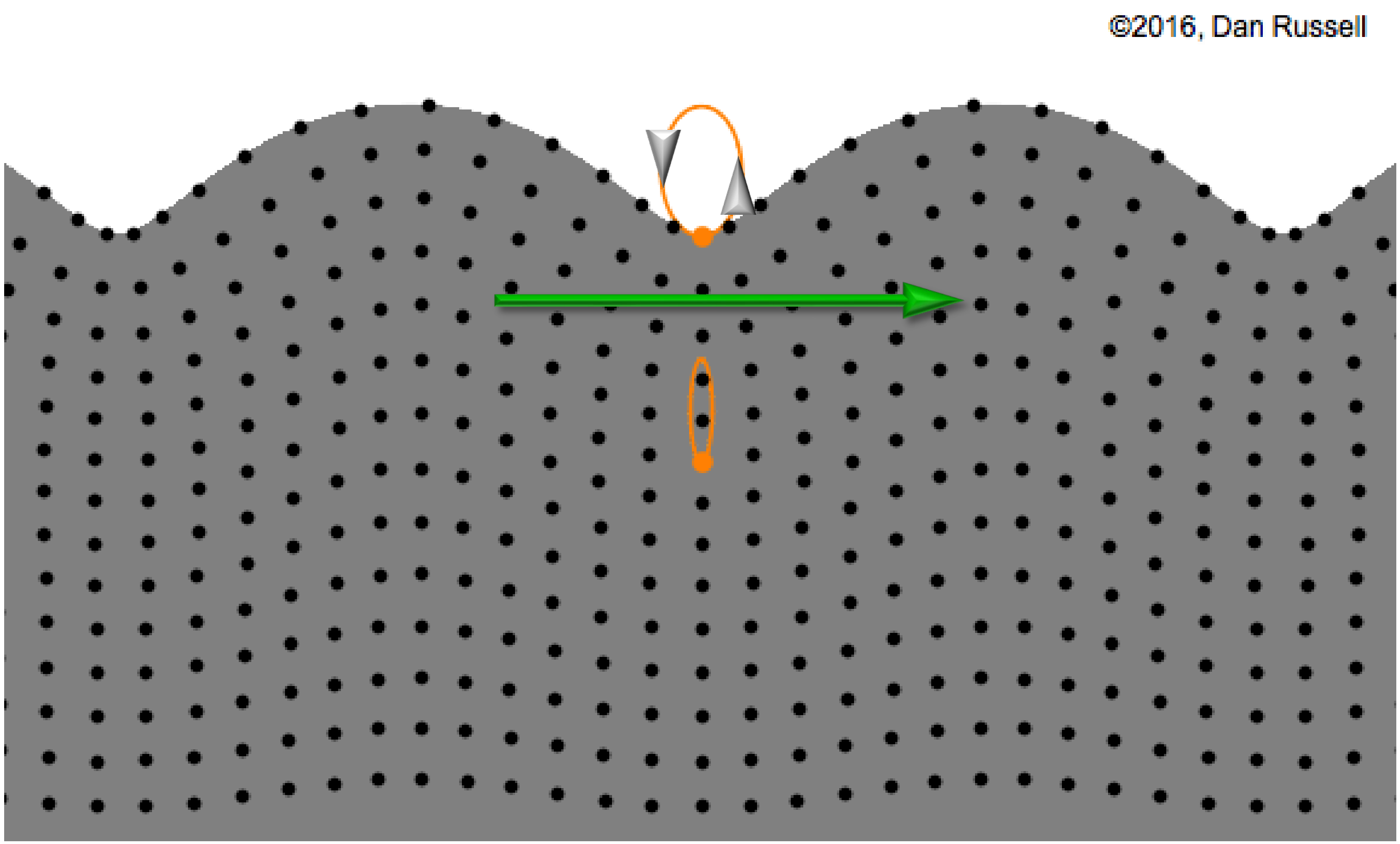
$$\mathbf{S} = \frac{\rho}{2\omega} \operatorname{Im}\left(\mathbf{v}^{*} \times \mathbf{v}\right)$$



Shi *et al. Nat. Sci. Rev.* (2019); Bliokh & Nori *PRB* (2019), *PRB* (2019), *arXiv* (2019)

