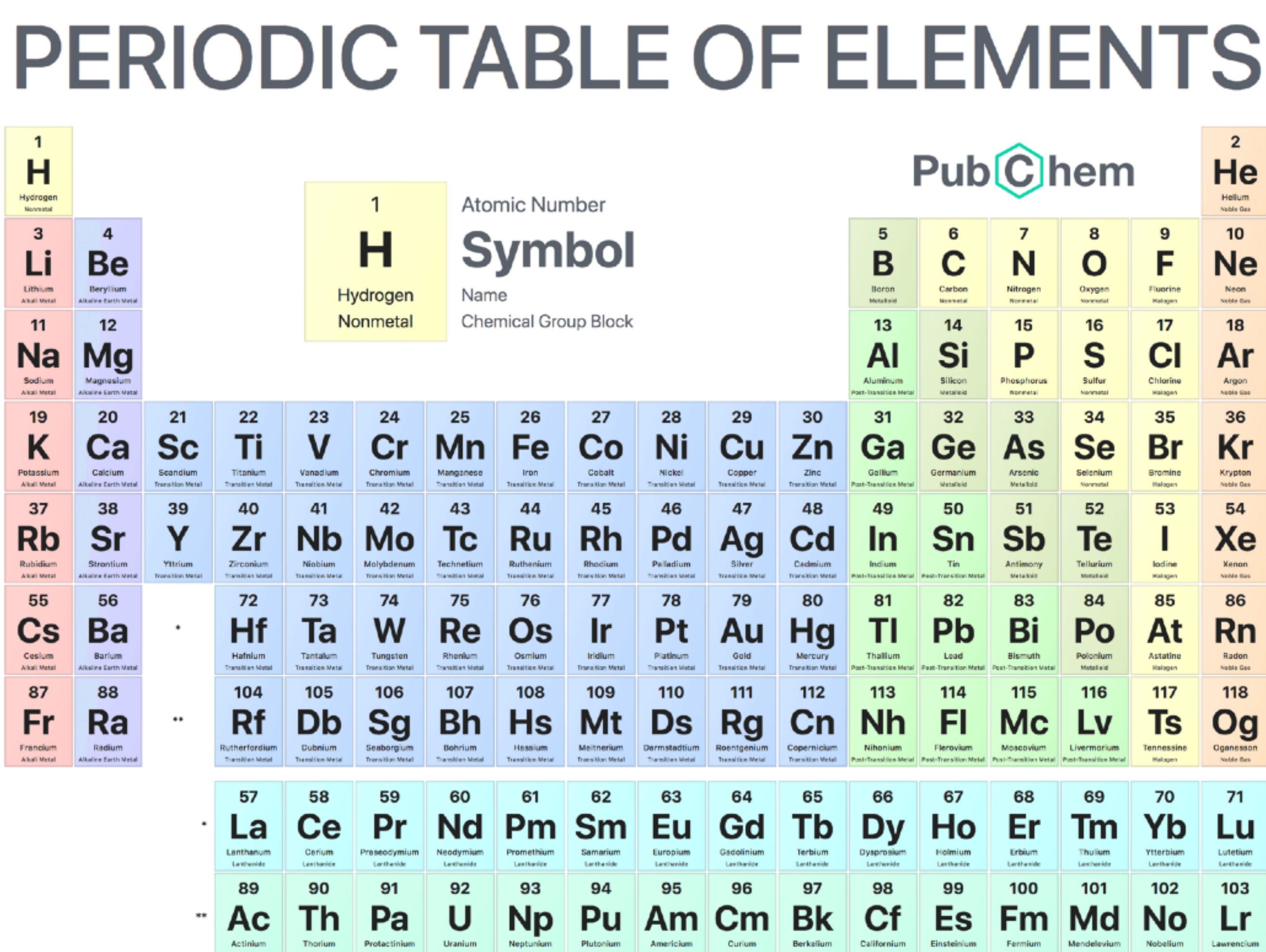
Topological Material Search via Symmetry Indicator and Filling Anomaly
+
Revisiting criteria for topological phases

Haruki Watanabe
University of Tokyo

Variety of materials



From https://pubchem.ncbi.nlm.nih.gov/

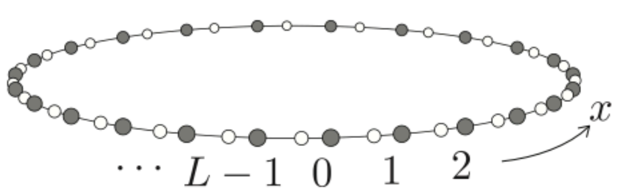
Unified classification?

Classification based on low-E spectrum

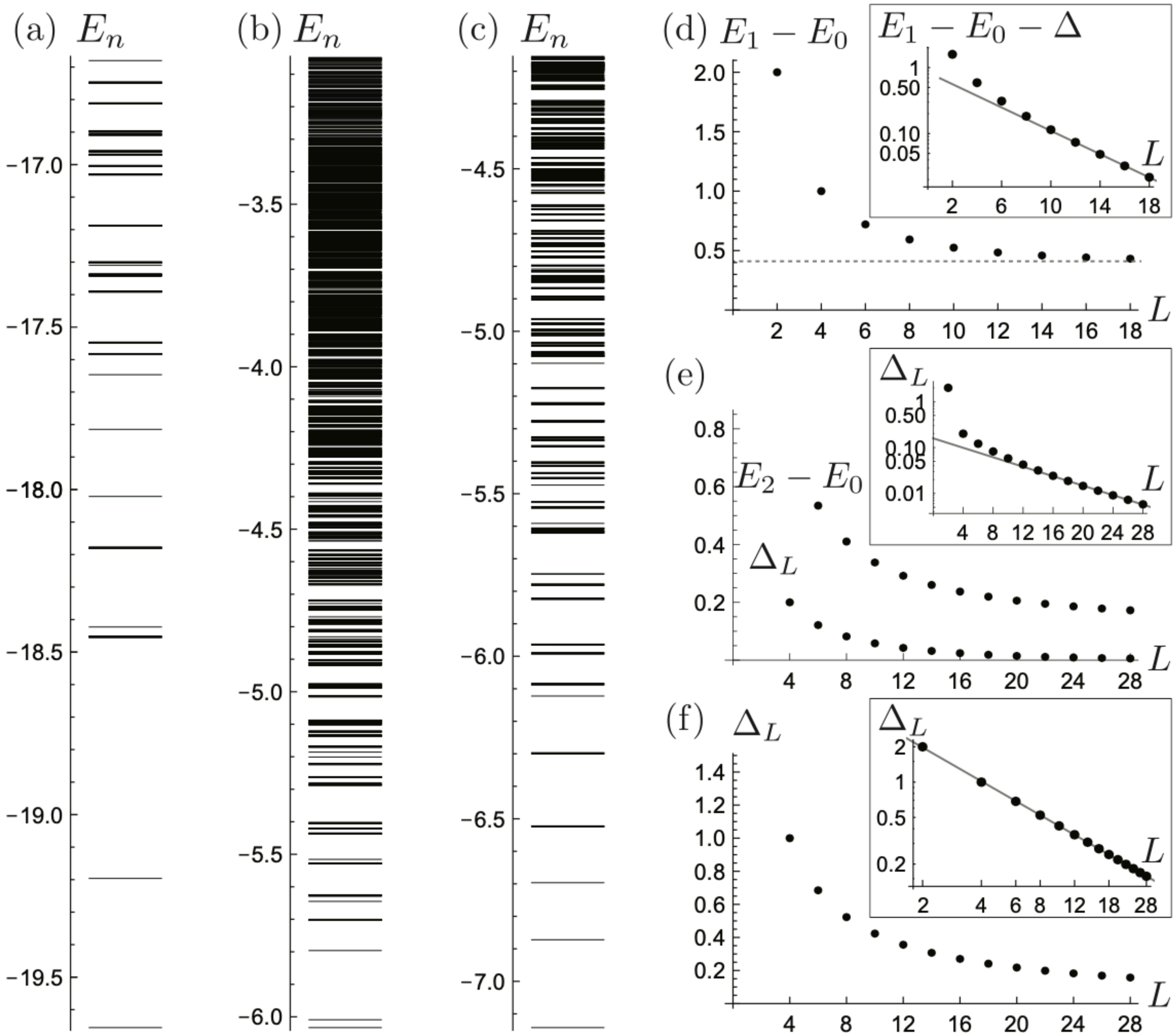
$$\hat{H}_{\text{Heisenberg}} := J \sum_{m=0}^{L-1} \hat{\boldsymbol{s}}_{m+1} \cdot \hat{\boldsymbol{s}}_m$$

$$\hat{H}_{\text{MG}} = \hat{H}_{\text{Heisenberg}} + \frac{J}{2} \sum_{m=0}^{L-1} \hat{\boldsymbol{s}}_{m+2} \cdot \hat{\boldsymbol{s}}_m$$

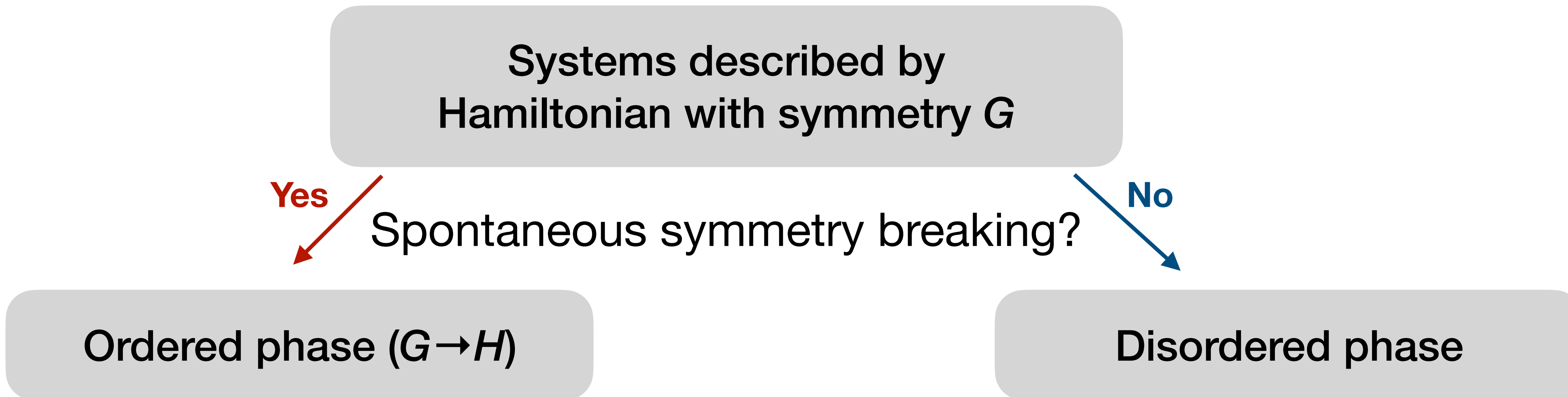
Periodic boundary condition



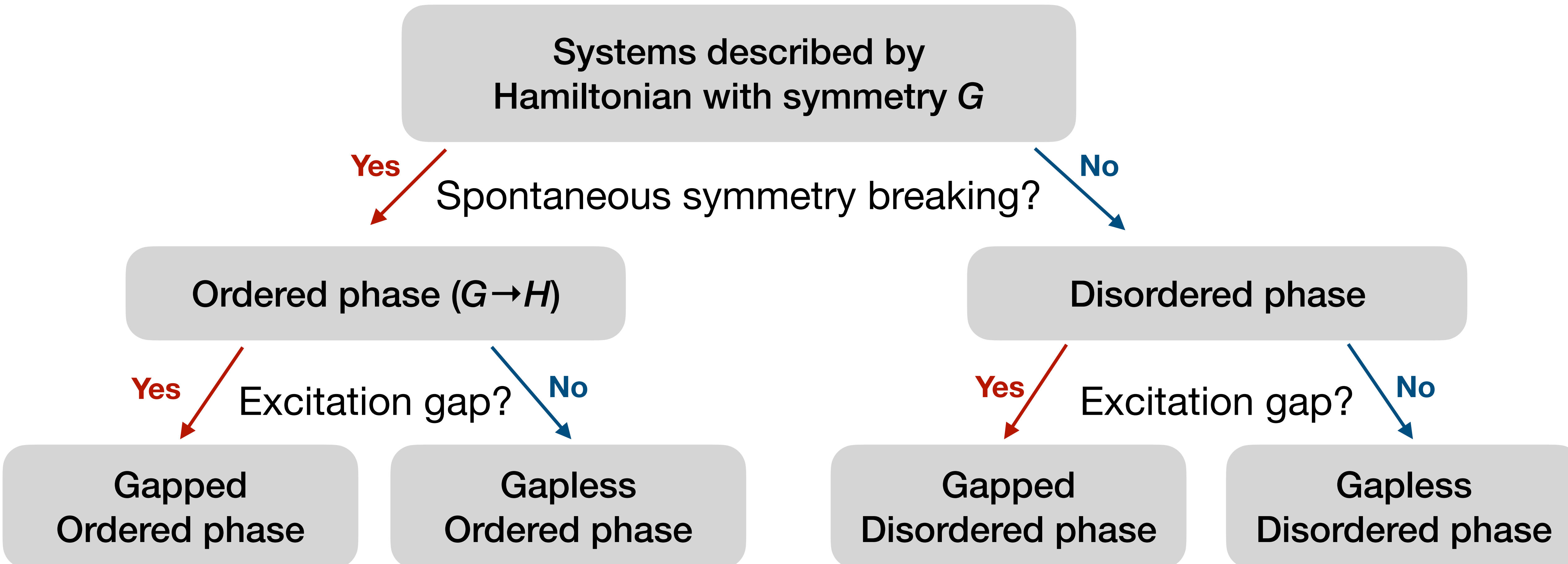
S=1 Heisenberg model | **S=1/2 MG model** | **S=1/2 Heisenberg model**



