IBS-APCTP Conference on Advances in The Physics of Topological and Correlated Matter
2022/09/22

*Non-equilibrium physics* *Many-body physics*

# Non-reciprocal frustration:
# Time crystalline order-by-disorder phenomenon and a spin-glass-like state

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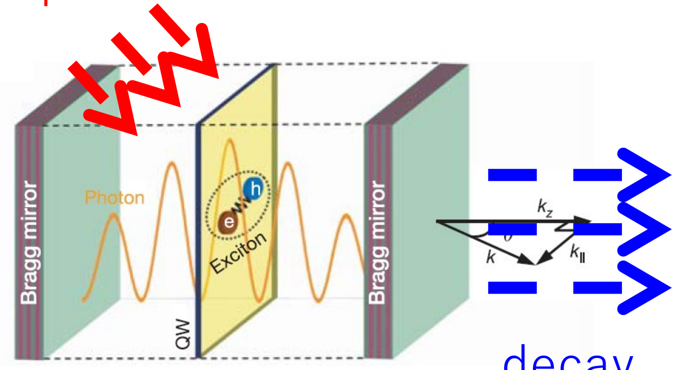
*Asia Pacific Center for Theoretical Physics (APCTP)*
*Pohang University of Science and Technology (POSTECH)*

# Nonequilibrium many-body physics

## Exciton polaritons

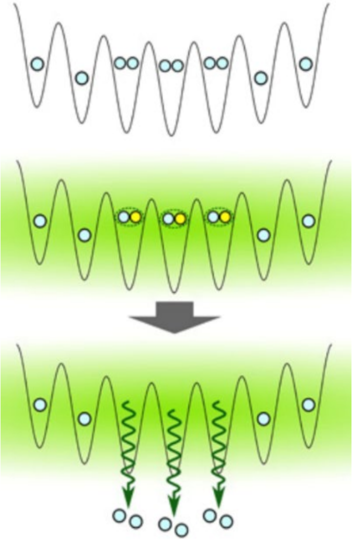
Pump



decay

J. Kasprzak, et al.,
Nature **443,** 409 (2006).

## Cold atoms

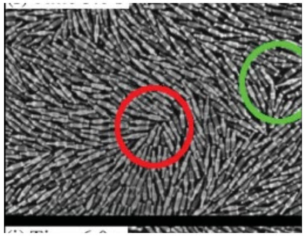


T. Tomita, et al.,
Sci. Adv. **3**, e1701513 (2017).

## Active matter

School of fish

Swarms of bacteria

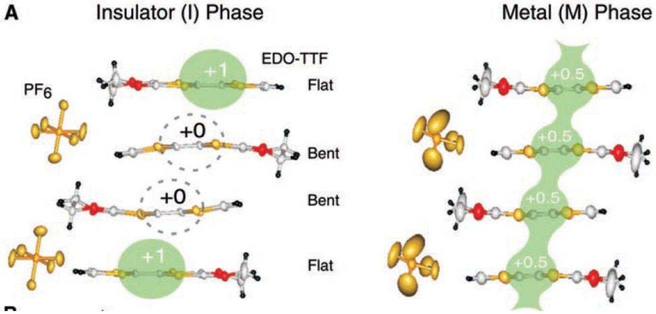


Narayan, Ramaswarmy,
Menon, Science (2007)

Potential to produce exotic states of matter beyond the equilibrium paradigm or design new devices

# Collective phenomena *unique* to nonequilibrium systems

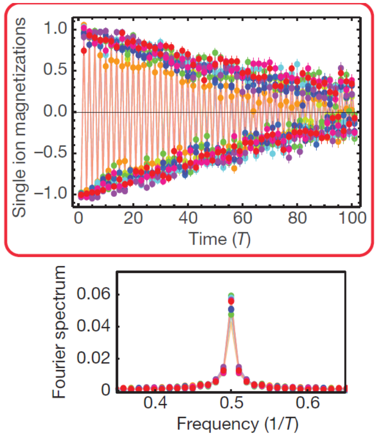
## Photoinduced phase transition



M. Chollet *et al.*, Science **307**, 86 (2006)

Phase transition induced by non-equilibrium drive

## Floquet time crystal



J. Zhang *et al.*, Nature **543**, 217 (2020)

No-go theorem in equilibrium

H. Watanabe and M. Oshikawa, PRL2015

## Bird flocking

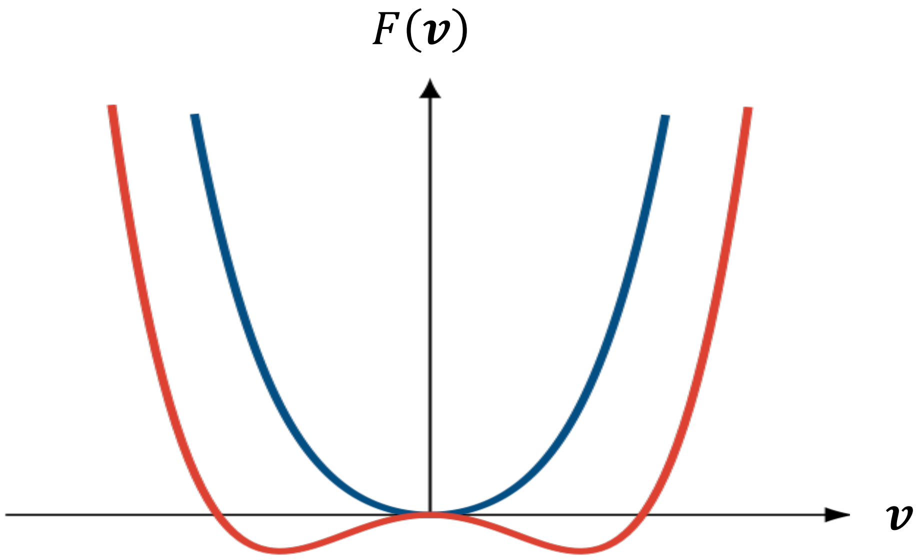
Vicsek et al., PRL1995,
Toner and Tu PRL1995, PRE1998

Long-ranged order in 2D

H. Tasaki PRL2020
L. P. Dadhichi, et al., PRE2020

Their full potential remains largely unexplored.