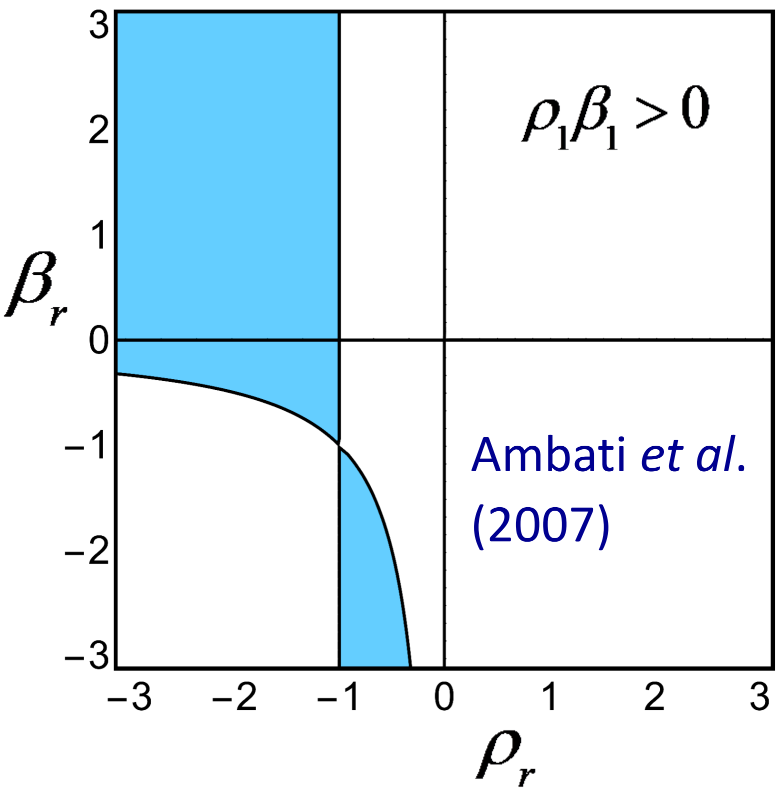
Types of waves

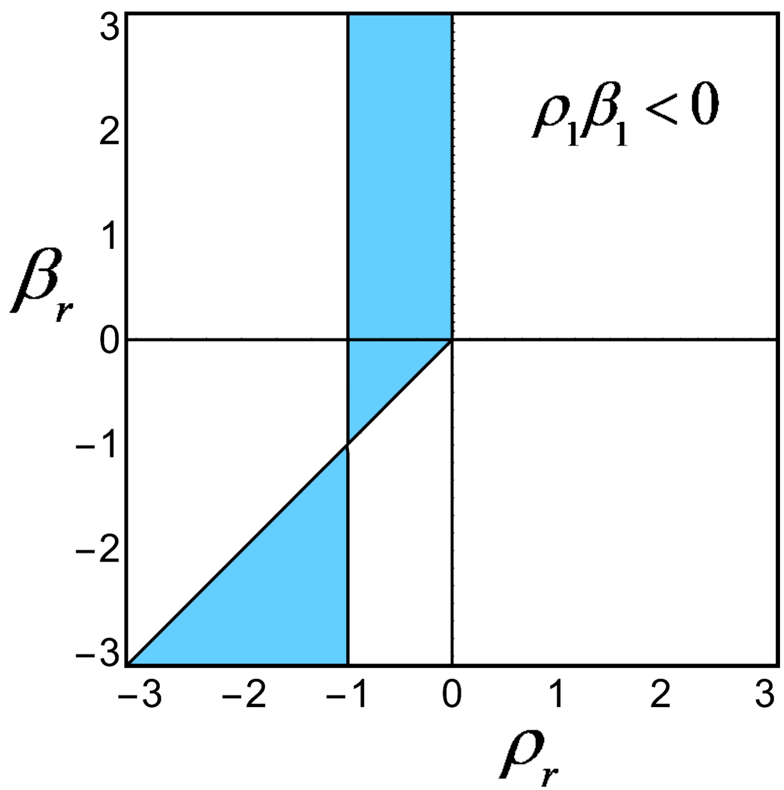
Surface waves (e.g., Rayleigh or ocean waves):



Acoustic (longitudinal) waves

There are also acoustic surface waves, analogous to surface plasmons. They appear at interfaces with negative–density media: $\mathrm{sgn}(\rho_r) = -1$.





Ambati *et al.* *PRB* (2007); Park *et al.* *PRL* (2011); Bliokh & Nori *PRB* (2019)

Relativistic formalisms for acoustic and Maxwell equations:

	Acoustics	**Electromagnetism**
Real fields	$\boldsymbol{\Psi}^{\mu} = \left(P, \mathbf{v}\right)$	$\boldsymbol{\Psi} = \left(\mathbf{E}, \mathbf{H}\right)$
Energy density and flux	$W = \frac{1}{2}\boldsymbol{\Psi}^{\mu} \bullet \boldsymbol{\Psi}^{\mu} = \frac{1}{2}\left(\beta P^2 + \rho \mathbf{v}^2\right)$ $\boldsymbol{\Pi} = \frac{1}{2}\boldsymbol{\Psi}^{\mu} \otimes \boldsymbol{\Psi}^{\mu} = P\mathbf{v}$	$W = \frac{1}{2}\boldsymbol{\Psi} \bullet \boldsymbol{\Psi} = \frac{1}{2}\left(\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2\right)$ $\boldsymbol{\Pi} = \frac{1}{2}\boldsymbol{\Psi} \otimes \boldsymbol{\Psi} = \mathbf{E} \times \mathbf{H}$
Connection with "relativistic wavefunctions"	$\boldsymbol{\Psi}^{\mu} = i\hat{p}^{\mu}\psi$ $= \left(-\sqrt{\rho}\,\partial_t \psi, \frac{\nabla\psi}{\sqrt{\rho}}\right)$	$\boldsymbol{\Psi} = \left(\frac{\operatorname{Re}\boldsymbol{\psi}}{\sqrt{\varepsilon}}, \frac{\operatorname{Im}\boldsymbol{\psi}}{\sqrt{\mu}}\right)$
Relativistic wave equations	$\left(\hat{p}^{\mu} \bullet \hat{p}_{\mu}\right)\psi = 0$	$\left(\hat{S}^{\mu}\hat{p}_{\mu}\right)\boldsymbol{\psi} = 0$
Four-momentum operator	$\hat{p}^{\mu} = \left(i\sqrt{\rho}\,\partial_t, \frac{-i\nabla}{\sqrt{\rho}}\right)$	$\hat{p}^{\mu} = \left(i\sqrt{\varepsilon\mu}\,\partial_t, -i\nabla\right)$
Topological indices	$w(\rho) = \frac{1}{2}\left[1 - \operatorname{sgn}(\rho)\right]$	$w(\varepsilon, \mu) = \frac{1}{2}\left[1 - \operatorname{sgn}(\varepsilon), 1 - \operatorname{sgn}(\mu)\right]$

Acoustic (longitudinal) waves

In quantum terms, acoustic waves correspond to massless and spinless particles: phonons ($\langle \mathbf{S} \rangle = 0$).