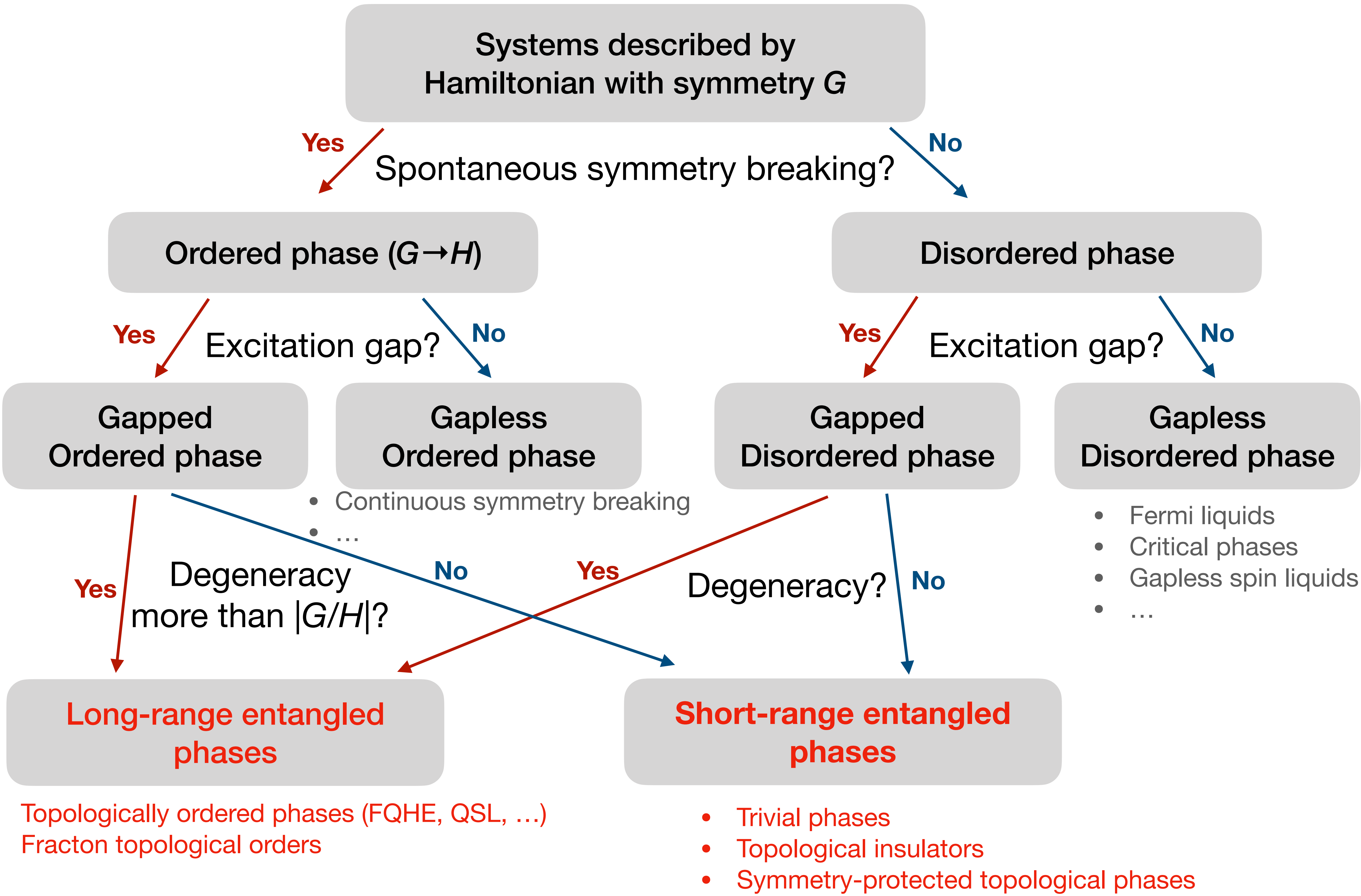
Classification of T=0 phases



Classification of T=0 phases



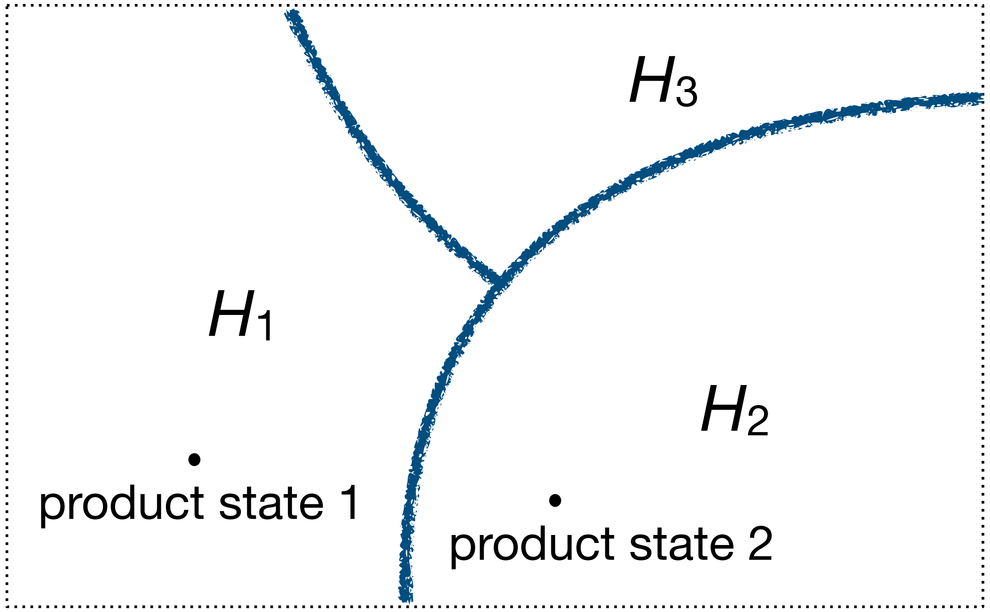
Classification of T=0 phases



Features of short-range entangled phases

- Unique ground state with excitation gap Δ.

- With symmetry G, $H_1 \sim H_2$ if H_1 and H_2 are connected without breaking symmetry G or closing gap Δ (with or without ancillas)

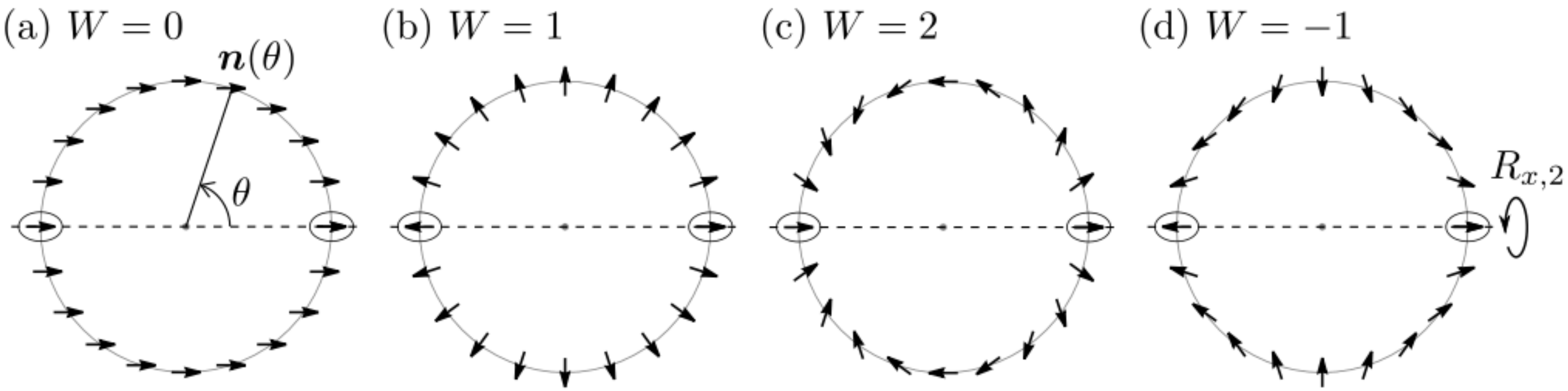


- *Trivial phases* are connected to a real-space **product state**.

- *Topological phases* contain irremovable **quantum entanglement.**

Symmetry indicators

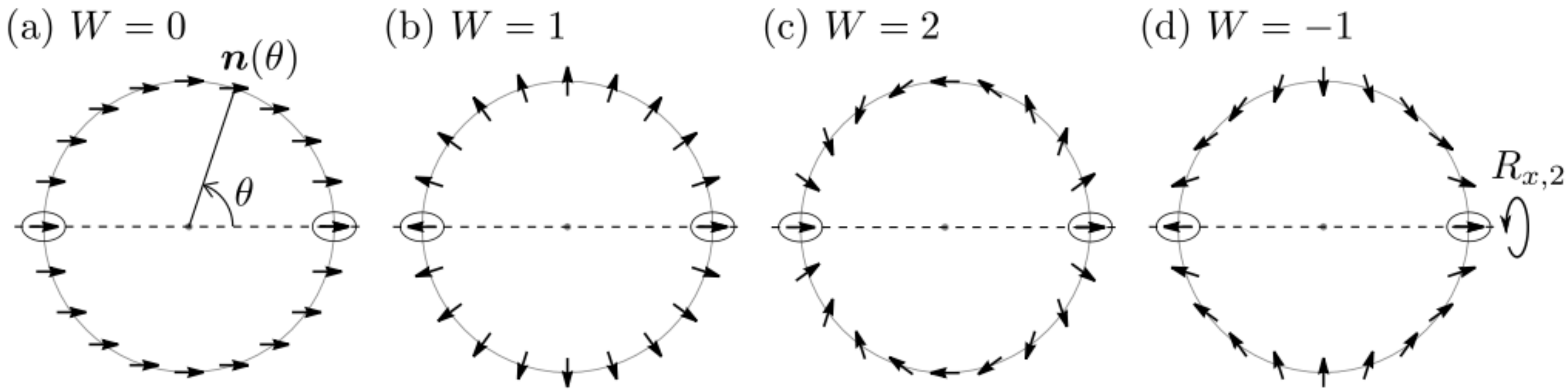
- Quick & partial diagnosis of topology
- Example: Winding number of in-plane spins on a ring



$$W[\boldsymbol{n}] := \int_0^{2\pi} \frac{d\theta}{2\pi} \left[\boldsymbol{n}(\theta) \times \partial_\theta \boldsymbol{n}(\theta)\right]_z$$

Symmetry indicators

- Quick & partial diagnosis of topology
- Example: Winding number of in-plane spins on a ring



$$W[\boldsymbol{n}] := \int_0^{2\pi} \frac{d\theta}{2\pi} \left[\boldsymbol{n}(\theta) \times \partial_\theta \boldsymbol{n}(\theta)\right]_z$$

- Winding number can be partially deduced by representation at symmetric points

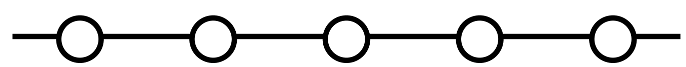
$$(-1)^{W[\boldsymbol{n}]} = \boldsymbol{n}(\pi) \cdot \boldsymbol{n}(0)$$

Lieb-Schultz-Mattis theorem

Necessary condition for unique ground state with excitation gap.
Violation implies degeneracy or gapless excitations.

- In U(1) symmetric systems with translation symmetry,
  filling ν (= change per unit cell) must be integral.
  Examples:
  - Band insulators with fractional filling are gapless.
  - Fractional quantum Hall states has topological degeneracy.

- In systems with discrete symmetry group *G*,
  all projective representations must be removable.
  Examples:
  - S=1/2 Heisenberg model with $Z_2 \times Z_2$ and translation is gapless / degeneracy
  - Toric code (with inversion) has topological degeneracy.



Violation of LSM conditions in symmetric gapped phases
→ Topological degeneracy on torus

Today's talk

- (brief) **Symmetry indicators for superconductors**

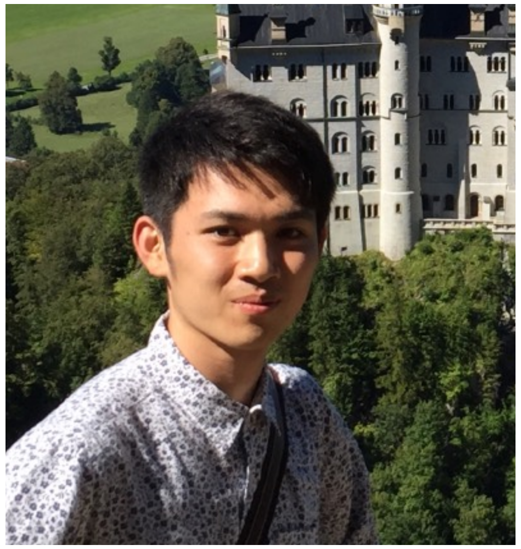
- **Revisiting criteria for topological phases**

  - **Symmetry indicators** → topological (crystalline) insulators
    The converse is certainly not true.
    - Chern number C=2n with C_2 rotation
    - fragile topology
    - …

  - **Product states** → not interesting.
    A counter example.

  - **Topological degeneracy on torus** → topological orders
    Is the converse true?