# Equilibrium paradigm: Minimization of (free) energy



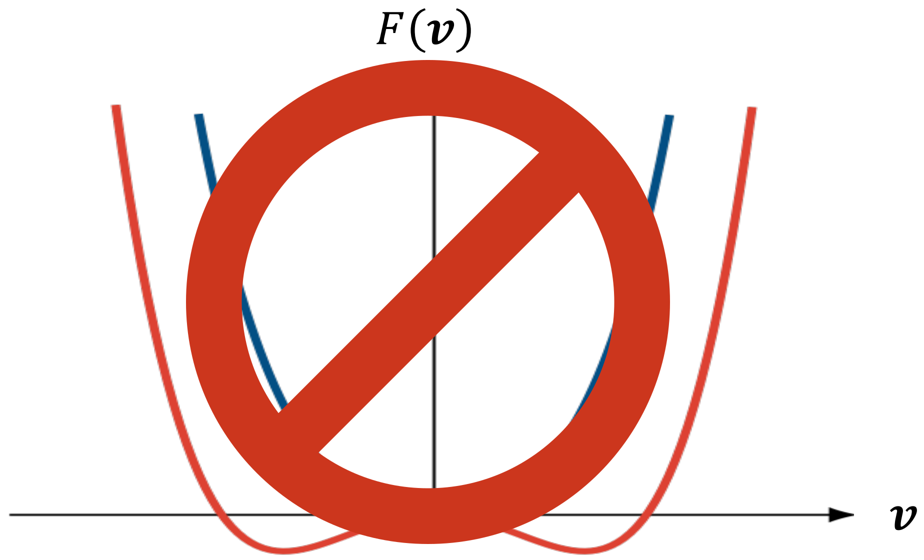
$$F(\boldsymbol{v}) = \alpha_{ab}\boldsymbol{v}_a \cdot \boldsymbol{v}_b + \beta_{abcd}(\boldsymbol{v}_a \cdot \boldsymbol{v}_b)(\boldsymbol{v}_c \cdot \boldsymbol{v}_d)$$

$$\partial_t \boldsymbol{v}_a = -\frac{\delta F(\boldsymbol{v})}{\delta \boldsymbol{v}_a}$$

Reciprocal coupling

$$\alpha_{ab} = \alpha_{ba}$$

# ~~Equilibrium~~ paradigm:
# Nonequilibrium



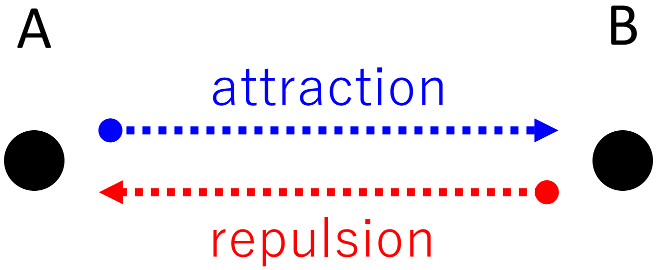
$$\cancel{F(\boldsymbol{v})} = \alpha_{ab}\boldsymbol{v}_a \cdot \boldsymbol{v}_b + \beta_{abcd}(\boldsymbol{v}_a \cdot \boldsymbol{v}_b)(\boldsymbol{v}_c \cdot \boldsymbol{v}_d)$$

$$\partial_t \boldsymbol{v}_a = -\cancel{\frac{\delta F(\boldsymbol{v})}{\delta \boldsymbol{v}_a}}$$