Therefore, the fundamental quantum–relativistic representation is the Klein–Gordon theory:

$$\mathcal{L} = \frac{1}{2}\left[c^{-2}\left(\partial_t \psi\right)^2 - \left(\nabla \psi\right)^2 \right]$$

$$\left(\hat{p}^{\mu}\hat{p}_{\mu}\right)\psi = \left(-c^{-2}\,\partial_t^2 + \nabla^2\right)\psi = 0$$

$$\left(W, \mathbf{\Pi}\right) = \left(\frac{1}{2}\left[c^{-2}\left(\partial_t \psi\right)^2 + \left(\nabla \psi\right)^2 \right], -\partial_t \psi \nabla \psi \right)$$

Berestetskii *et al. QED*

Acoustic (longitudinal) waves

The acoustic to Klein–Gordon mapping is:

$$\Psi^{\mu} = \left(P, \mathbf{v}\right) = i p^{\mu} \psi$$

$$a^{\mu} b_{\mu} \equiv \beta\, a^{(P)} b^{(P)} - \rho\, \mathbf{a}^{(\mathbf{v})} \cdot \mathbf{b}^{(\mathbf{v})}$$

$$\hat{p}^{\mu} = \left[i\sqrt{\rho}\,\partial_t, -\left(i / \sqrt{\rho}\right)\nabla \right]$$

$$c^2 = \left(\rho\beta\right)^{-1}$$

The Klein–Gordon wavefunction ψ is similar to the velocity potential:

$$\mathbf{v} = \left(1 / \sqrt{\rho}\right) \nabla \psi$$

Bliokh & Nori, *PRL* (2019)

Acoustic (longitudinal) waves

Most importantly, the $\rho \leftrightarrow \beta$ symmetry is broken and only $\sqrt{\rho}$ appears in this representation.

Moreover, the key operator of the problem, the **four–momentum**, is generally **non–Hermitian**:

$$\hat{p}^{\mu} = \left[i\sqrt{\rho}\,\partial_t, -\left(i / \sqrt{\rho} \right) \nabla \right]$$

Hence, the $\rho > 0$ and $\rho < 0$ zones are topologically different: real and imaginary four–momentum eigenvalues. $\rho = 0$ is exceptional.

Bliokh & Nori, *PRL* (2019)

Acoustic (longitudinal) waves

Entirely similar to the electromagnetic helicity, we introduce the topological Z_2 number:

$$w(\rho) = \frac{1}{2}\left[1 - \mathrm{sgn}(\rho)\right] = (0,1)$$

and the bulk–boundary correspondence:

$$N = \left| w(\rho_2) - w(\rho_1) \right| = w(\rho_r) = (0,1)$$

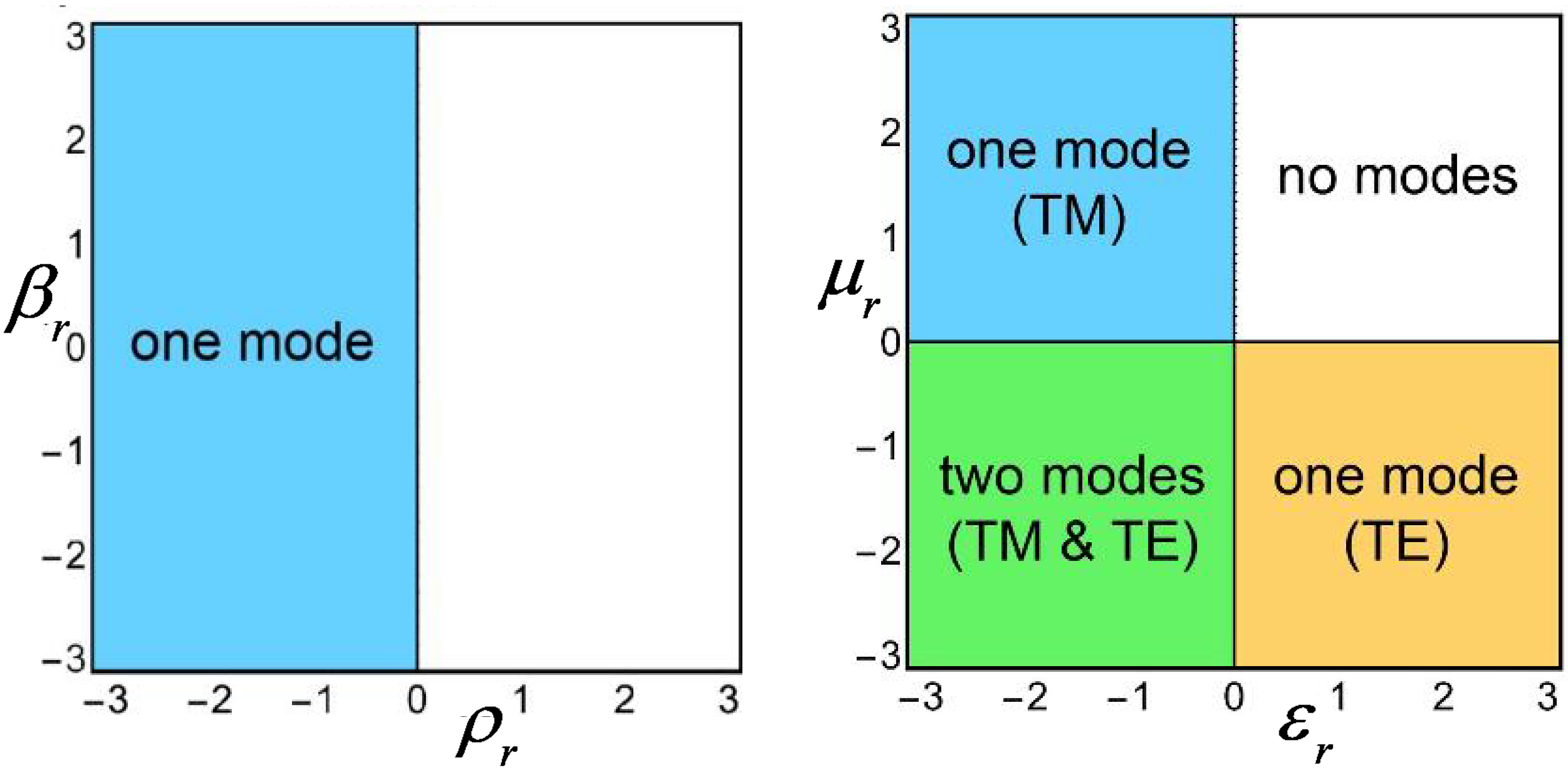
which explains the $\mathrm{sgn}(\rho_r) = -1$ condition for acoustic surface waves.