Symmetry Indicators for topological superconductors



Seishiro Ono

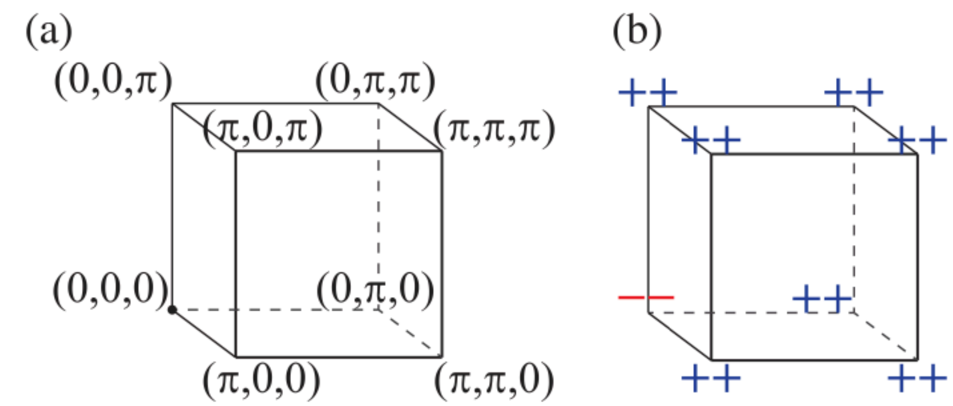
U Tokyo

Getting PhD soon!

F. Tang*, S. Ono*, X. Wan, HW, *High-Throughput Investigations of Topological and Nodal Superconductors*, PRL (2022). Editors' Suggestion

Symmetry indicators for topological insulators

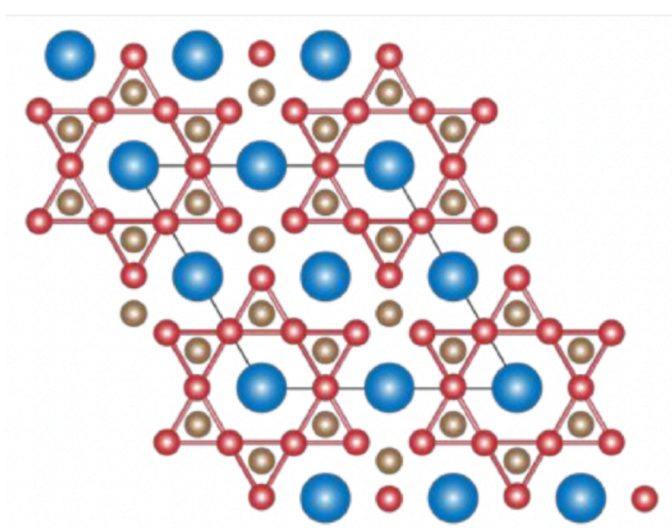
- Topological properties via representations of crystalline symmetries
  e.g. Fu-Kane formula for Z2 index
  via inversion parities



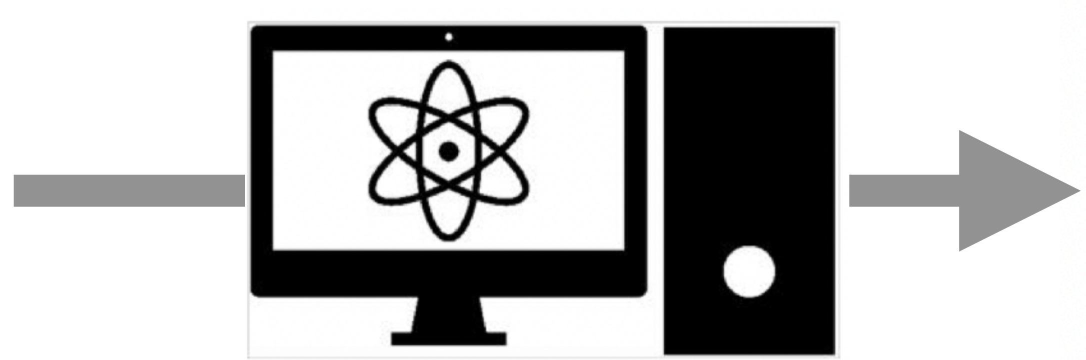
- All (magnetic) space groups & Various topology

- H. C. Po, A. Vishwanath, HW, Nat. Commun. (2017)
- B. Bradlyn, L. Elcoro, J. Cano, M. G. Vergniory, Z. Wang, C. Felser, M. I. Aroyo & B. A. Bernevig, Nature (2017)
- J. Kruthoff, J. de Boer, J. van Wezel, C. L. Kane, and R.-J. Slager, PRX (2017)
- Z. Song, T. Zhang, Z. Fang, C. Fang, Nat Comm (2018)
- …

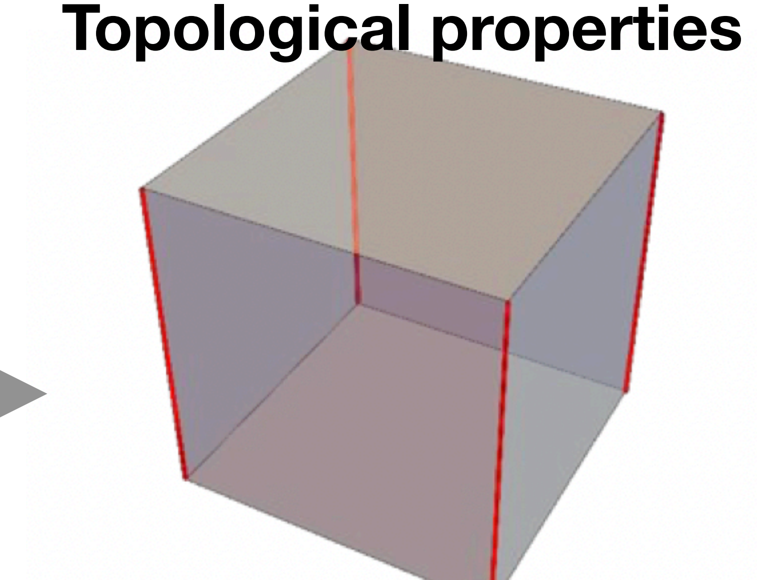
Input: Material data

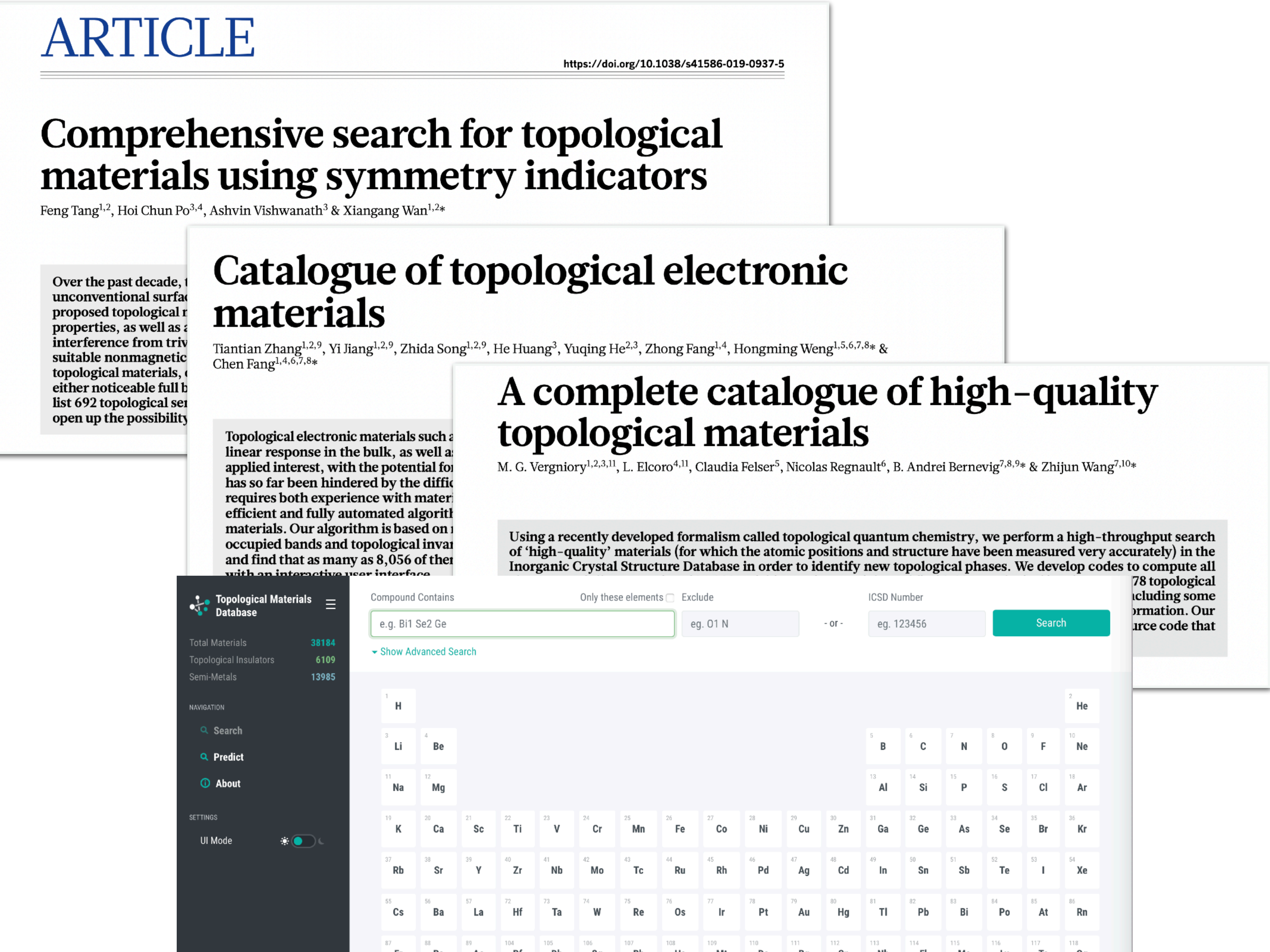


DFT calculations



Output: Topological properties




ARTICLE
https://doi.org/10.1038/s41586-019-0937-5
Comprehensive search for topological materials using symmetry indicators
Feng Tang[1,2], Hoi Chun Po[3,4], Ashvin Vishwanath[3] & Xiangang Wan[1,2]*
Over the past decade, t
unconventional surfac
proposed topological r
properties, as well as a
interference from triv
suitable nonmagnetic
topological materials, c
either noticeable full b
list 692 topological ser
open up the possibility
Catalogue of topological electronic materials
Tiantian Zhang[1,2,9], Yi Jiang[1,2,9], Zhida Song[1,2,9], He Huang[3], Yuqing He[2,3], Zhong Fang[1,4], Hongming Weng[1,5,6,7,8]* & Chen Fang[1,4,6,7,8]*
Topological electronic materials such a
linear response in the bulk, as well a
applied interest, with the potential fo
has so far been hindered by the diffic
requires both experience with materi
efficient and fully automated algorith
materials. Our algorithm is based on
occupied bands and topological invar
and find that as many as 8,056 of the
A complete catalogue of high-quality topological materials
M. G. Vergniory[1,2,3,11], L. Elcoro[4,11], Claudia Felser[5], Nicolas Regnault[6], B. Andrei Bernevig[7,8,9]* & Zhijun Wang[7,10]*
Using a recently developed formalism called topological quantum chemistry, we perform a high-throughput search of 'high-quality' materials (for which the atomic positions and structure have been measured very accurately) in the Inorganic Crystal Structure Database in order to identify new topological phases. We develop codes to compute all
78 topological
ncluding some
rmation. Our
urce code that
Topological Materials Database
Total Materials 38184
Topological Insulators 6109
Semi-Metals 13985
NAVIGATION
Search
Predict
About
SETTINGS
UI Mode
Compound Contains
e.g. Bi1 Se2 Ge
Only these elements
Exclude
eg. O1 N
- or -
ICSD Number
eg. 123456
Search
Show Advanced Search
H
He
Li
Be
B
C
N
O
F
Ne
Na
Mg
Al
Si
P
S
Cl
Ar
K
Ca
Sc
Ti
V
Cr
Mn
Fe
Co
Ni
Cu
Zn
Ga
Ge
As
Se
Br
Kr
Rb
Sr
Y
Zr
Nb
Mo
Tc
Ru
Rh
Pd
Ag
Cd
In
Sn
Sb
Te
I
Xe
Cs
Ba
La
Hf
Ta
W
Re
Os
Ir
Pt
Au
Hg
Tl
Pb
Bi
Po
At
Rn

Extension to topological superconductors?

- Symmetry indicator method itself needs to be modified (Z2 index in 0D)

AZ	0	1	2	3	4	5	6	7
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

- Pairing symmetry (s-wave, p-wave, …) becomes an additional data.

M. Geier, P. W. Brouwer, and L. Trifunovic, Symmetry-based indicators for topological Bogoliubov–de Gennes Hamiltonians, PRB (2020)
S. Ono, H. Chun Po, K. Shiozaki, $\mathbb{Z}$2-enriched symmetry indicators for topological superconductors in the 1651 magnetic space groups, PRR (2021)

Pairing symmetries

- Pairing symmetry

$$\hat{H} \simeq (\hat{\boldsymbol{c}}_{\boldsymbol{k}}^{\dagger}\ \hat{\boldsymbol{c}}_{-\boldsymbol{k}}) H_{\boldsymbol{k}}^{\mathrm{BdG}} \begin{pmatrix} \hat{\boldsymbol{c}}_{\boldsymbol{k}} \\ \hat{\boldsymbol{c}}_{-\boldsymbol{k}}^{\dagger} \end{pmatrix}$$

$$H_{\boldsymbol{k}}^{\mathrm{BdG}} = \begin{pmatrix} h_{\boldsymbol{k}} & \Delta_{\boldsymbol{k}} \\ \Delta_{\boldsymbol{k}}^{\dagger} & -h_{-\boldsymbol{k}}^{*} \end{pmatrix},$$

$$U_{\boldsymbol{k}}^{\mathrm{BdG}}(g) = \begin{pmatrix} U_{\boldsymbol{k}}(g) & 0 \\ 0 & \chi_g U_{-\boldsymbol{k}}^{*}(g) \end{pmatrix},$$

$$U_{\boldsymbol{k}}(g) h_{\boldsymbol{k}} U_{\boldsymbol{k}}^{\dagger}(g) = h_{p_g \boldsymbol{k}},$$

$$U_{\boldsymbol{k}}(g) \Delta_{\boldsymbol{k}} U_{-\boldsymbol{k}}^{T}(g) = \chi_g \Delta_{p_g \boldsymbol{k}}\ (\chi_g \in \mathrm{U}(1)).$$

$\chi_g = 1$: conventional

$\chi_g \neq 1$: unconventional

- Unconventional pairing symmetry is recipe for TSC.
- Symmetry indicators are all trivial for conventional pairing symmetries in time-reversal symmetric case.
- We investigate symmetry indicators for materials for each unconventional pairing symmetry.