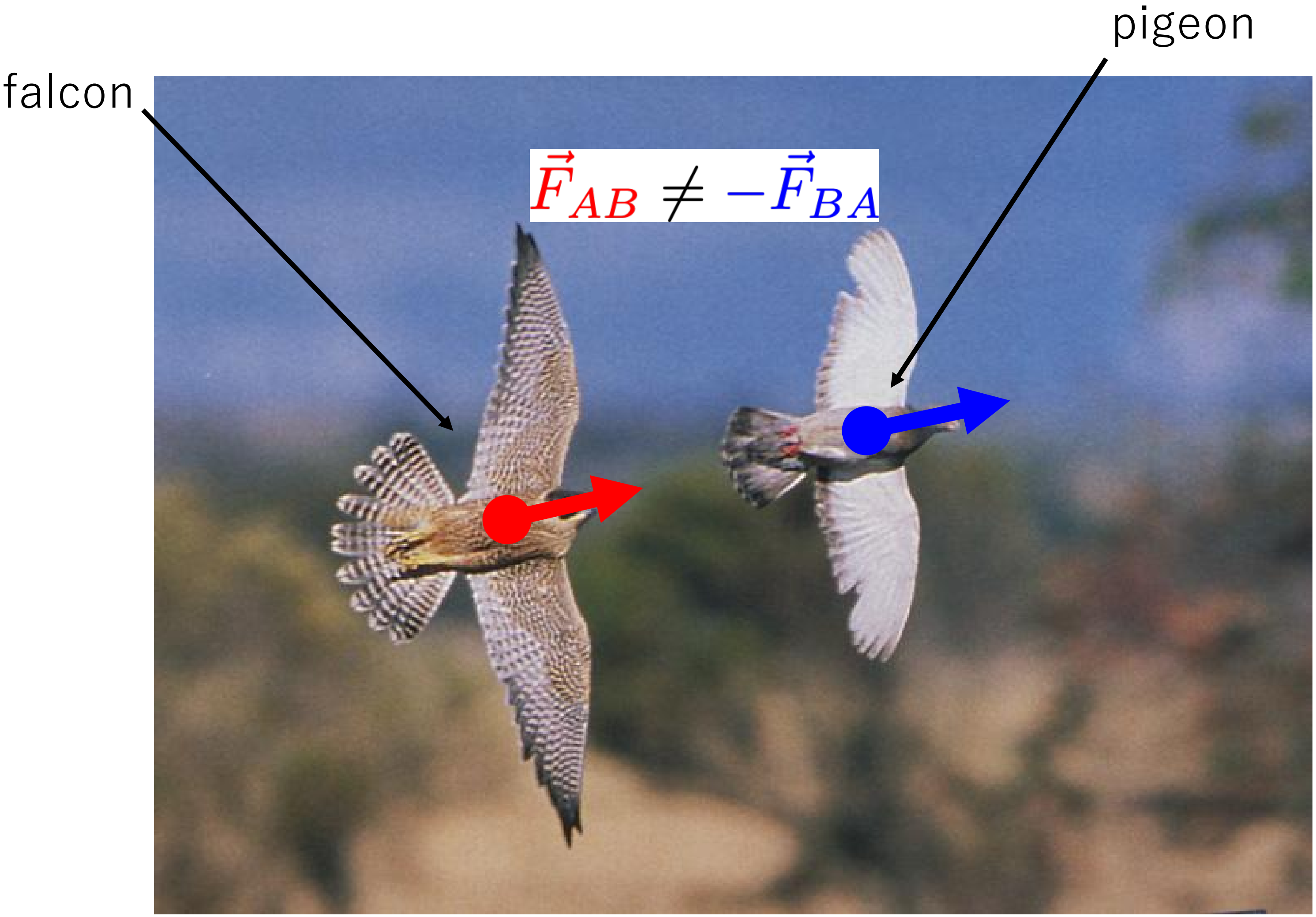
Reciprocal coupling

$$\cancel{\alpha_{ab} = \alpha_{ba}}$$

# Non-reciprocal interaction



# Non-reciprocal interaction



$\vec{F}_{AB} \neq -\vec{F}_{BA}$

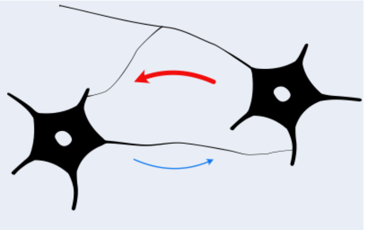
Image from: http://animal.memozee.com/view.php?tid=3&did=3513

# Non-reciprocally interacting systems

## Prey and predators

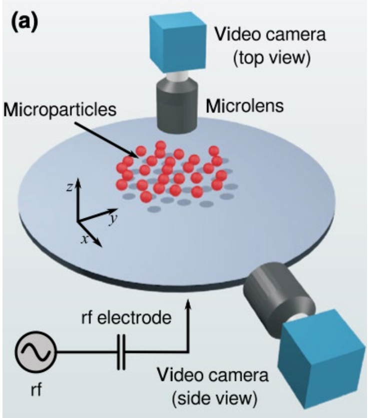
http://animal.memozee.com/view.php?tid=3&did=3513

## Inhibitory and exhibitory neurons



e.g., J. W. Krakauer, et al., Neuron 93, 480 (2017).

## Complex plasma

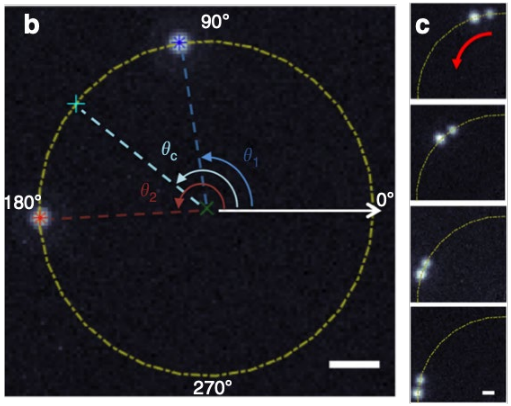


Ivlev, et al., PRX 5, 011035 (2015).

## Social network

e.g., A. Pluchino, et al., Inter. J. Mod. Phys. C 16, 515 (2005).

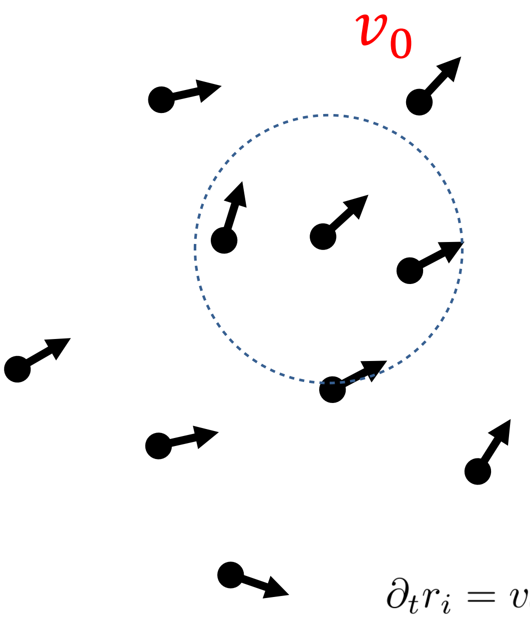
## Reactive optical matter



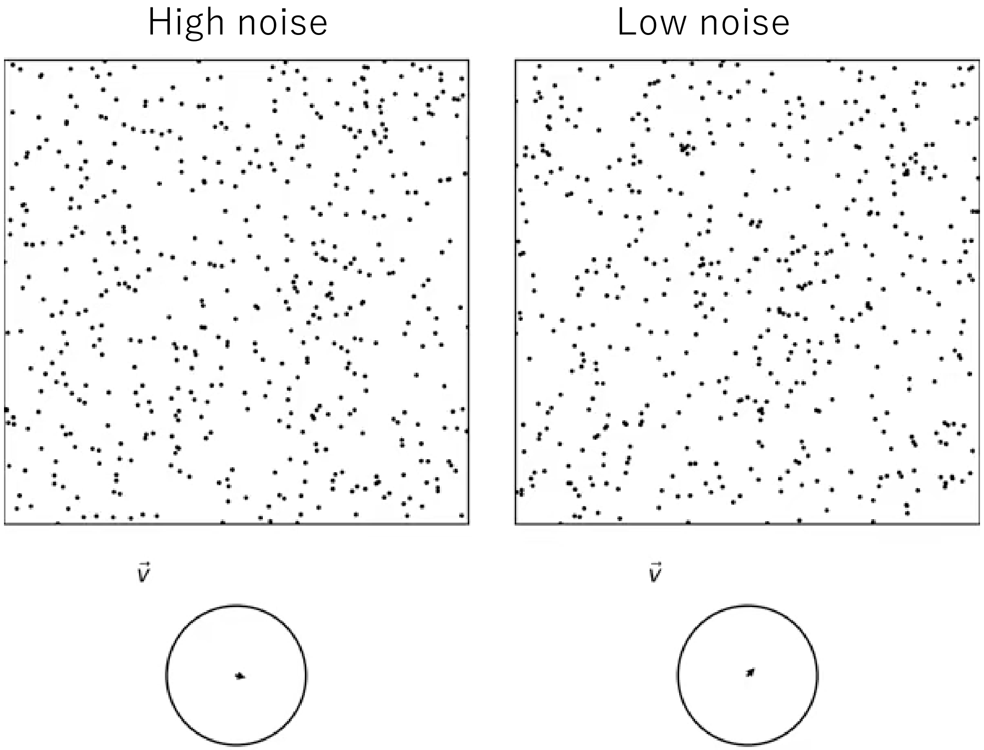
Yifat, et al., Light: Science and Applications 7, 105 (2018).