Bliokh & Nori, *PRL* (2019)

Acoustic (longitudinal) waves

Acoustic vs. electromagnetic problems:

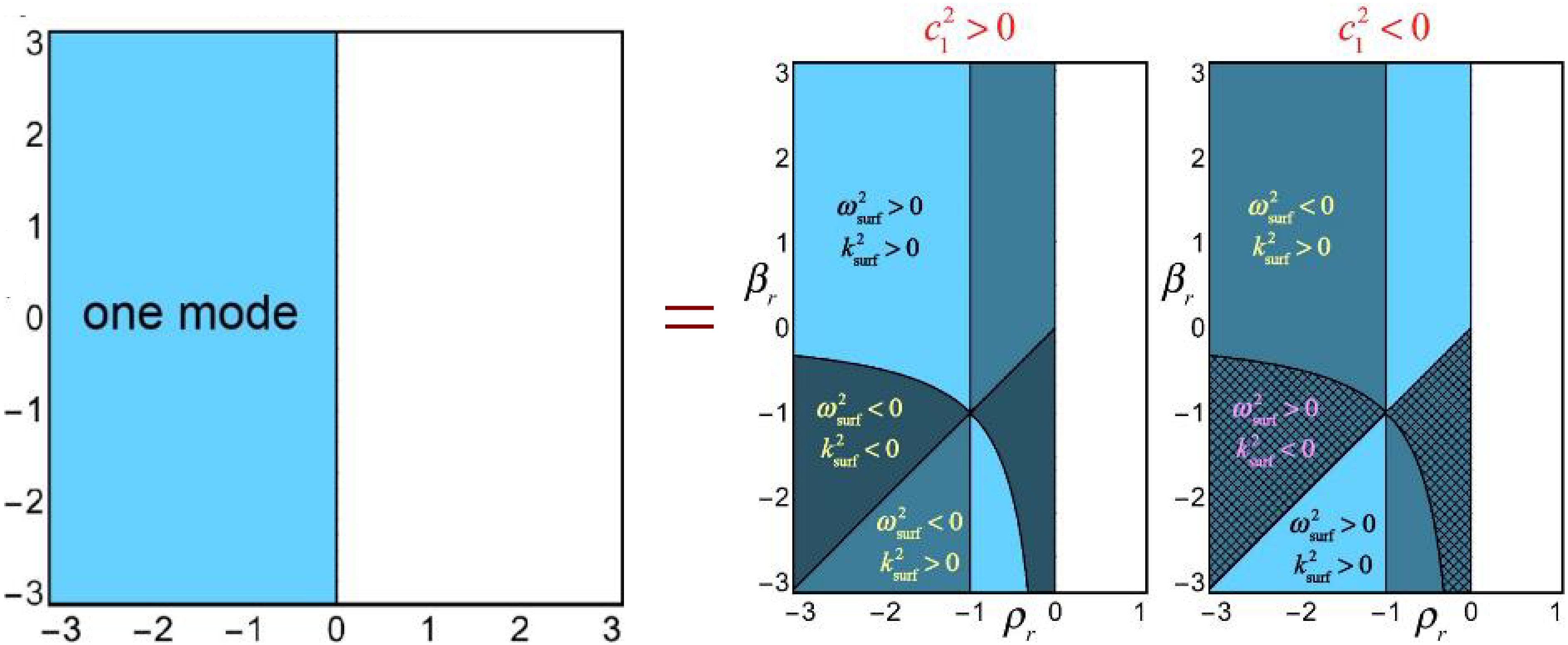
$$\text{topology of } \hat{p}^{\mu} \leftrightarrow \text{topology of } \hat{\mathfrak{S}}$$



Bliokh & Nori *PRL* (2019)

Acoustic (longitudinal) waves

Akin to the electromagnetic case, the "white spots" in the acoustic phase diagrams are filled by "dark" non–Hermitian modes with imaginary frequency/wavevector:



Bliokh & Nori, *PRL* (2019)

Acoustic (longitudinal) waves

This can be seen in the explicit solutions of the acoustic boundary problem for a planar interface:

$$\boxed{\frac{\kappa_1}{\rho_1} + \frac{\kappa_2}{\rho_2} = 0} \quad \text{– topological condition}$$

$$\boxed{k_{\text{surf}}^2} = \kappa_1^2 \frac{\rho_r(\beta_r - \rho_r)}{\rho_r \beta_r - 1}, \qquad \boxed{\omega_{\text{surf}}^2} = c_1^2 \kappa_1^2 \frac{(1 - \rho_r^2)}{\rho_r \beta_r - 1}$$

real or imagnary

Ambati *et al*. *PRB* (2007); Bliokh & Nori *PRB* (2019), *PRL* (2019)

Conclusions

- ✓ Surface acoustic modes between isotropic lossless media have a topological origin, similar to the electromagnetic surface modes.
- ✓ The topology of the four-momentum in the Klein-Gordon representation rather than helicity.
- ✓ Complex four-momentum spectrum.
  Z_2 and ρ instead of $Z_4=Z_2\times Z_2$ and (ε,μ).
- ✓ The bulk-boundary correspondence yields the conditions $\mathrm{sgn}(\rho_r)=-1$ for the acoustic surface modes.
- ✓ The four-momentum is non-Hermitian. So are the surface modes. Can be "dark" (imaginary frequency/wavevector).