PHYSICAL REVIEW LETTERS

Editors' Suggestion Open Access

High-Throughput Investigations of Topological and Nodal Superconductors

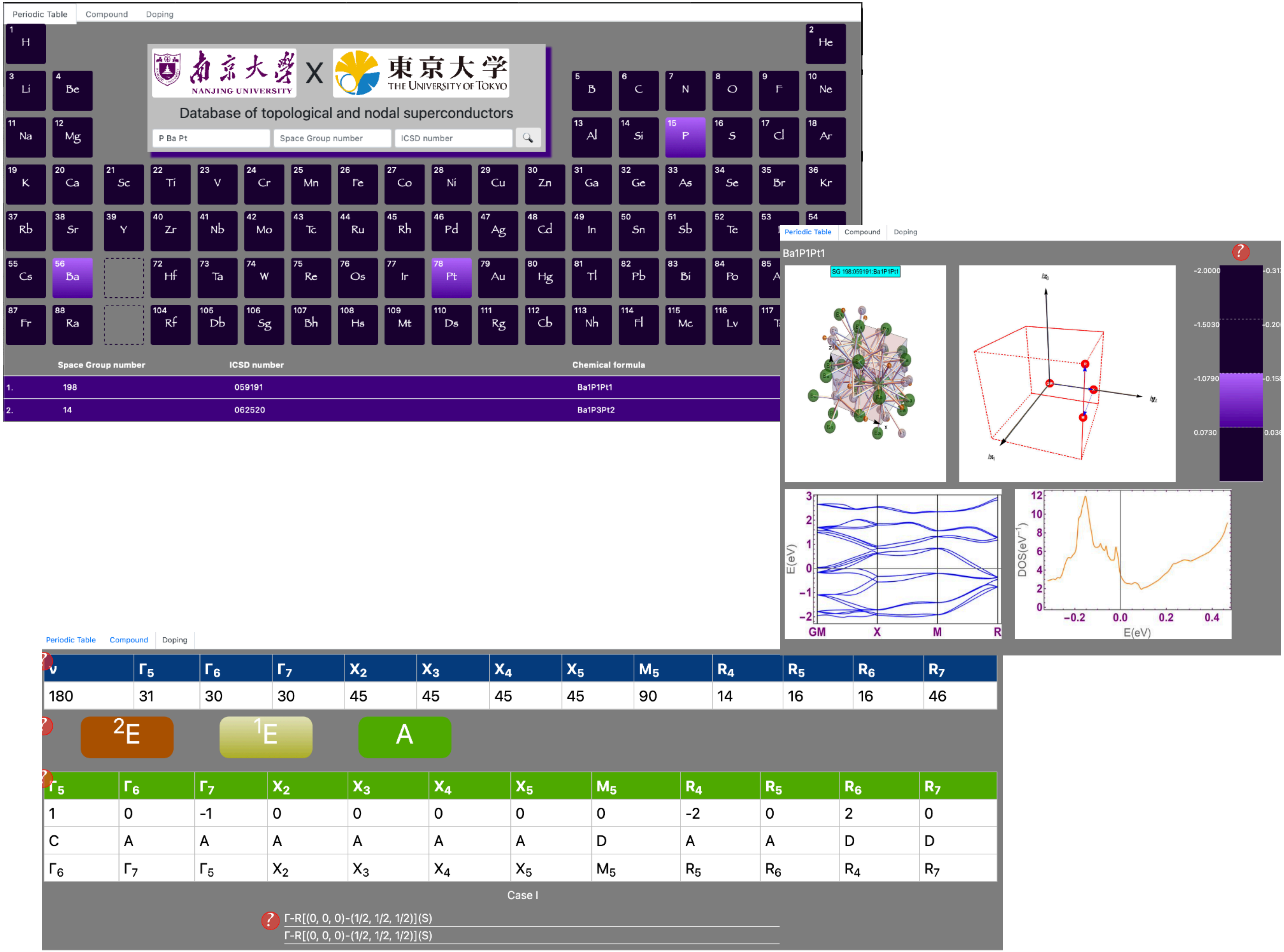
Feng Tang, Seishiro Ono, Xiangang Wan, and Haruki Watanabe
Phys. Rev. Lett. **129**, 027001 – Published 6 July 2022

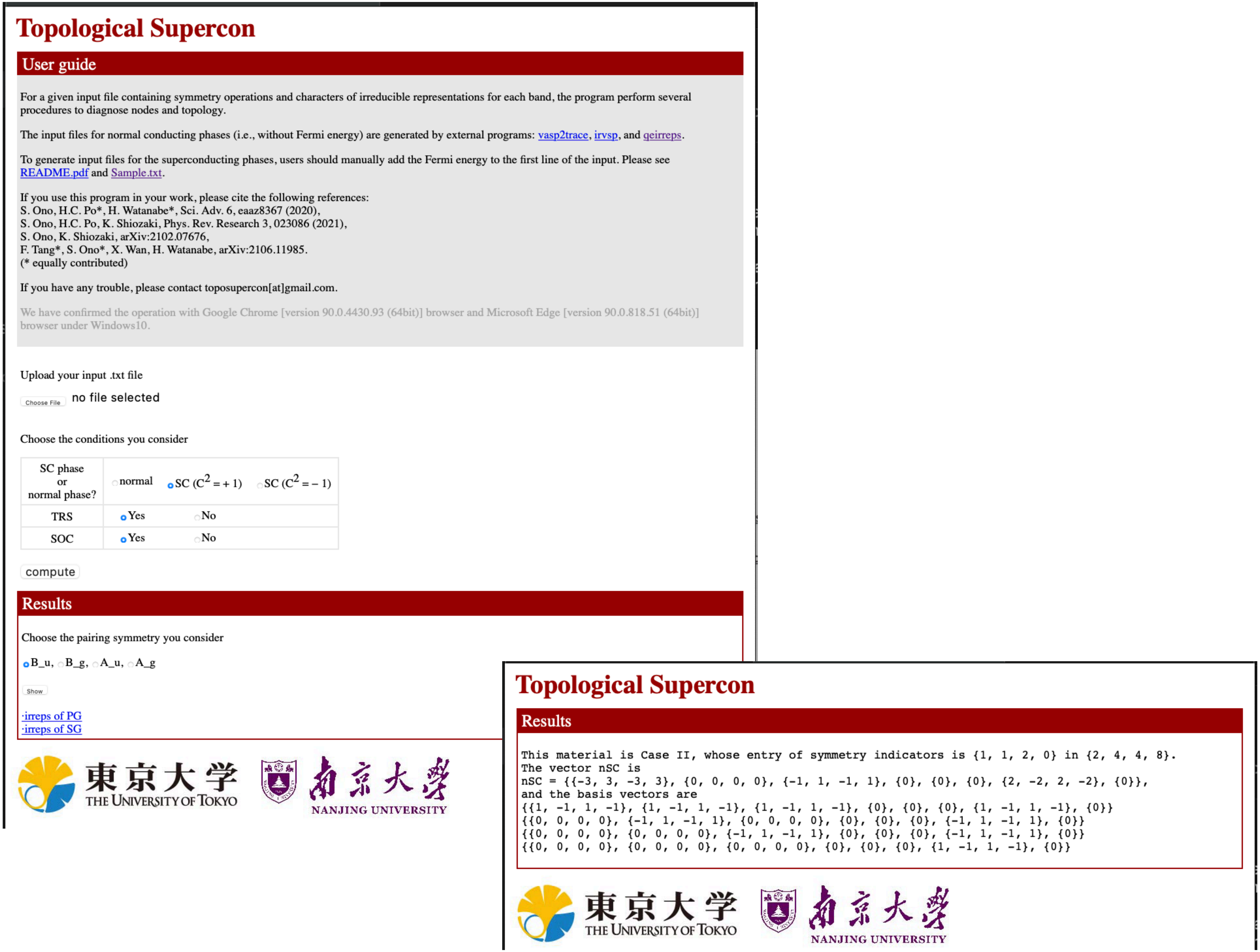
Article PDF HTML Export Citation

ABSTRACT

The theory of symmetry indicators has enabled database searches for topological materials in normal conducting phases, which has led to several encyclopedic topological material databases. To date, such a database for topological superconductors is yet to be achieved because of the lack of information about pairing symmetries of realistic materials. In this Letter, sidestepping this issue, we tackle an alternative problem: the predictions of topological and nodal superconductivity in materials for each single-valued representation of point groups. Based on recently developed symmetry indicators for superconductors, we provide comprehensive mappings from pairing symmetries to the

Issue
Vol. 129, Iss. 2 — 8 July 2022


Periodic Table
Compound
Doping
NANJING UNIVERSITY
X
東京大学
THE UNIVERSITY OF TOKYO
Database of topological and nodal superconductors
P Ba Pt
Space Group number
ICSD number
Chemical formula
1. 198 059191 Ba1P1Pt1
2. 14 062520 Ba1P3Pt2
Ba1P1Pt1
SG 198:059191:Ba1P1Pt1
E(eV)
GM X M R
DOS(eV^{-1})
E(eV)
ν Γ5 Γ6 Γ7 X2 X3 X4 X5 M5 R4 R5 R6 R7
180 31 30 30 45 45 45 45 90 14 16 16 46
^{2}E
^{1}E
A
Γ5 Γ6 Γ7 X2 X3 X4 X5 M5 R4 R5 R6 R7
1 0 -1 0 0 0 0 0 -2 0 2 0
C A A A A A A D A A D D
Γ6 Γ7 Γ5 X2 X3 X4 X5 M5 R5 R6 R4 R7
Case I
Γ-R[(0, 0, 0)-(1/2, 1/2, 1/2)](S)
Γ-R[(0, 0, 0)-(1/2, 1/2, 1/2)](S)


Topological Supercon
User guide
For a given input file containing symmetry operations and characters of irreducible representations for each band, the program perform several procedures to diagnose nodes and topology.
The input files for normal conducting phases (i.e., without Fermi energy) are generated by external programs: vasp2trace, irvsp, and qeirreps.
To generate input files for the superconducting phases, users should manually add the Fermi energy to the first line of the input. Please see README.pdf and Sample.txt.
If you use this program in your work, please cite the following references:
S. Ono, H.C. Po*, H. Watanabe*, Sci. Adv. 6, eaaz8367 (2020),
S. Ono, H.C. Po, K. Shiozaki, Phys. Rev. Research 3, 023086 (2021),
S. Ono, K. Shiozaki, arXiv:2102.07676,
F. Tang*, S. Ono*, X. Wan, H. Watanabe, arXiv:2106.11985.
(* equally contributed)
If you have any trouble, please contact toposupercon[at]gmail.com.
We have confirmed the operation with Google Chrome [version 90.0.4430.93 (64bit)] browser and Microsoft Edge [version 90.0.818.51 (64bit)] browser under Windows10.
Upload your input .txt file
Choose File
no file selected
Choose the conditions you consider
SC phase or normal phase?
normal
SC ($C^2 = +1$)
SC ($C^2 = -1$)
TRS
Yes
No
SOC
Yes
No
compute
Results
Choose the pairing symmetry you consider
B_u, B_g, A_u, A_g
Show
·irreps of PG
·irreps of SG
東京大学
THE UNIVERSITY OF TOKYO
南京大學
NANJING UNIVERSITY
Topological Supercon
Results
This material is Case II, whose entry of symmetry indicators is {1, 1, 2, 0} in {2, 4, 4, 8}.
The vector nSC is
nSC = {{-3, 3, -3, 3}, {0, 0, 0, 0}, {-1, 1, -1, 1}, {0}, {0}, {0}, {2, -2, 2, -2}, {0}},
and the basis vectors are
{{1, -1, 1, -1}, {1, -1, 1, -1}, {1, -1, 1, -1}, {0}, {0}, {0}, {1, -1, 1, -1}, {0}}
{{0, 0, 0, 0}, {-1, 1, -1, 1}, {0, 0, 0, 0}, {0}, {0}, {0}, {-1, 1, -1, 1}, {0}}
{{0, 0, 0, 0}, {0, 0, 0, 0}, {-1, 1, -1, 1}, {0}, {0}, {0}, {-1, 1, -1, 1}, {0}}
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0}, {0}, {0}, {1, -1, 1, -1}, {0}}
東京大学
THE UNIVERSITY OF TOKYO
南京大學
NANJING UNIVERSITY

Topological superconductors with conventional pairing symmetries

- **Symmetry indicators** → topological (crystalline) insulators.
  The converse is not true.

- Time-reversal & inversion symmetric superconductors can be topological even with conventional pairing symmetry

S. Qin, C. Fang, F.-C. Zhang, J. Hu, Topological Superconductivity in an Extended s-Wave Superconductor and Its Implication to Iron-Based Superconductor, PRX (2022)

  This study examined only a few examples

- The majority of SCs have conventional pairing symmetries
  → There might be TSCs in known materials
  overlooked because they have conventional pairing symmetries.