# Flocking

Vicsek PRL 1995, Toner and Tu PRL 1995



$$\partial_t r_i = v_0 \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix}$$



**Disordered phase**

**Flocking**

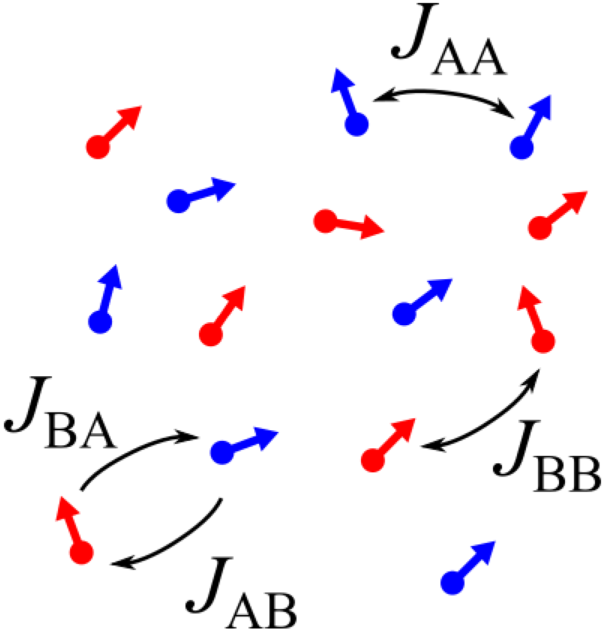
Reciprocal coupling $J_{ij} = J_{ji}$

$$\partial_t \theta_i = \sum_j J_{ij} \sin(\theta_j - \theta_i) + \eta_i(t)$$

$$= -\frac{\partial V}{\partial \theta_i}$$

# Non-reciprocal flocking

Generalized Vicsek model with two groups of non-reciprocally interacting agents



$$J_{AB} \neq J_{BA}$$

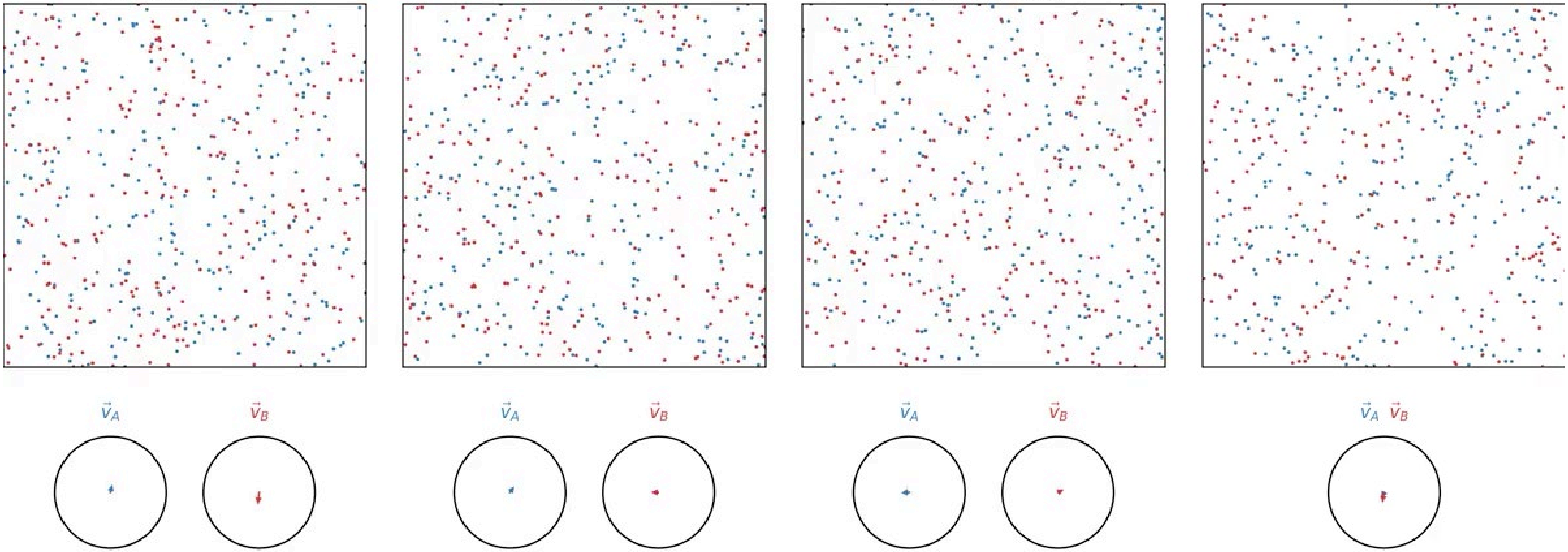
$$\partial_t \theta_i = \sum_j J_{ij} \sin(\theta_j - \theta_i) + \eta(t)$$

$$\neq -\frac{\partial V}{\partial \theta_i}$$

# Non-reciprocal flocking

Reciprocal ($J_{\mathrm{AB}} = J_{\mathrm{BA}} > 0$) Reciprocal ($J_{\mathrm{AB}} = J_{\mathrm{BA}} < 0$) Non-reciprocal ($J_{\mathrm{AB}}J_{\mathrm{BA}} < 0$)

High noise Low noise Low noise Low noise



**Disordered phase** **Aligned phase** **Anti-aligned phase** **Non-reciprocity induced chiral phase!**

**Spontaneous $Z_2$ symmetry breaking**

How do we understand the emergence of chiral phase?