Thank you!

Additional slides

	Acoustics	Electromagnetism
Real fields	$\Psi^{\mu} = \left(P, \mathbf{v}\right)$	$\boldsymbol{\Psi} = \left(\mathbf{E}, \mathbf{H}\right)$
Energy density and flux	$W = \frac{1}{2}\Psi^{\mu} \bullet \Psi^{\mu} = \frac{1}{2}\left(\beta P^2 + \rho \mathbf{v}^2\right)$ $\boldsymbol{\Pi} = \frac{1}{2}\Psi^{\mu} \otimes \Psi^{\mu} = P\mathbf{v}$	$W = \frac{1}{2}\boldsymbol{\Psi} \bullet \boldsymbol{\Psi} = \frac{1}{2}\left(\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2\right)$ $\boldsymbol{\Pi} = \frac{1}{2}\boldsymbol{\Psi} \otimes \boldsymbol{\Psi} = \mathbf{E} \times \mathbf{H}$
Connection with “relativistic wavefunctions”	$\Psi^{\mu} = i\hat{p}^{\mu}\psi$ $= \left(-\sqrt{\rho}\,\partial_t \psi, \frac{\nabla \psi}{\sqrt{\rho}}\right)$	$\boldsymbol{\Psi} = \left(\frac{\mathrm{Re}\,\boldsymbol{\psi}}{\sqrt{\varepsilon}}, \frac{\mathrm{Im}\,\boldsymbol{\psi}}{\sqrt{\mu}}\right)$
Relativistic wave equations	$\left(\hat{p}^{\mu} \bullet \hat{p}_{\mu}\right)\psi = 0$	$\left(\hat{S}^{\mu}\hat{p}_{\mu}\right)\boldsymbol{\psi} = 0$
Four-momentum operator	$\hat{p}^{\mu} = \left(i\sqrt{\rho}\,\partial_t, \frac{-i\nabla}{\sqrt{\rho}}\right)$	$\hat{p}^{\mu} = \left(i\sqrt{\varepsilon\mu}\,\partial_t, -i\nabla\right)$
Topological indices	$w(\rho) = \frac{1}{2}\left[1 - \mathrm{sgn}(\rho)\right]$	$w(\varepsilon, \mu) = \frac{1}{2}\left[1 - \mathrm{sgn}(\varepsilon), 1 - \mathrm{sgn}(\mu)\right]$

Relativistic formalisms for acoustic and Maxwell equations

	Acoustics	Electromagnetism
Real fields	$\Psi^{\mu}=\left(P,\mathbf{v}\right)$	$\boldsymbol{\Psi}=\left(\mathbf{E},\mathbf{H}\right)$
Energy density and flux	$W=\frac{1}{2}\Psi^{\mu}\cdot\Psi^{\mu}=\frac{1}{2}\left(\beta P^2+\rho\mathbf{v}^2\right)$ $\boldsymbol{\Pi}=\frac{1}{2}\Psi^{\mu}\otimes\Psi^{\mu}=P\mathbf{v}$	$W=\frac{1}{2}\boldsymbol{\Psi}\cdot\boldsymbol{\Psi}=\frac{1}{2}\left(\varepsilon\mathbf{E}^2+\mu\mathbf{H}^2\right)$ $\boldsymbol{\Pi}=\frac{1}{2}\boldsymbol{\Psi}\otimes\boldsymbol{\Psi}=\mathbf{E}\times\mathbf{H}$
Connection with “relativistic wavefunctions”	$\Psi^{\mu}=i\hat{p}^{\mu}\psi$ $=\left(-\sqrt{\rho}\,\partial_t\psi,\ \frac{\nabla\psi}{\sqrt{\rho}}\right)$	$\boldsymbol{\Psi}=\left(\frac{\operatorname{Re}\boldsymbol{\psi}}{\sqrt{\varepsilon}},\frac{\operatorname{Im}\boldsymbol{\psi}}{\sqrt{\mu}}\right)$
Relativistic wave equations	$\left(\hat{p}^{\mu}\cdot\hat{p}_{\mu}\right)\psi=0$	$\left(\hat{S}^{\mu}\hat{p}_{\mu}\right)\boldsymbol{\psi}=0$
Four-momentum operator	$\hat{p}^{\mu}=\left(i\sqrt{\rho}\,\partial_t,\frac{-i\nabla}{\sqrt{\rho}}\right)$	$\hat{p}^{\mu}=\left(i\sqrt{\varepsilon\mu}\,\partial_t,-i\nabla\right)$
Topological indices	$w(\rho)=\frac{1}{2}\left[1-\operatorname{sgn}(\rho)\right]$	$w(\varepsilon,\mu)=\frac{1}{2}\left[1-\operatorname{sgn}(\varepsilon),1-\operatorname{sgn}(\mu)\right]$

II. General properties of acoustic wave fields

We start with the linear equations for acoustic (sound) waves in a homogeneous dense medium, fluid or gas [41]:

$$\beta \frac{\partial P}{\partial t} = -\nabla \cdot \mathbf{v}, \quad \rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla P. \tag{1}$$

Here the variables are the velocity $\mathbf{v}(\mathbf{r}, t)$ and the pressure $P(\mathbf{r}, t)$ fields, while the real-valued medium parameters are the mass density ρ and the compressibility $\beta = 1/B$ (B is the bulk modulus). Equations (1) obey the energy conservation law, an acoustic analog of the electromagnetic Poynting theorem:

$$\frac{\partial}{\partial t}\left(\frac{\beta P^2}{2} + \frac{\rho \mathbf{v}^2}{2}\right) + \nabla \cdot (P\mathbf{v}) = 0, \tag{2}$$

where the expressions in the first and second parentheses determine the acoustic energy density and energy flux density, respectively [41].