Classification of topological superconductors with conventional pairing symmetries

K. Shiozaki, et al arXiv:1810.00801
Z. Song, et al, Sci. (2019)

K-theory type classification via
Real-space Atiyah-Hirzebruch spectral sequence (AHSS)
Also known as “topological crystals”

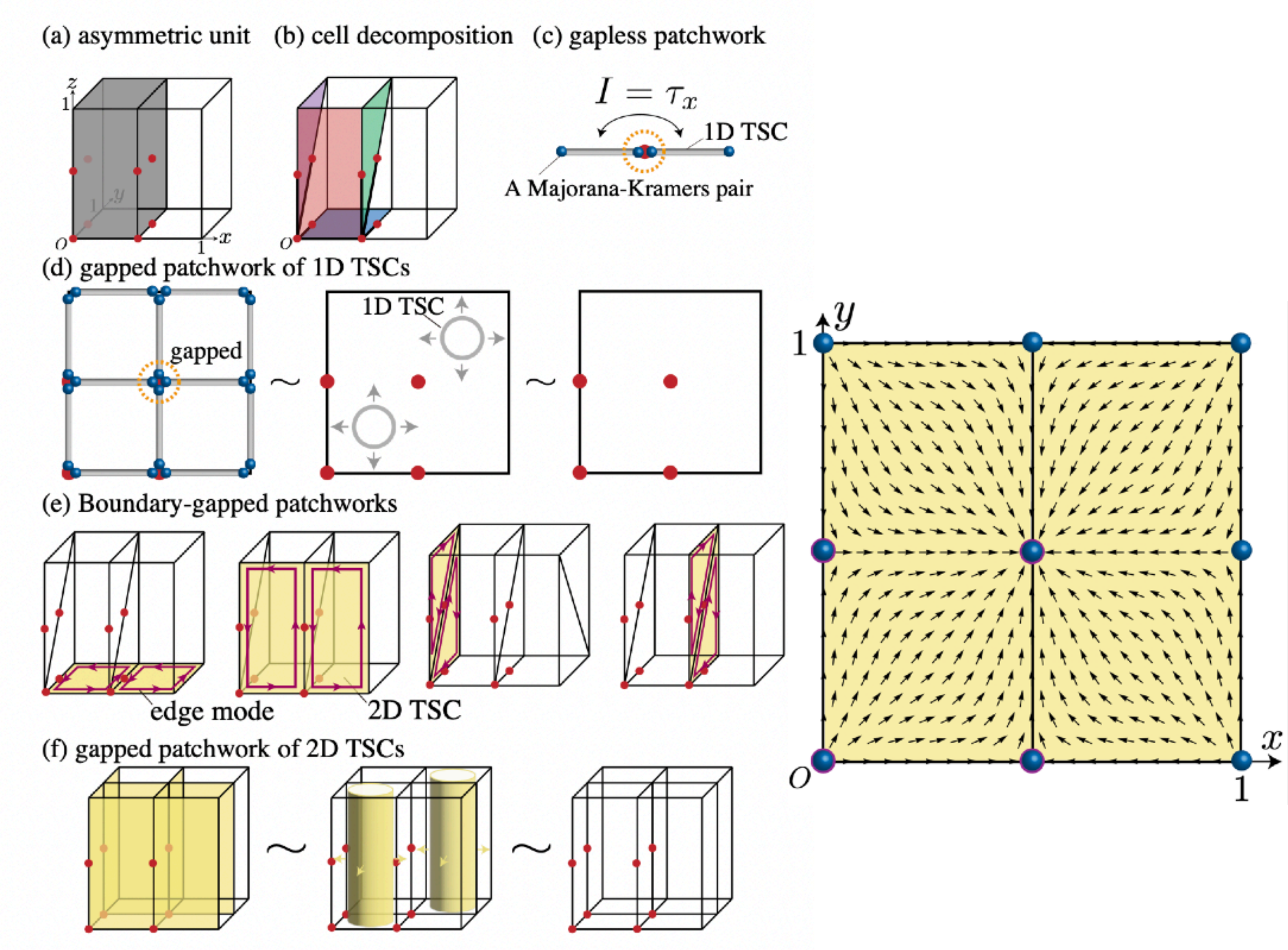


FIG. 1. **Illustration of 3D patchworks in space group $P\bar{1}$.** (a) A

TABLE XVIII: Classification table of topological phases in space groups.

Space group	pairing symmetry	$E^{\infty}_{3,-3}$	$E^{\infty}_{2,-2}$	$E^{\infty}_{1,-1}$	${}^{\phi}K_G^{(z,c)-n}(T^3)$
1	A	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^6 \times \mathbb{Z}$
2	A_g	0	0	0	0
3	A	$\mathbb{Z}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3 \times \mathbb{Z}^4$	??
4	A	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	??
5	A	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	??
6	A'	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}^4$	$\mathbb{Z}_2 \times \mathbb{Z}^4$
7	A'	0	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$
8	A'	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$
9	A'	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
10	A_g	0	0	0	0
11	A_g	0	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$
12	A_g	0	0	0	0
13	A_g	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	$\mathbb{Z}_2 \times \mathbb{Z}^2$
14	A_g	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
15	A_g	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
16	A_1	$\mathbb{Z}$	$\mathbb{Z}_2^5$	$\mathbb{Z}^{12}$	??
17	A_1	$\mathbb{Z}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3 \times \mathbb{Z}^4$	??
18	A_1	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	??
19	A_1	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	??
20	A_1	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^2$	??
21	A_1	$\mathbb{Z}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2 \times \mathbb{Z}^7$	??
22	A_1	$\mathbb{Z}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2 \times \mathbb{Z}^6$	??
23	A_1	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}^6$	??
24	A_1	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}^3$	??
25	A_1	0	$\mathbb{Z}_2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$
26	A_1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}^2$	$\mathbb{Z}_2 \times \mathbb{Z}^2$

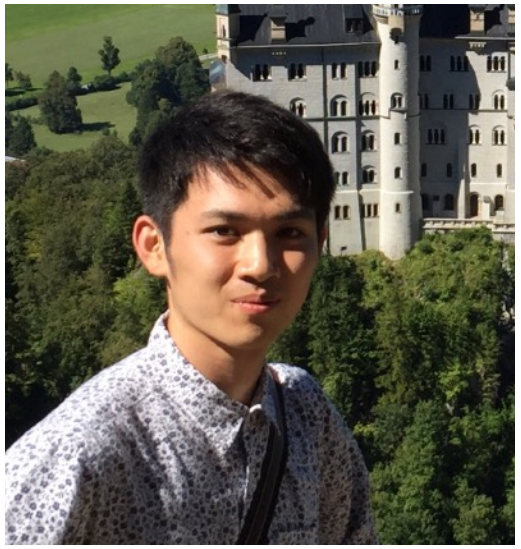
S. Ono, K. Shiozaki, HW, arXiv:2206.02489

Please invite Ono-kun for the detailed talk!

2. Fractional corner charges in trivial insulators

Hoi Chun Po

The Hong Kong University
of Science and Technology



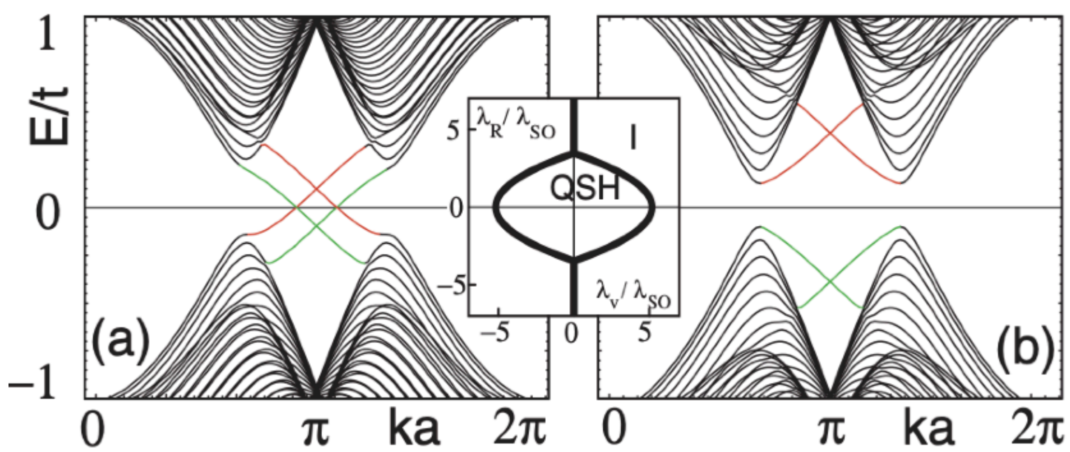
Seishiro Ono

U Tokyo

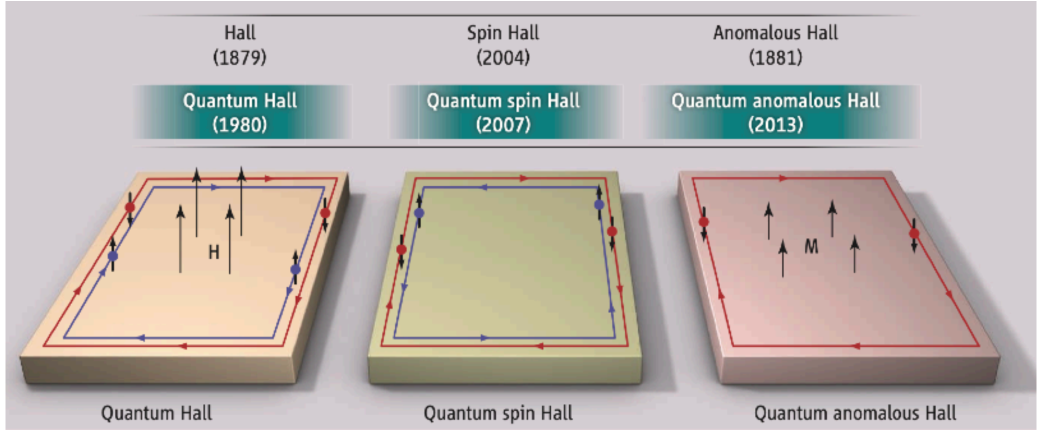
- HW, S. Ono, Corner charge and bulk multipole moment in periodic systems, PRB (2020).
- HW, H. C. Po, *Fractional Corner Charge of Sodium Chloride,* PRX (2021).