M. Fruchart*, RH*****, P. B. Littlewood, and V. Vitelli, Nature **592**, 363 (2021).

# *Nonequilibrium generalization* of Landau theory

**Dynamical system**

$$\partial_t \boldsymbol{v}_a = -[\alpha_{ab}\boldsymbol{v}_b + \beta_{abcd}(\boldsymbol{v}_b \cdot \boldsymbol{v}_c)\boldsymbol{v}_d] \left(\neq -\frac{\delta F(\phi)}{\delta \boldsymbol{v}_a}\text{!!!}\right)$$

*Asymmetric* coefficients

$$\alpha_{ab} \neq \alpha_{ba}$$

**Nonequilibrium steady state** (Two components, $\beta_{abcd} = \beta\delta_{ab}\delta_{cd}$)**:**

$$0 = \partial_t \begin{pmatrix} \boldsymbol{v}_A \\ \boldsymbol{v}_B \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_{AA} + \beta|\boldsymbol{v}_A|^2 & \alpha_{AB} \\ \alpha_{BA} & \alpha_{BB} + \beta|\boldsymbol{v}_B|^2 \end{pmatrix}}_{A \neq A^\dagger} \begin{pmatrix} \boldsymbol{v}_A \\ \boldsymbol{v}_B \end{pmatrix}$$

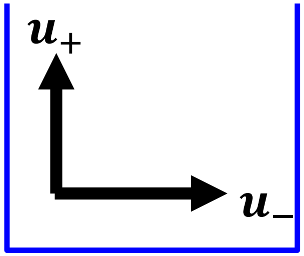
➡ Nonlinear, non-Hermitian eigenvalue problem

$$\partial_t \begin{pmatrix} \delta\phi_A \\ \delta\phi_B \end{pmatrix} = L \begin{pmatrix} \delta\phi_A \\ \delta\phi_B \end{pmatrix} \qquad L \neq L^\dagger$$

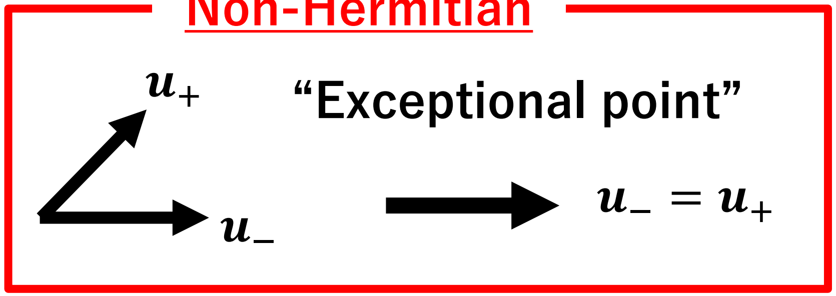
M. Fruchart*, RH*****, P. B. Littlewood, and V. Vitelli, Nature **592**, 363 (2021).

# Non-reciprocal phase transition

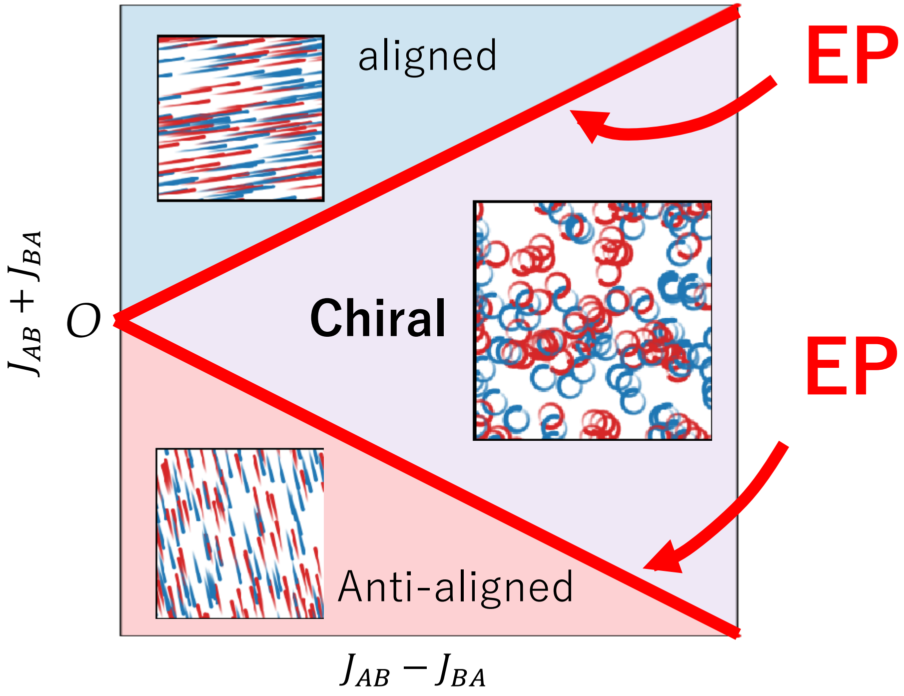
**Hermitian**



**Non-Hermitian**



A phase transition point marked by **exceptional points (EPs)** emerges!



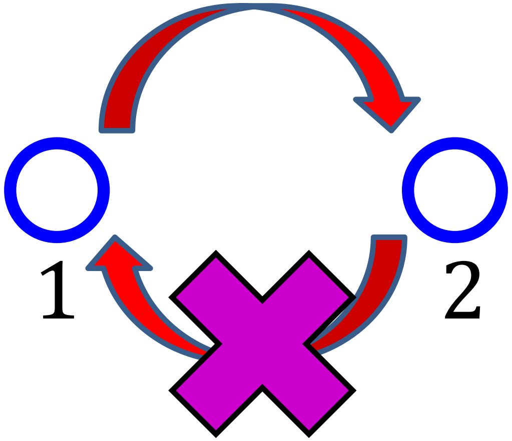
M. Fruchart*, RH*****, P. B. Littlewood, and V. Vitelli, Nature **592**, 363 (2021).