From now on, we consider monochromatic acoustic waves of frequency ω. Making the substitution $\mathbf{v}(\mathbf{r}, t) \to \mathrm{Re}[\mathbf{v}(\mathbf{r})e^{-i\omega t}]$ and $P(\mathbf{r}, t) \to \mathrm{Re}[P(\mathbf{r})e^{-i\omega t}]$, Eqs. (1) are reduced to the following equations for the complex velocity and pressure fields $\mathbf{v}(\mathbf{r})$ and $P(\mathbf{r})$:

$$\nabla \cdot \mathbf{v} = i\beta\omega P, \quad \nabla P = i\rho\,\omega\,\mathbf{v}. \tag{3}$$

Equations (1) or (3) support only longitudinal (i.e., curl-free) waves: $\nabla \times \mathbf{v} = 0$. For plane waves with the wave vector $\mathbf{k}$, $\nabla \to i\mathbf{k}$, the dispersion relation and the "longitudinality" condition follow from Eqs. (3):

$$\omega^2 = k^2 c^2 \equiv \frac{k^2}{\rho\beta}, \quad \mathbf{k} \times \mathbf{v} = 0, \tag{4}$$

where c is the speed of sound.

TABLE I. Comparison of acoustic and electromagnetic quantities and properties.

	Acoustics	Electromagnetism
Fields	Velocity $\mathbf{v}$, pressure P	Electric $\mathbf{E}$, magnetic $\mathbf{H}$
Constraints	$\nabla \times \mathbf{v} = 0$	$\nabla \cdot \mathbf{E} = 0, \nabla \cdot \mathbf{H} = 0$
Energy density	$\frac{1}{4}\left(\rho \lvert\mathbf{v}\rvert^2 + \beta \lvert P\rvert^2\right)$	$\frac{1}{4}\left(\varepsilon \lvert\mathbf{E}\rvert^2 + \mu \lvert\mathbf{H}\rvert^2\right)$
Canonical momentum density	$\frac{1}{4\omega}\mathrm{Im}\left[\beta\, P^* \nabla P + \rho\, \mathbf{v}^* \cdot (\nabla)\mathbf{v}\right]$	$\frac{1}{4\omega}\mathrm{Im}\left[\varepsilon\, \mathbf{E}^* \cdot (\nabla)\mathbf{E} + \mu\, \mathbf{H}^* \cdot (\nabla)\mathbf{H}\right]$
Kinetic momentum density	$\frac{1}{2c^2}\mathrm{Re}(P^*\mathbf{v}) = \mathbf{p} + \frac{1}{4}\nabla \times \mathbf{S}$	$\frac{1}{2c^2}\mathrm{Re}(\mathbf{E}^* \times \mathbf{H}) = \mathbf{p} + \frac{1}{2}\nabla \times \mathbf{S}$
Spin AM density	$\frac{1}{2\omega}\rho\, \mathrm{Im}(\mathbf{v}^* \times \mathbf{v})$	$\frac{1}{4\omega}\left[\varepsilon\, \mathrm{Im}(\mathbf{E}^* \times \mathbf{E}) + \mu\, \mathrm{Im}(\mathbf{H}^* \times \mathbf{H})\right]$
Orbital AM density	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$
Integral AM values	$\langle\mathbf{M}\rangle = \langle\mathbf{L}\rangle, \langle\mathbf{S}\rangle = 0$	$\langle\mathbf{M}\rangle = \langle\mathbf{L}\rangle + \langle\mathbf{S}\rangle, \langle\mathbf{S}\rangle \neq 0$
Helicity	$\mathfrak{S} \equiv 0$	$\mathfrak{S} \not\equiv 0$

TABLE I. Comparison of acoustic and electromagnetic quantities and properties.

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Fields	Velocity $\mathbf{v}$, pressure P	Electric $\mathbf{E}$, magnetic $\mathbf{H}$
Constraints	$\nabla \times \mathbf{v} = 0$	$\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{H} = 0$
Energy density	$\frac{1}{4}\left(\rho\lvert\mathbf{v}\rvert^2 + \beta\lvert P\rvert^2\right)$	$\frac{1}{4}\left(\varepsilon\lvert\mathbf{E}\rvert^2 + \mu\lvert\mathbf{H}\rvert^2\right)$
Canonical momentum density	$\frac{1}{4\omega}\mathrm{Im}\left[\beta\, P^*\nabla P + \rho\, \mathbf{v}^* \cdot (\nabla)\mathbf{v}\right]$	$\frac{1}{4\omega}\mathrm{Im}\left[\varepsilon\, \mathbf{E}^* \cdot (\nabla)\mathbf{E} + \mu\, \mathbf{H}^* \cdot (\nabla)\mathbf{H}\right]$
Kinetic momentum density	$\frac{1}{2c^2}\mathrm{Re}(P^*\mathbf{v}) = \mathbf{p} + \frac{1}{4}\nabla \times \mathbf{S}$	$\frac{1}{2c^2}\mathrm{Re}(\mathbf{E}^* \times \mathbf{H}) = \mathbf{p} + \frac{1}{2}\nabla \times \mathbf{S}$
Spin AM density	$\frac{1}{2\omega}\rho\,\mathrm{Im}(\mathbf{v}^* \times \mathbf{v})$	$\frac{1}{4\omega}\left[\varepsilon\,\mathrm{Im}(\mathbf{E}^* \times \mathbf{E}) + \mu\,\mathrm{Im}(\mathbf{H}^* \times \mathbf{H})\right]$
Orbital AM density	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$
Integral AM values	$\langle\mathbf{M}\rangle = \langle\mathbf{L}\rangle$, $\langle\mathbf{S}\rangle = 0$	$\langle\mathbf{M}\rangle = \langle\mathbf{L}\rangle + \langle\mathbf{S}\rangle$, $\langle\mathbf{S}\rangle \neq 0$
Helicity	$\mathfrak{S} \equiv 0$	$\mathfrak{S} \not\equiv 0$