Bulk-boundary correspondence of topological phases

- 2D topological insulator

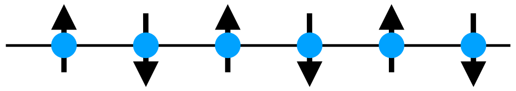


C. L. Kane and E. J. Mele, PRL (2005)



S. Oh, Science (2013)

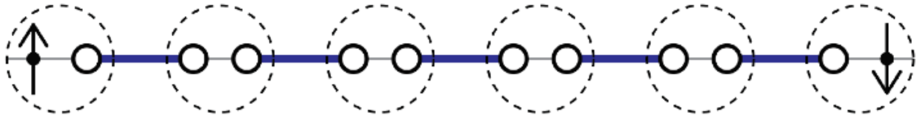
- Haldane phase

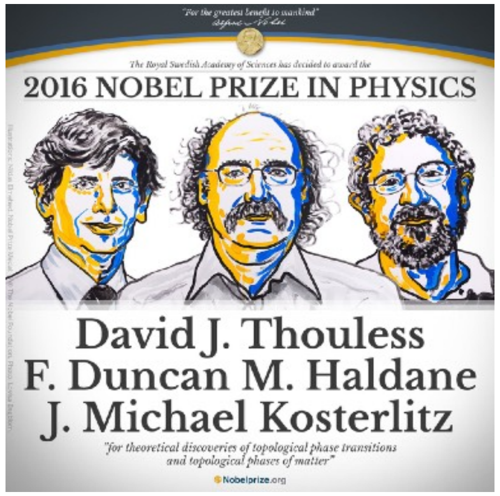


S = 1 Heisenberg model

$$\hat{H} = J \sum_n \hat{\boldsymbol{s}}_n \cdot \hat{\boldsymbol{s}}_{n+1}$$

S = 1/2 edge spin

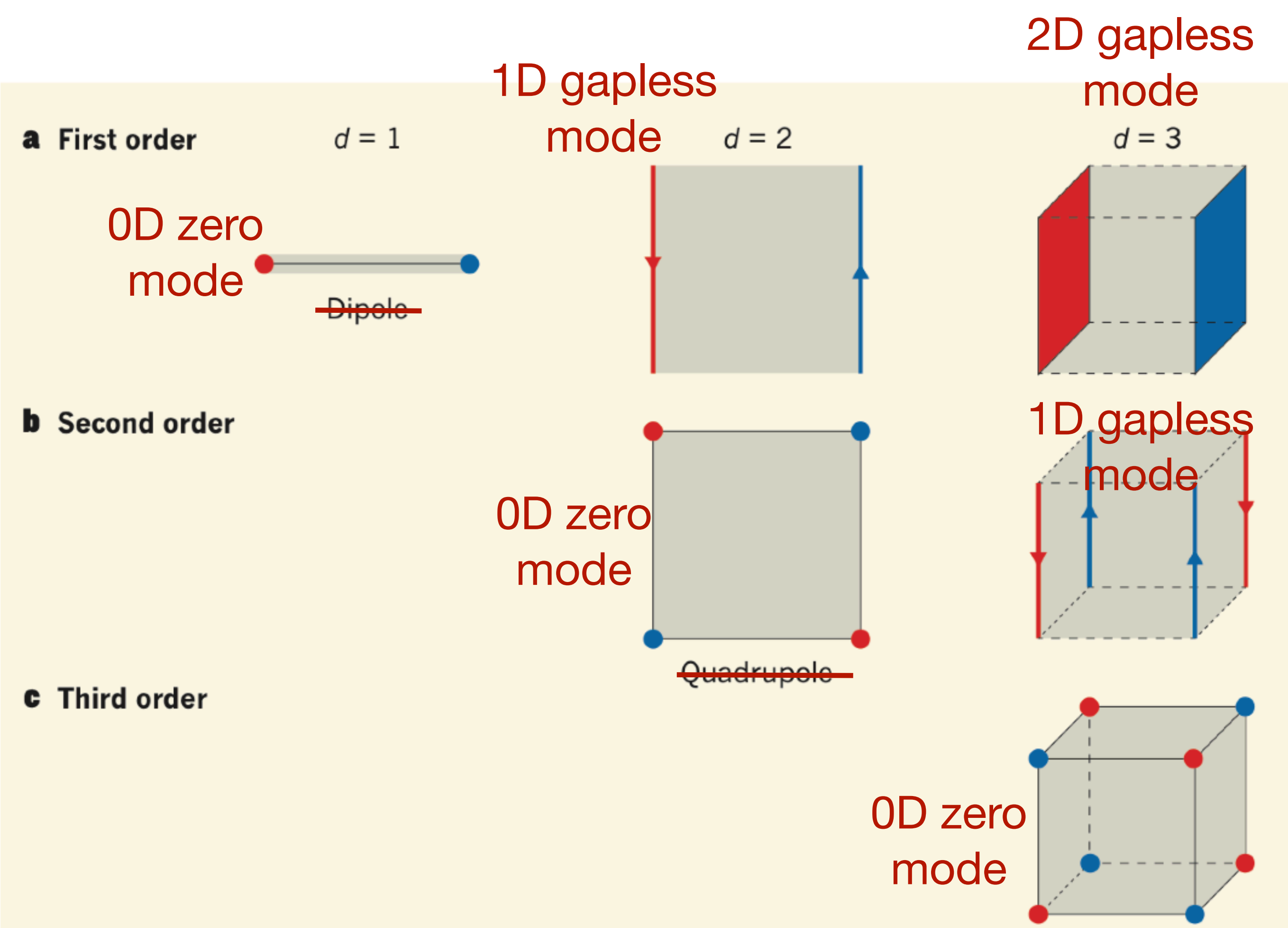




***Bulk* topology implies nontrivial *boundary*.**

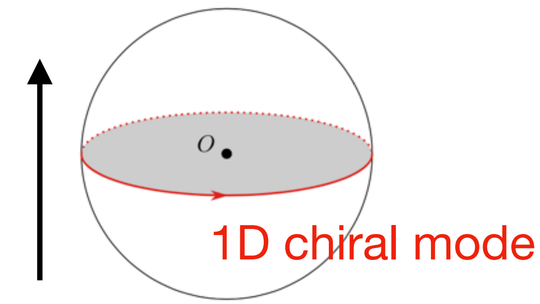
Boundary states(=degrees of freedom) / surface topological order

Bulk-boundary correspondence for higher-order topology

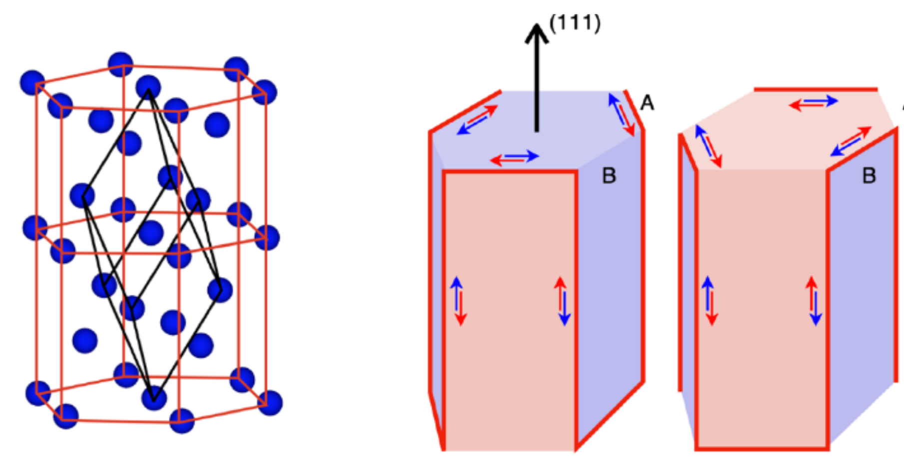


M. Fruchart et al, Nature (2018)

Inversion symmetric 3D topological insulator under magnetic field



Higher order topology in Bismuth

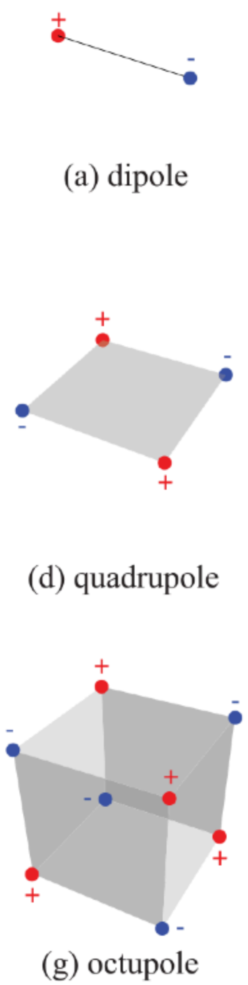


F. Schindler et al, Nature Physics (2018)
F. Schindler et al, Science Advances (2018)

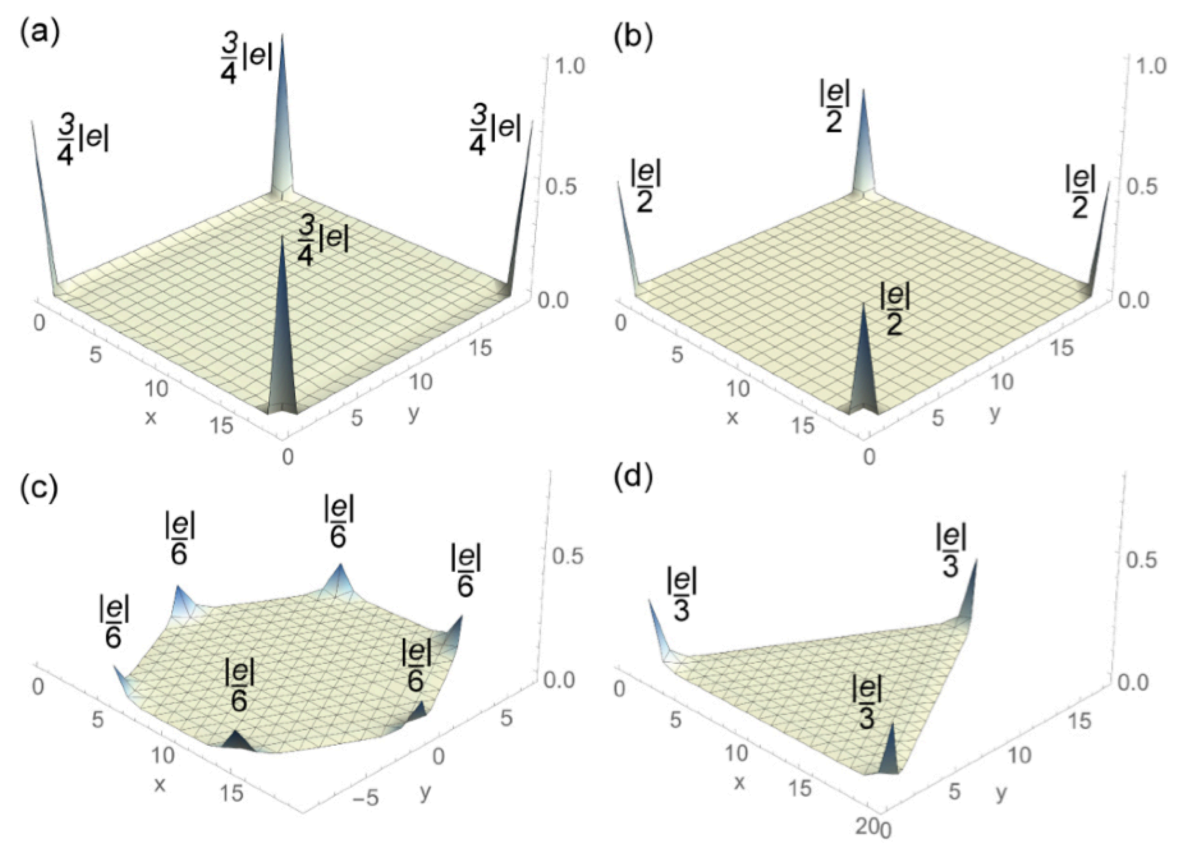
Boundary states can be localized to **corners** and **hinges** depending on the bulk topology.

Bulk-boundary correspondence of *trivial* insulators

- Even trivial insulators may have interesting signature on their boundary.
- No known material realization of fractional corner charge.



W. A. Benalcazar, B. A. Bernevig, T. L. Hughes, Science (2017) / PRB (2017)



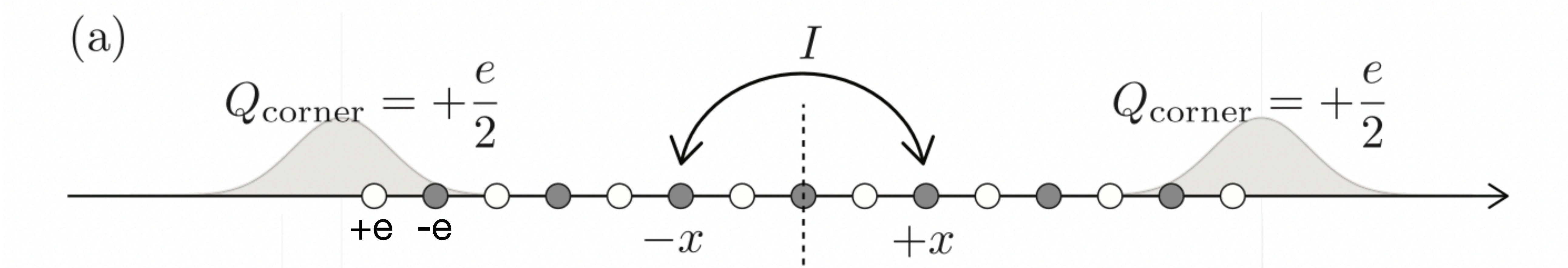
W. A. Benalcazar, T. Li, T. L. Hughes, PRB (2019)

Charges are "frozen" i.e., *not* degrees of freedom

Filling anomaly

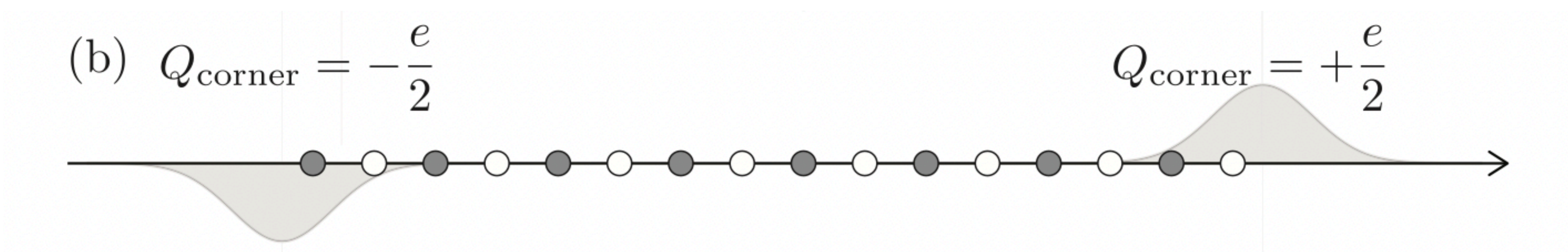
W. A. Benalcazar, T. Li, T. L. Hughes, PRB (2019)

- Sometimes, point group symmetry and charge neutrality cannot be simultaneously respected.

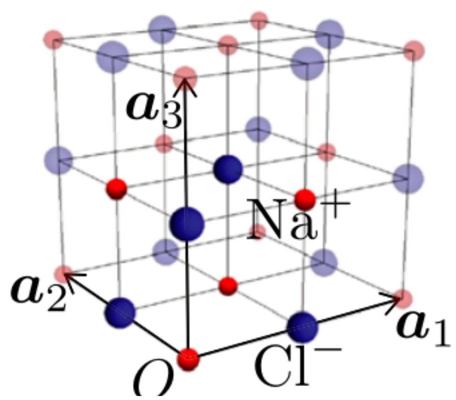


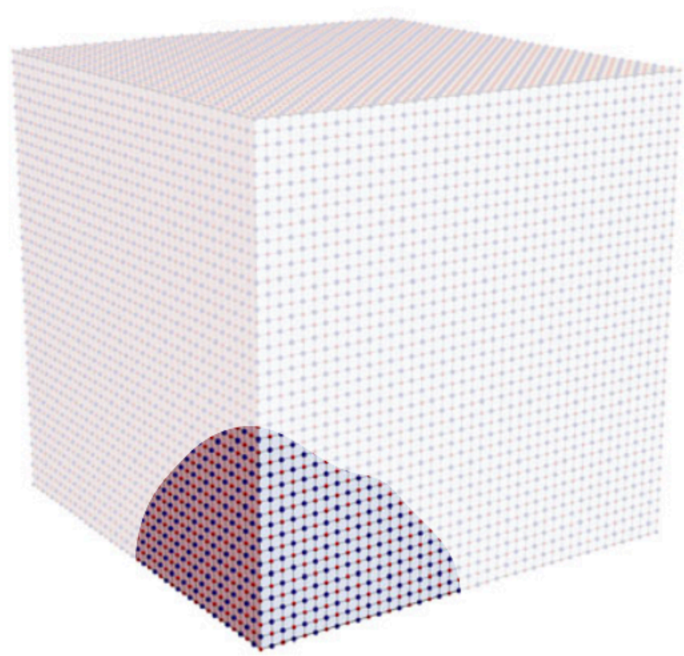
$$Q_{\rm corner} = \frac{1}{2} Q_{\rm tot} = \frac{e}{2} \mod e$$

- Surface charge is a local property.