# Non-reciprocal phase transition

**Exceptional point** ⟷ Non-diagonalizable matrix

- One-way (non-reciprocal) coupling of the collective modes

$$\partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = L \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

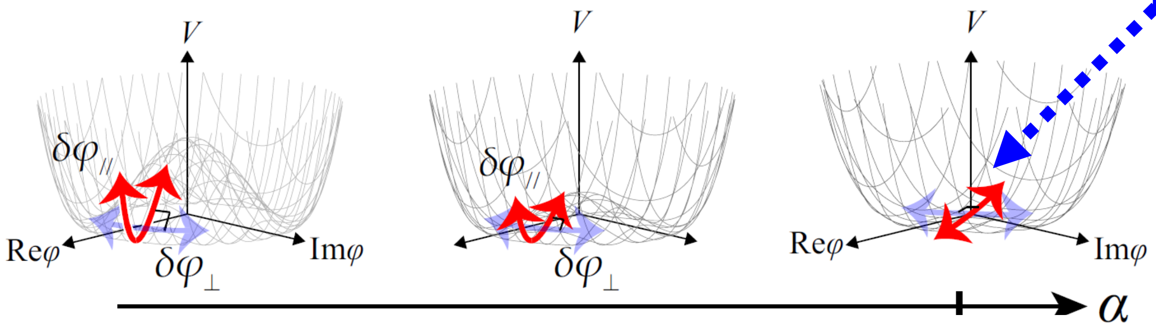


*No* equilibrium counterpart exists

# Non-reciprocal phase transition

Conventional (equilibrium) critical phenomena

Amplitude mode softens by
*flattening of free energy landscape*

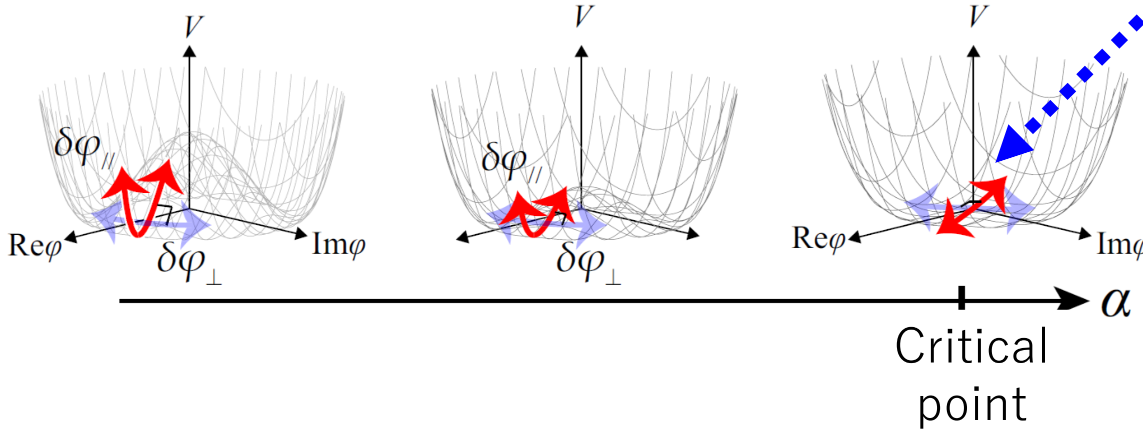


Critical
point

RH and P.B. Littlewood, PRResearch **2**, 033018 (2020).

# Non-reciprocal phase transition

Conventional (equilibrium) critical phenomena



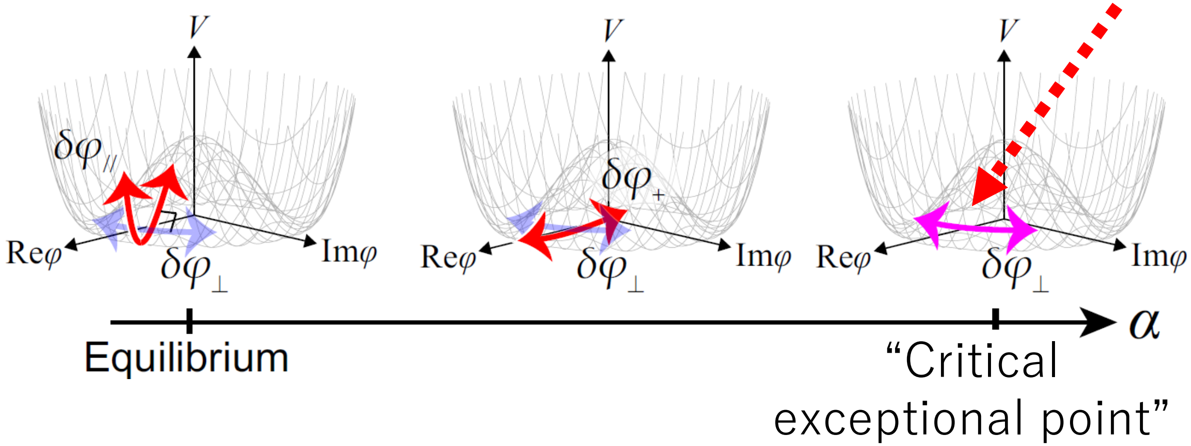
Amplitude mode softens by *flattening of free energy landscape*

$$\langle(\delta\theta)^2\rangle \sim \int dk k^{d-1}\frac{1}{\boldsymbol{k^2}}$$

$$\to \infty\ (d \le \mathbf{2})$$

(Mermin-Wagner's theorem)

**Our result**



*Coalescence* of the collective modes to the Goldstone mode

- **Anomalously large fluctuations**

$$\langle(\delta\theta)^2\rangle \sim \int dk k^{d-1}\frac{1}{\boldsymbol{k^4}}$$

$$\to \infty\ (d \le \mathbf{4})$$

**New universality class**

$$\chi = \frac{4-d}{2} - \frac{\epsilon}{10}, z = 1\ (\epsilon = 8-d)$$

RH and P.B. Littlewood, PRResearch **2**, 033018 (2020).

# Non-reciprocal phase transition

Occurs quite *generally* in nonequilibrium systems!