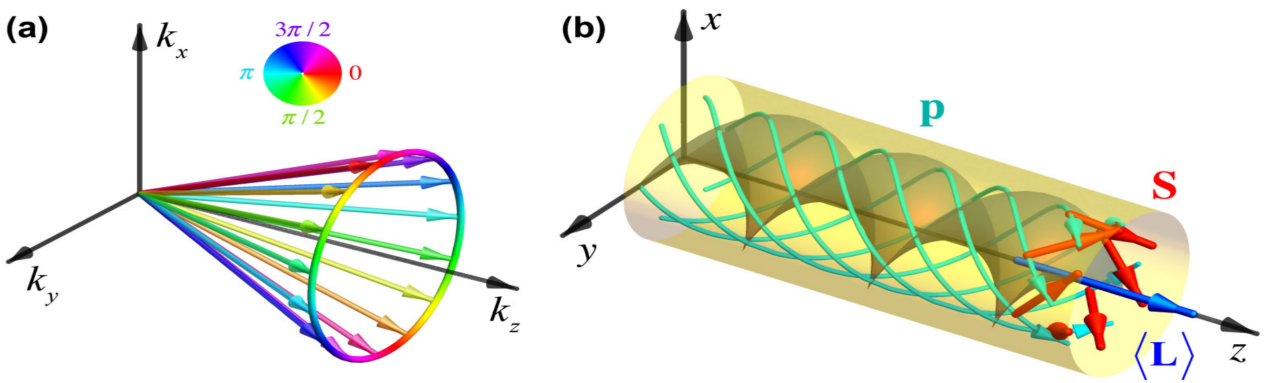

FIG. 1. Schematics of the acoustic Bessel beams. (a) The momentum (plane-wave) spectrum of the beam is a circle with fixed polar angle θ_0. The mutual phases of the plane waves (color-marked) have an azimuthal gradient and the $2\pi\ell$ increment around the circle ($\ell = 2$ is shown here). (b) The real-space field forms a cylindrically symmetric vortex beam possessing the helical phase front and carrying the orbital AM $\langle\mathbf{L}\rangle \propto \ell\bar{\mathbf{z}}$. This angular momentum is produced by the spiraling canonical momentum density $\mathbf{p}$ in the beam (shown in cyan). Although all plane waves in the spectrum (a) are longitudinally polarized (i.e., the Fourier components of the velocity $\tilde{\mathbf{v}}(\mathbf{k}) \parallel \mathbf{k}$), the local polarization in real space, $\mathbf{v}(\mathbf{r})$, becomes elliptical, which produces a nonzero spin AM density $\mathbf{S} \propto \mathrm{Im}(\mathbf{v}^* \times \mathbf{v})$ (shown by red arrows).

Importantly, although commonly classified as "scalar waves", sound waves also have inherent vector properties [38–40]. Indeed, these waves are described by one scalar (pressure) and one vector (velocity) fields, which determine the qualitatively different degrees of freedom in the acoustic field. These scalar and vector degrees of freedom are equally important, as can be seen from their equal contributions to the energy conservation law (2). In quantum-like terms, one can say that acoustic waves are described by the four-component "wave function" $\psi = (P, \mathbf{v})^T$. In what follows, we will use a fruitful analogy with electromagnetic waves described by Maxwell equations. The main difference is that Maxwell waves are described by two vector fields (electric and magnetic), $\psi = (\mathbf{E}, \mathbf{H})^T$, and these are transverse (i.e., divergence-free) rather than longitudinal: $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0$.

Similarly to electromagnetic waves [5,6,42,43], the main dynamical properties of acoustic wave fields are the energy, momentum, and angular momentum. The time-averaged energy density and energy flux density (an acoustic counterpart of the Poynting vector) in a monochromatic acoustic field follow from Eq. (2):

$$W = \frac{1}{4}\left(\beta |P|^2 + \rho |\mathbf{v}|^2\right), \quad \boldsymbol{\Pi} = \frac{1}{2}\mathrm{Re}(P^* \mathbf{v}). \tag{5}$$

TABLE I. Comparison of acoustic and electromagnetic quantities and properties.

	Acoustics	Electromagnetism
Fields	Velocity $\mathbf{v}$, pressure P	Electric $\mathbf{E}$, magnetic $\mathbf{H}$
Constraints	$\nabla \times \mathbf{v} = 0$	$\nabla \cdot \mathbf{E} = 0, \nabla \cdot \mathbf{H} = 0$
Energy density	$\frac{1}{4}\left(\rho\|\mathbf{v}\|^2 + \beta\|P\|^2\right)$	$\frac{1}{4}\left(\varepsilon\|\mathbf{E}\|^2 + \mu\|\mathbf{H}\|^2\right)$
Canonical momentum density	$\frac{1}{4\omega}\mathrm{Im}\left[\beta\, P^*\nabla P + \rho\, \mathbf{v}^* \cdot (\nabla)\mathbf{v}\right]$	$\frac{1}{4\omega}\mathrm{Im}\left[\varepsilon\, \mathbf{E}^* \cdot (\nabla)\mathbf{E} + \mu\, \mathbf{H}^* \cdot (\nabla)\mathbf{H}\right]$
Kinetic momentum density	$\frac{1}{2c^2}\mathrm{Re}(P^*\mathbf{v}) = \mathbf{p} + \frac{1}{4}\nabla \times \mathbf{S}$	$\frac{1}{2c^2}\mathrm{Re}(\mathbf{E}^* \times \mathbf{H}) = \mathbf{p} + \frac{1}{2}\nabla \times \mathbf{S}$
Spin AM density	$\frac{1}{2\omega}\rho\,\mathrm{Im}(\mathbf{v}^* \times \mathbf{v})$	$\frac{1}{4\omega}\left[\varepsilon\,\mathrm{Im}(\mathbf{E}^* \times \mathbf{E}) + \mu\,\mathrm{Im}(\mathbf{H}^* \times \mathbf{H})\right]$
Orbital AM density	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$
Integral AM values	$\langle\mathbf{M}\rangle = \langle\mathbf{L}\rangle, \langle\mathbf{S}\rangle = 0$	$\langle\mathbf{M}\rangle = \langle\mathbf{L}\rangle + \langle\mathbf{S}\rangle, \langle\mathbf{S}\rangle \neq 0$
Helicity	$\mathfrak{S} \equiv 0$	$\mathfrak{S} \not\equiv 0$