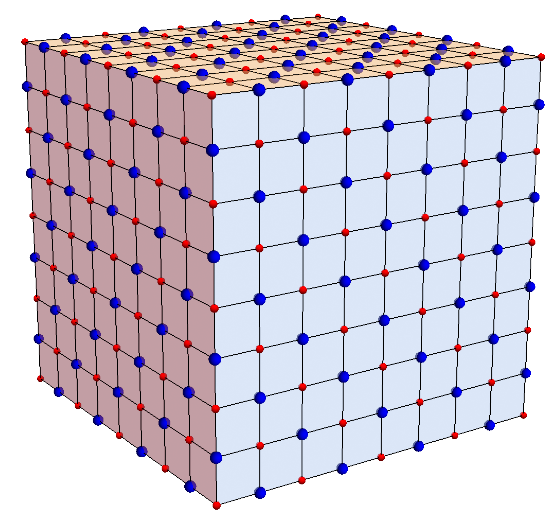
Fractional corner charge in sodium chloride (NaCl)

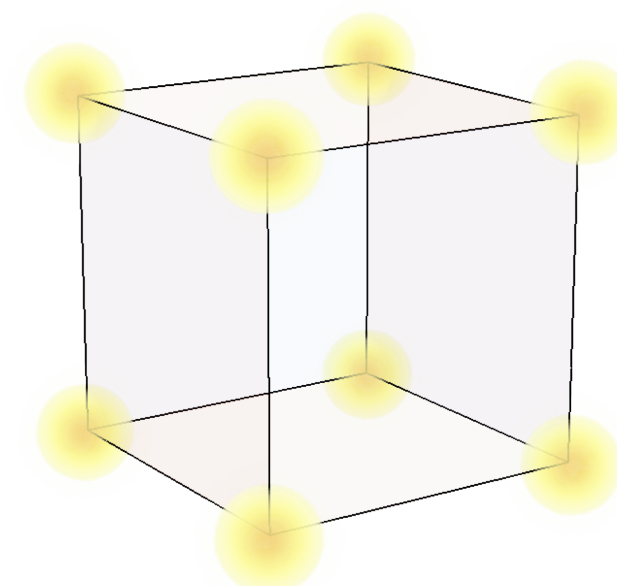




Filling anomaly

$$Q_{\mathrm{corner}} = Q_{\mathrm{tot}}/N_{\mathrm{corner}}$$

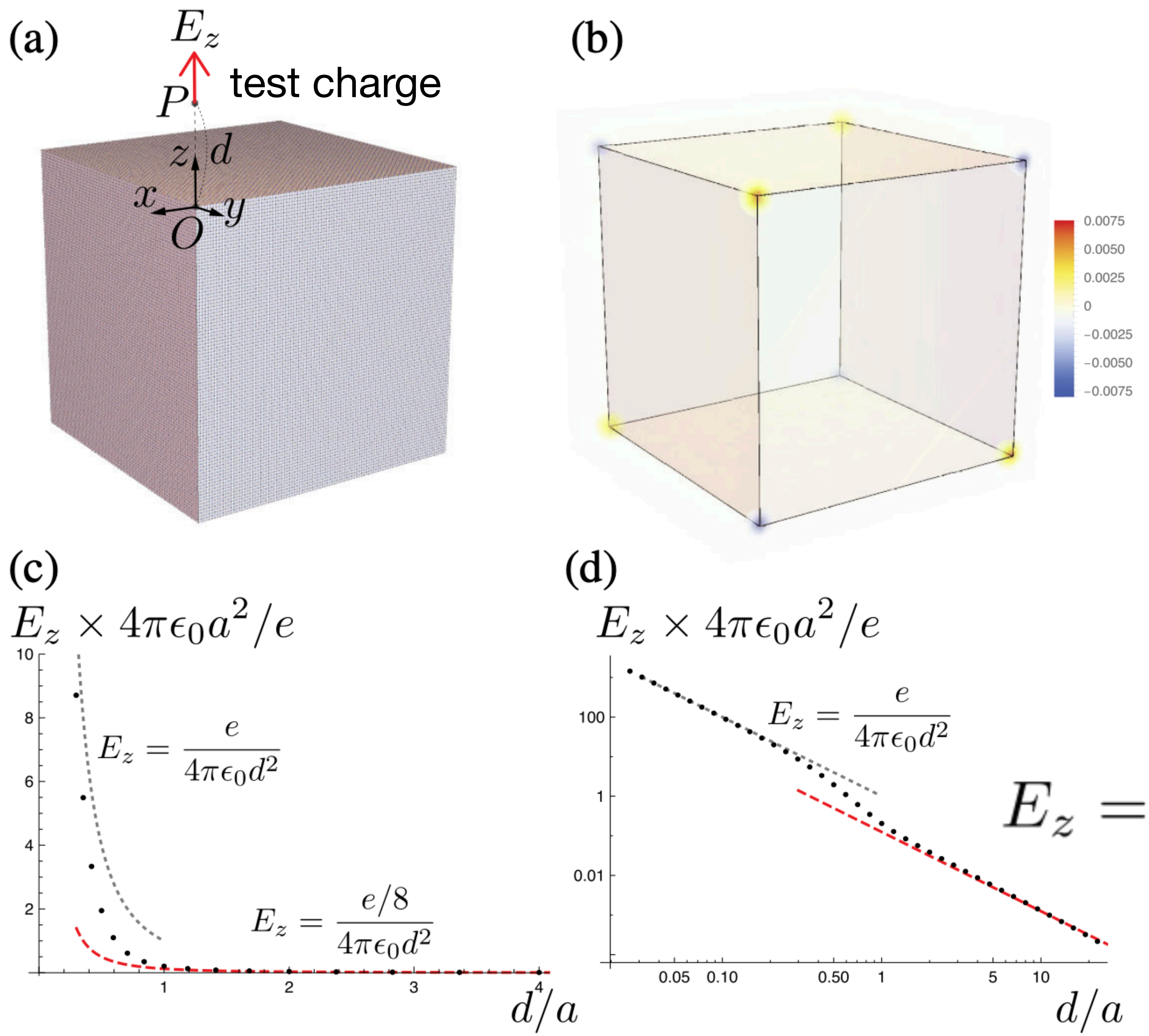




$$Q_{\mathrm{tot}} = +e,\ Q_{\mathrm{corner}} = \tfrac{1}{8}e\ \ (\mathrm{mod}\ \tfrac{1}{4}e)$$

- HW, S. Ono, Corner charge and bulk multipole moment in periodic systems, PRB (2020).
- HW, H. C. Po, *Fractional Corner Charge of Sodium Chloride,* PRX (2021).
- K. Naito, R. Takahashi, HW, S. Murakami, *Fractional hinge and corner charges in various crystal shapes with cubic symmetry, PRB* (2022).

Possible direct measurement via Coulomb force



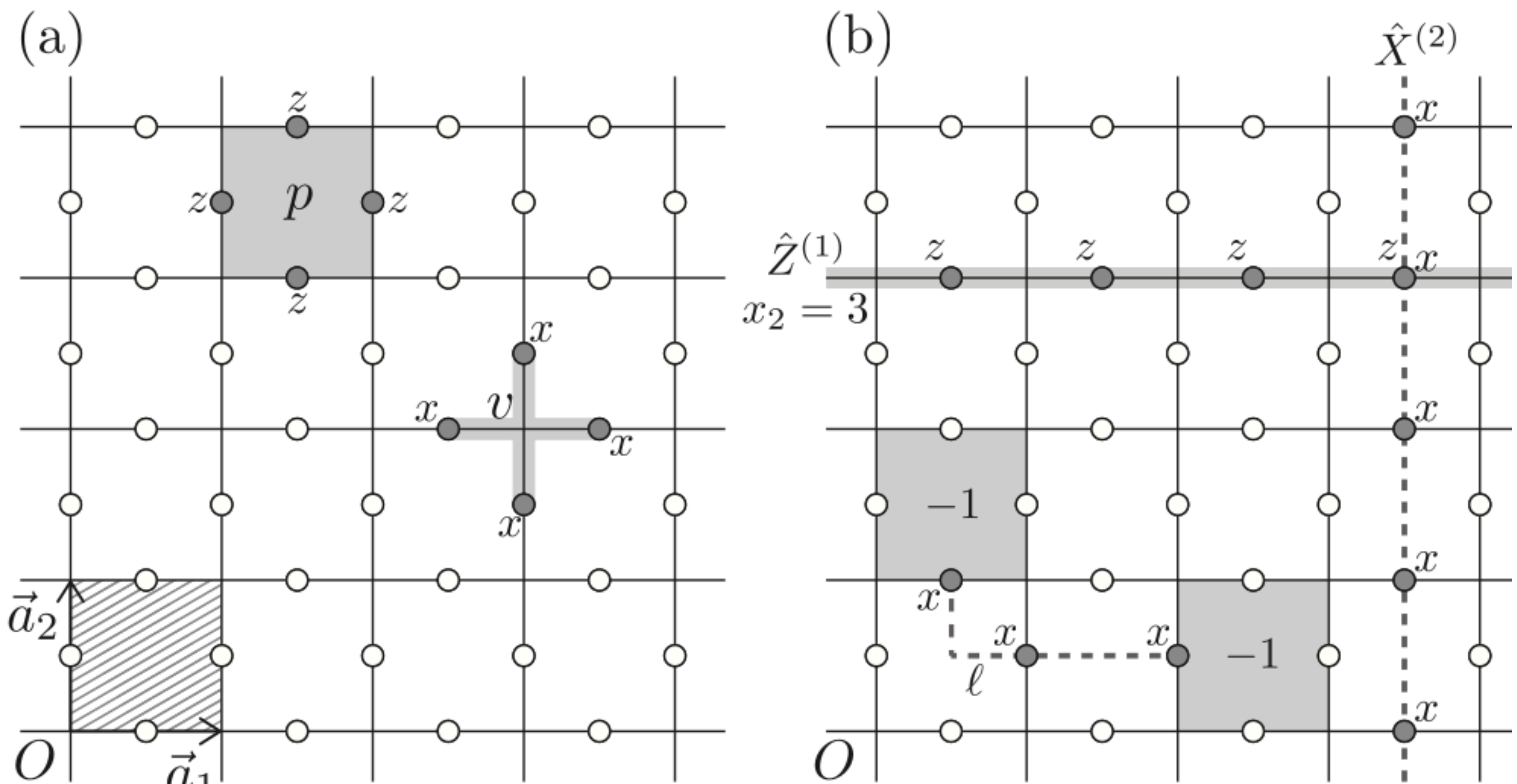
$$E_z = \frac{e/8}{4\pi\epsilon_0 d^2}$$

3. Topological orders “without” ground state degeneracy

Meng Cheng, Yohei Fuji, HW, in prep

Features of topologically ordered phases

- Topological ground state degeneracy
- Fractional excitations (anyons) and their statistics
- Topological entanglement entropy
- …



Z_N toric code (1)
N-level spins

$$X \equiv \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix},$$

$$Z \equiv \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix},$$

$$\omega \equiv e^{\frac{2\pi i}{N}},$$

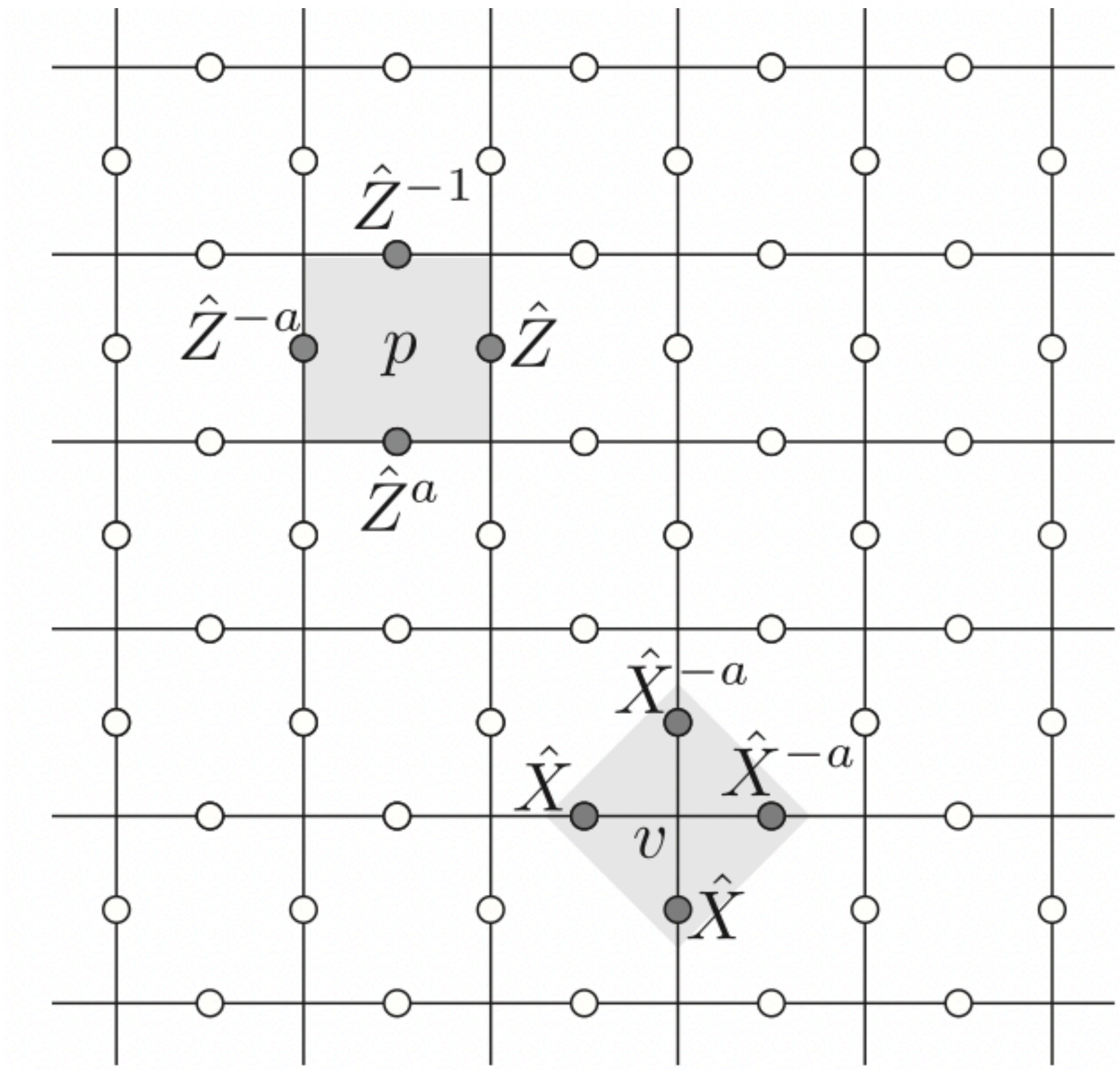
$$Z^N = X^N = 1,$$

$$ZX = \omega XZ = \begin{pmatrix} & & & & 1 \\ \omega & & & & \\ & \omega^2 & & & \\ & & \ddots & & \\ & & & \omega^{N-1} & \end{pmatrix}$$