(A) Nonequilibrium

(B) Spontaneous continuous symmetry breaking

(C) Consist of two (or more) order parameters

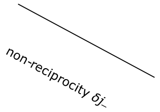
Pattern formation

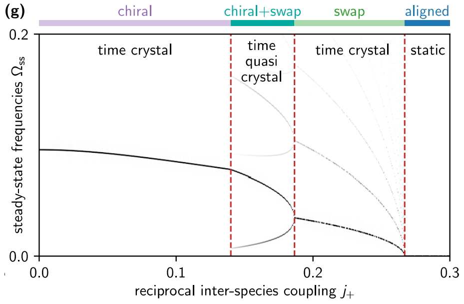
Synchronization

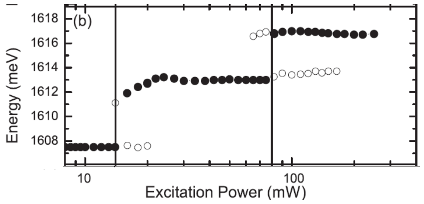
Flocking

Exciton-polaritons

L Pan and J. R. de Bruyn, PRE1994







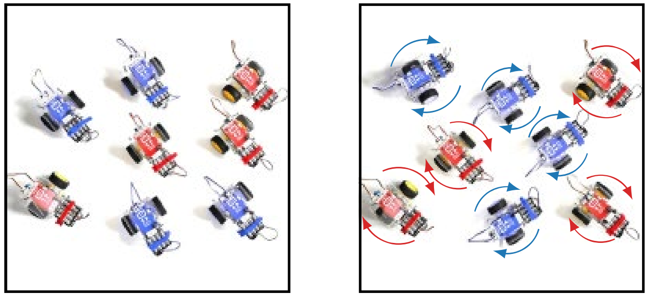
J. Tempel, et al PRB2012

RH, et al PRL2019

M. Fruchart*, RH*****, P. B. Littlewood, and V. Vitelli, Nature **592**, 363 (2021).

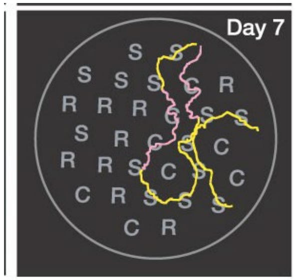
# Collective phenomena in non-reciprocal many-body systems

## Non-reciprocal phase transitions



M. Fruchart*, RH*, P. B. Littlewood, and V. Vitelli, Nature 2021

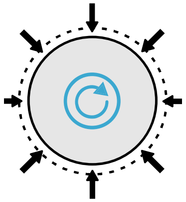
## Biodiversity in ecosystems



B. Kerr, et al., Nature 2002

S. Allesina and S. Tang, Nature 2012

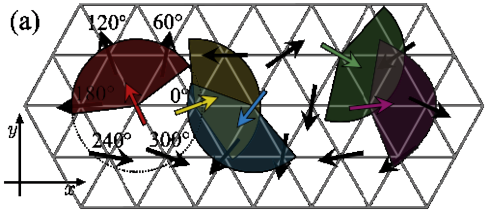
## Odd elasticity



C. Scheibner, et al., , Nat. Phys. 2020

T. H. Tan, et al., , Nature 2022

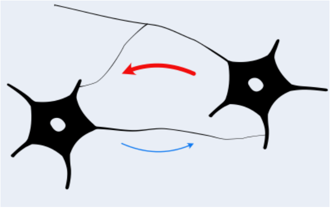
## Long-ranged order in 2D



Loos, Klapp, and Martynec, arXiv:2206.10519

Dadhichi, et al., Phys. Rev. E 2020

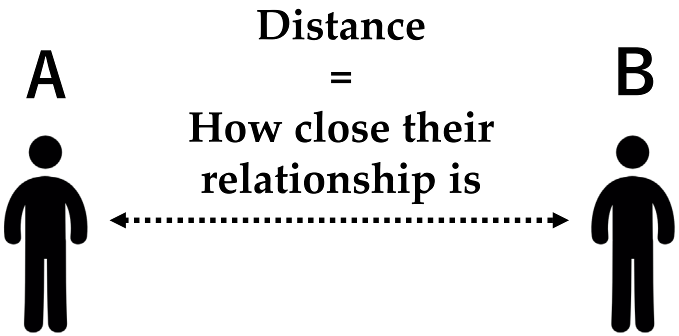
## Controlling Neuron dynamics



H. Wilson and J. Cowan,
Biophys. J. 1972

G. Parisi, J. Phys. A 1986

# Non-reciprocal friendship: source of frustration



**A**

**Distance**

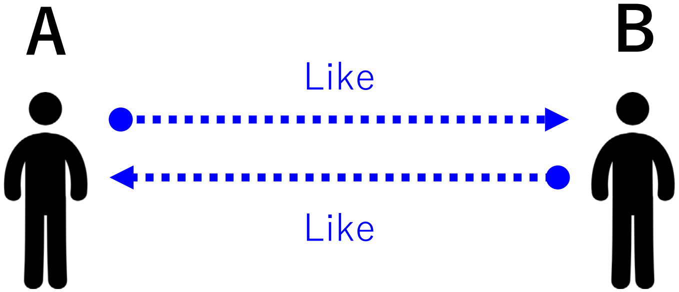
**=**

**How close their relationship is**

**B**

# Non-reciprocal friendship: source of frustration

(a) Reciprocal relationship (like)



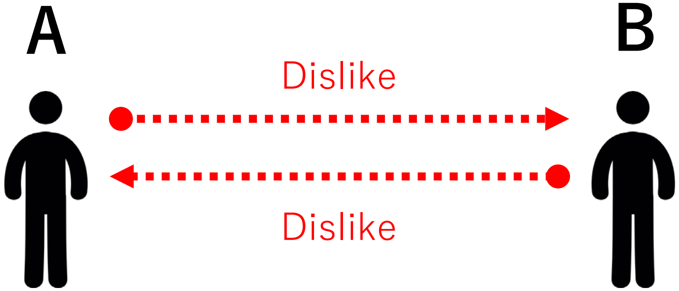
A

B

Like

Like

(b) Reciprocal relationship (dislike)