Employing the quantum-like formalism [5,42,43], the energy density can be regarded as the local expectation value of the energy (frequency) operator ω, $W = (\psi|\omega|\psi)$, where the inner product $(\psi|\psi)$ is defined with the scaling coefficients $\beta/4\omega$ and $\rho/4\omega$ at the pressure and velocity degrees of freedom, respectively. Using this formalism, similarly to the electromagnetic case [5,6,8,10,42–44], we introduce the *canonical momentum density* of the acoustic field as the local expectation value of the momentum operator $\hat{\mathbf{p}} = -i\nabla$:

$$\mathbf{p} = \frac{1}{4\omega}\mathrm{Im}\left[\beta\, P^{*}\nabla P + \rho\, \mathbf{v}^{*}\cdot(\nabla)\mathbf{v}\right], \qquad (6)$$

where $[\mathbf{v}^{*}\cdot(\nabla)\mathbf{v}]_i \equiv \sum_j v_j^{*}\nabla_i v_j$. The momentum density (6) represents the natural definition of the local phase gradient (i.e., the local wave vector) in a multicomponent field ψ (for a single-component scalar field it would be proportional to $\nabla \mathrm{Arg}(\psi)$) [44].

In analogy with electromagnetism, the energy flux density (5) can also be associated with the momentum density (multiplied by c^2), but this should be regarded as the *kinetic momentum density* $\mathbf{\Pi}/c^2$ [41]. Using some vector algebra involving the "longitudinality" condition $\nabla \times \mathbf{v} = 0$, the difference between the kinetic and canonical momentum can be written as

$$\frac{\mathbf{\Pi}}{c^2} = \mathbf{p} + \frac{1}{4}\nabla \times \mathbf{S}, \quad \mathbf{S} = \frac{\rho}{2\omega}\mathrm{Im}(\mathbf{v}^* \times \mathbf{v}). \tag{7}$$

Here, $\mathbf{S}$ is the *spin AM density* of the acoustic waves [39,40]. Thus, entirely similar to the electromagnetic case [5,6,8,10,42,43], the difference between the kinetic and canonical momentum densities in the acoustic field is related to the presence of the spin AM density. This difference can be regarded as the *spin momentum* density $\mathbf{p}_S = \frac{1}{4}\nabla \times \mathbf{S}$ [8,10,42,44,45]. The only distinction as compared to electromagnetism is the prefactor $1/4$ instead of $1/2$; this is because the scalar (pressure) part of the "wave function" ψ does not contribute to the difference between the kinetic and canonical momentum. Remarkably, the spin density (7) can also be presented as the local expectation value of the spin-1 operator $\hat{\mathbf{S}}$ acting on the vector (velocity) degrees of freedom such that $\mathbf{v}^* \cdot (\hat{\mathbf{S}})\mathbf{v} = \mathrm{Im}(\mathbf{v}^* \times \mathbf{v})$ [5,42–44]: $\mathbf{S} = 2(\psi|\hat{\mathbf{S}}|\psi)$, where a factor of 2 originates from the same asymmetry between the scalar and vector degrees of freedom [40].

TABLE I. Comparison of acoustic and electromagnetic quantities and properties.

	Acoustics	Electromagnetism
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Energy density	$\frac{1}{4}\left(\rho\lvert\mathbf{v}\rvert^2 + \beta\lvert P\rvert^2\right)$	$\frac{1}{4}\left(\varepsilon\lvert\mathbf{E}\rvert^2 + \mu\lvert\mathbf{H}\rvert^2\right)$
Canonical momentum density	$\frac{1}{4\omega}\mathrm{Im}\left[\beta\, P^*\nabla P + \rho\, \mathbf{v}^* \cdot (\nabla)\mathbf{v}\right]$	$\frac{1}{4\omega}\mathrm{Im}\left[\varepsilon\, \mathbf{E}^* \cdot (\nabla)\mathbf{E} + \mu\, \mathbf{H}^* \cdot (\nabla)\mathbf{H}\right]$
Kinetic momentum density	$\frac{1}{2c^2}\mathrm{Re}(P^*\mathbf{v}) = \mathbf{p} + \frac{1}{4}\nabla \times \mathbf{S}$	$\frac{1}{2c^2}\mathrm{Re}(\mathbf{E}^* \times \mathbf{H}) = \mathbf{p} + \frac{1}{2}\nabla \times \mathbf{S}$
Spin AM density	$\frac{1}{2\omega}\rho\, \mathrm{Im}(\mathbf{v}^* \times \mathbf{v})$	$\frac{1}{4\omega}\left[\varepsilon\, \mathrm{Im}(\mathbf{E}^* \times \mathbf{E}) + \mu\, \mathrm{Im}(\mathbf{H}^* \times \mathbf{H})\right]$
Orbital AM density	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$
Integral AM values	$\langle\mathbf{M}\rangle = \langle\mathbf{L}\rangle$, $\langle\mathbf{S}\rangle = 0$	$\langle\mathbf{M}\rangle = \langle\mathbf{L}\rangle + \langle\mathbf{S}\rangle$, $\langle\mathbf{S}\rangle \neq 0$
Helicity	$\mathfrak{S} \equiv 0$	$\mathfrak{S} \not\equiv 0$