Z_N toric code (2) Hamiltonian

Kitaev (2003)

$$\hat{H}^{(L_1,L_2)} = -\sum_{v\in\mathscr{V}} \frac{1}{2}(\hat{A}_v + \text{h.c.}) - \sum_{p\in\mathscr{P}} \frac{1}{2}(\hat{B}_p + \text{h.c.}).$$

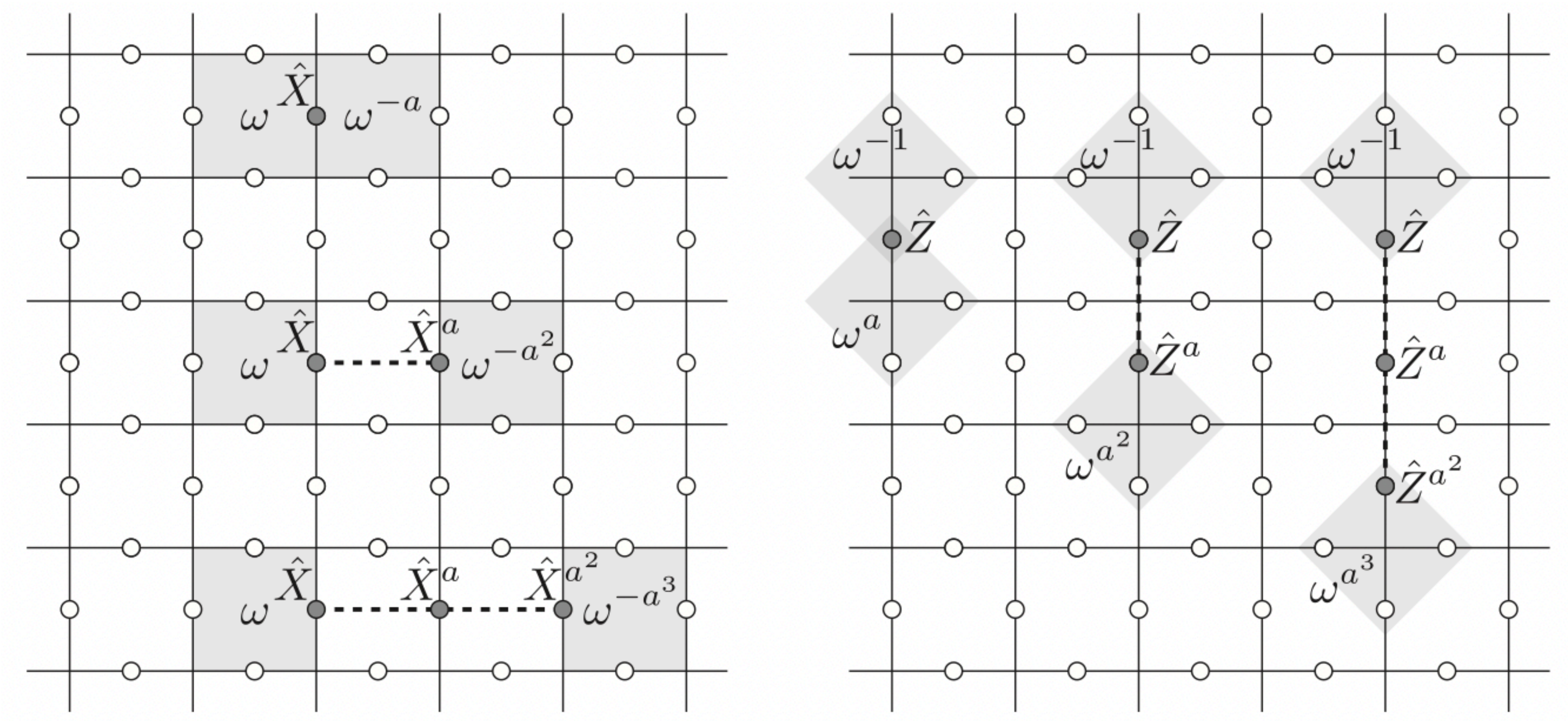


- a=1 is the standard model
- a=-1 is the simplest nontrivial example

M. Barkeshli, et al CMP (2020) (preprint in 2016)
J. C. Bridgeman et al, PRB (2017)
Y. Fuji, PRB (2019)
…

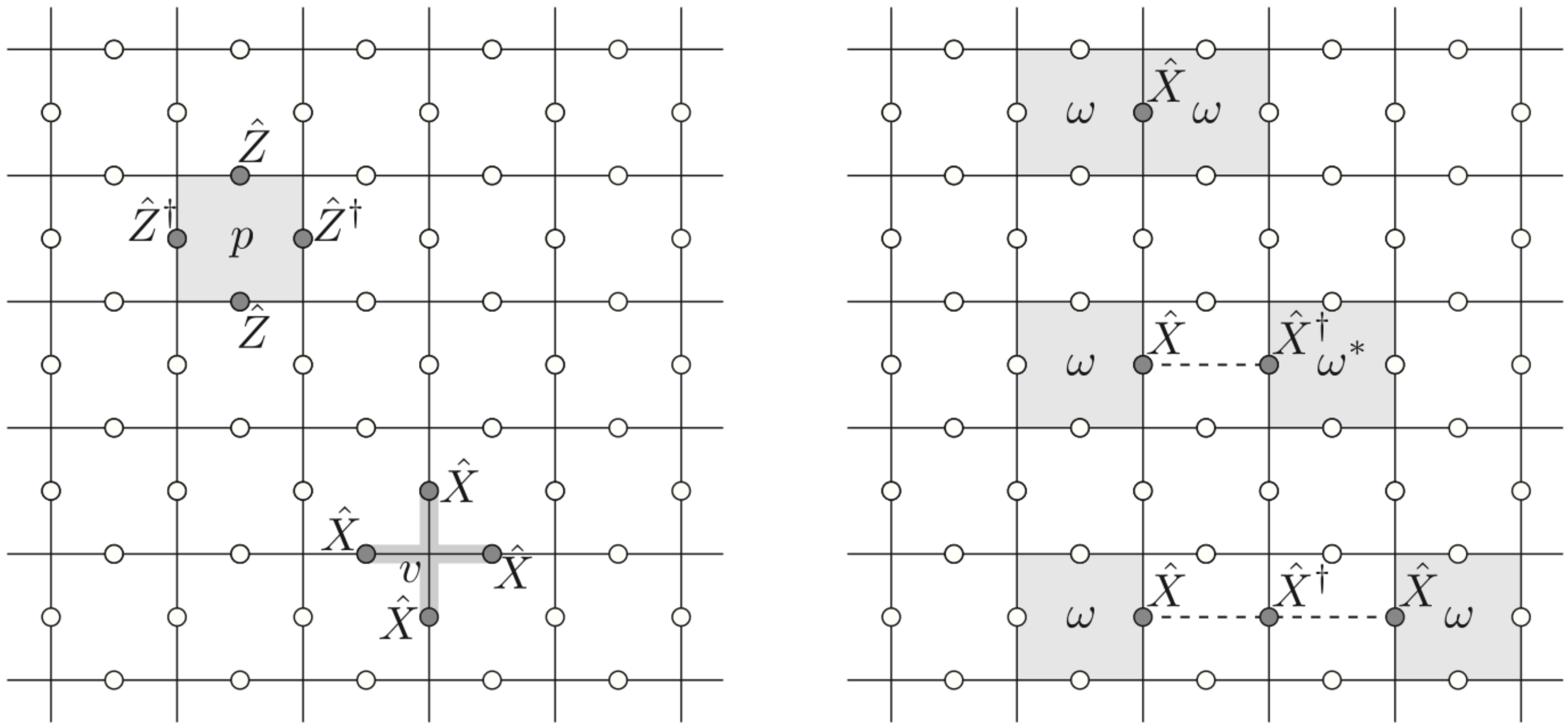
- Other values of a are also interesting.

Z_N toric code (3)
String operators



- Strings can be closed only when L_1, L_2 are multiple of the period.
- Translation permutes anyons.

Ground state degeneracy on torus for *a*=-1



$$N_{\text{deg}}^{(L_1,L_2)} = \begin{cases} N^2 & (L_1 \text{ and } L_2 \text{ are even}) \\ 4 & (L_1 \text{ or } L_2 \text{ is odd \& } N \text{ is even}) \\ 1 & (L_1 \text{ or } L_2 \text{ is odd \& } N \text{ is odd}) \end{cases}$$

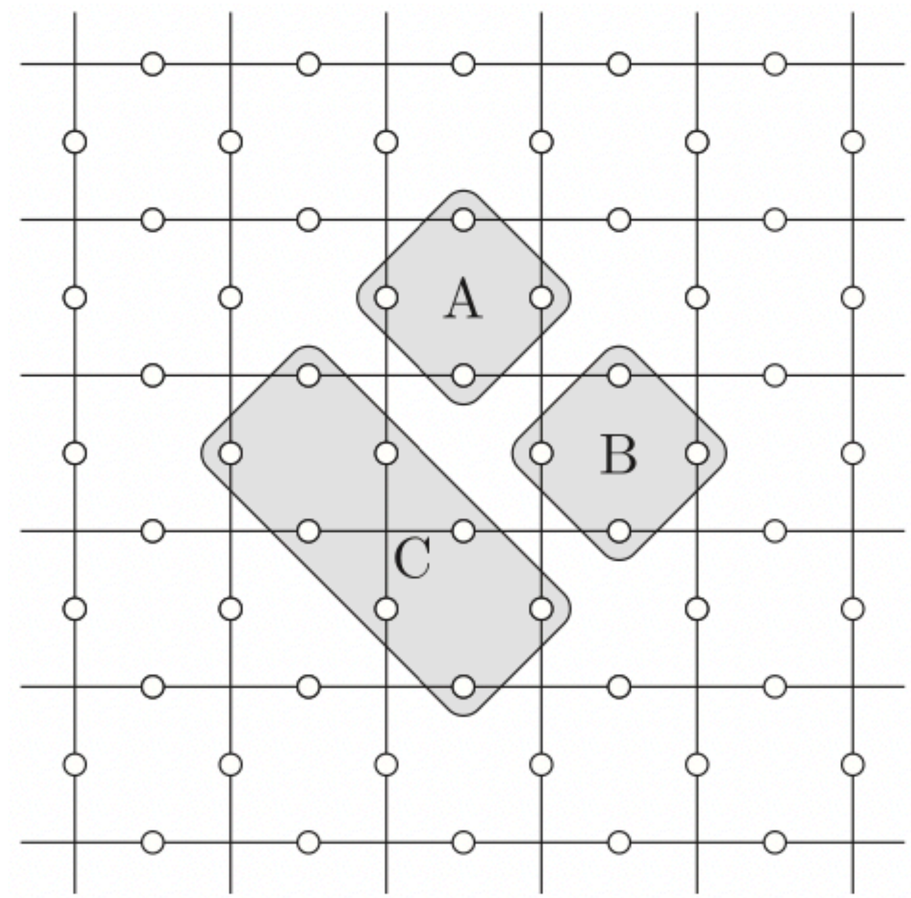
- Ground state can be unique and gapped!!

Topological entanglement entropy

- Entanglement entropy

$$S_{\mathrm{R}} \equiv -\mathrm{tr}[\hat{\rho}_{\mathrm{R}} \log \hat{\rho}_{\mathrm{R}}]$$

$$\hat{\rho}_{\mathrm{R}} \equiv \mathrm{tr}_{\bar{\mathrm{R}}} |\Phi_0\rangle\langle\Phi_0|$$



- Topological entanglement entropy

$$S_{\mathrm{R}} = \alpha \partial \mathrm{R} + S_{\mathrm{topo}}^{(L_1,L_2)}$$

$$S_{\mathrm{topo}}^{(L_1,L_2)} = S_{\mathrm{A}} + S_{\mathrm{B}} + S_{\mathrm{C}} - (S_{\mathrm{AB}} + S_{\mathrm{BC}} + S_{\mathrm{CA}}) + S_{\mathrm{ABC}}$$

$$S_{\mathrm{topo}}^{(L_1,L_2)} = -\log N$$ regardless of a, L_1, L_2

Conclusions

- **Symmetry indicators for superconductors**
  New database & subroutine
  F. Tang*, S. Ono*, X. Wan, HW, PRL (2022)

- **Revisiting criteria for topological phases**

  - **Symmetry indicators** → topological (crystalline) insulators
    The converse is not true.
    For example, many TSCs with conventional pairing symmetries.
    S. Ono, K. Shiozaki, HW, arXiv:2206.02489

  - **Product states** → not interesting.
    NaCl as a counter example. HW, H. C. Po, PRX (2021).

  - **Topological degeneracy on torus** → topological orders
    There can be topological order without degeneracy on torus for a sequence of L_1, L_2.