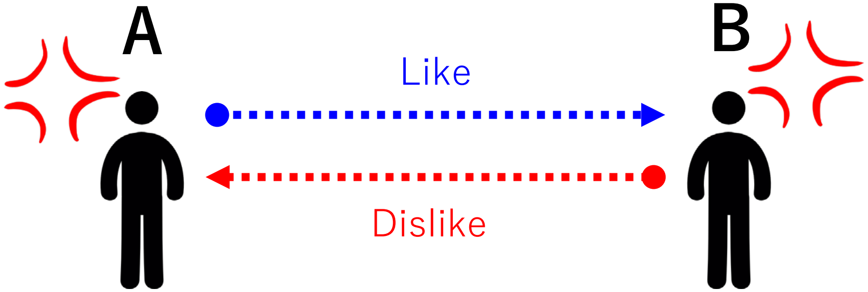
A

B

Dislike

Dislike

(c) Non-reciprocal relationship



A

B

Like

Dislike

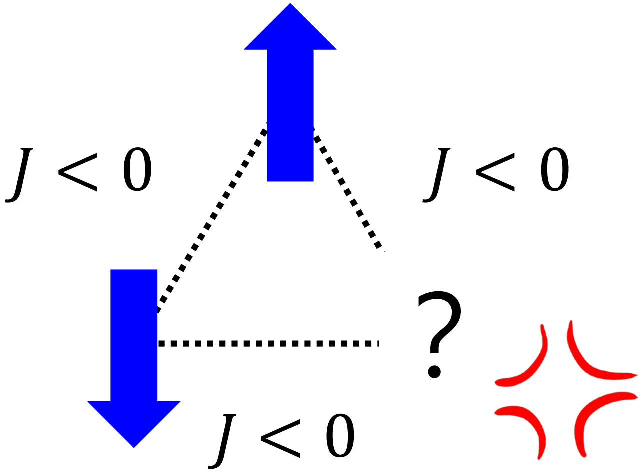
No configuration that makes both happy

**FRUSTRATION!**

# Geometrical frustration

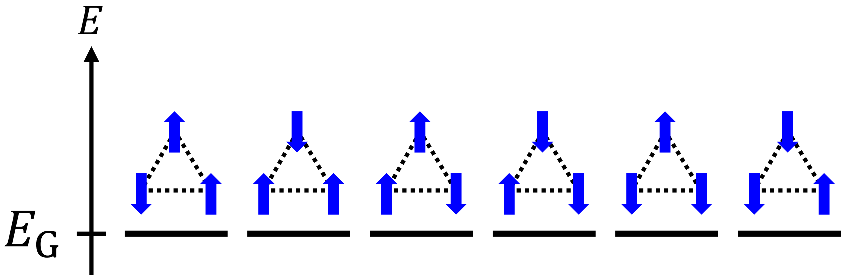
**Geometrically frustrated systems**

Systems that cannot satisfy all the constitutes' "desire" to minimize all interactions



No configuration can make all spins happy

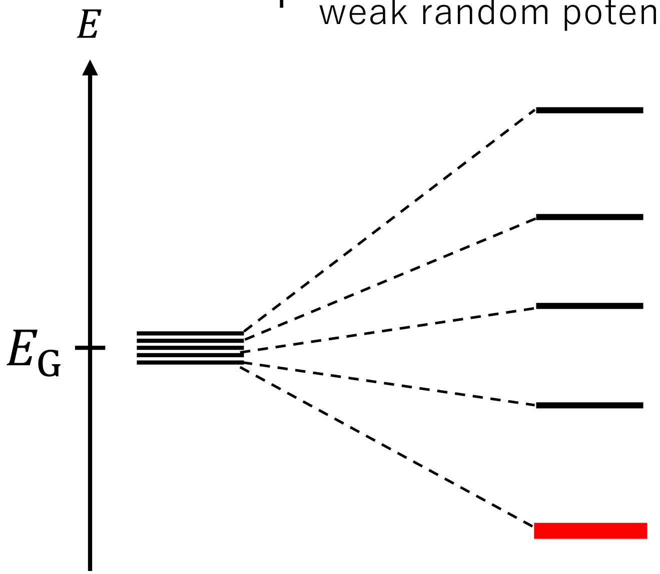
**Accidental degeneracy of ground states**



# Order by disorder phenomena

Villain, et al., J. Physique (1980)

**Accidental degeneracy**: Not protected by symmetry nor topology



+ Thermal or quantum noise,
weak random potential, etc.

**Selected** ground state

(ordered state in many cases)

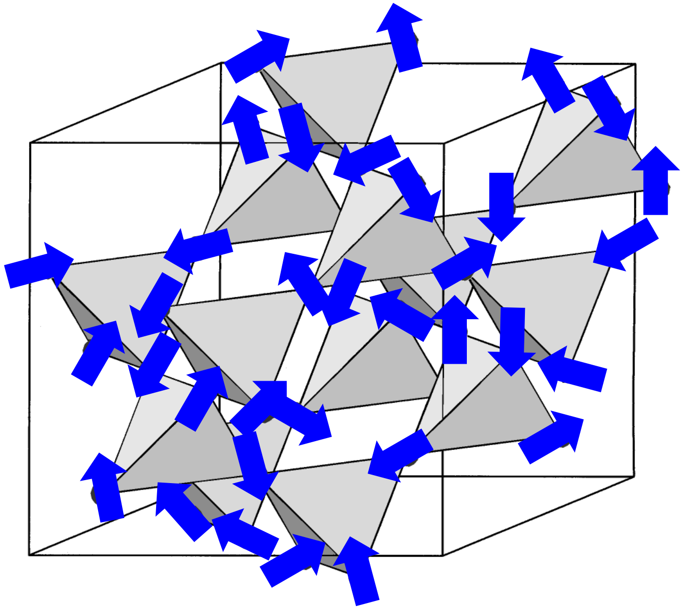
"Order by disorder"

# Order by disorder phenomena

## (Example) XY spins on a pyrochlore lattice

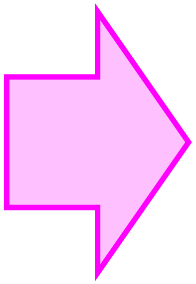
Moessner and Chalker, PRL1998, PRB1998

Ground state

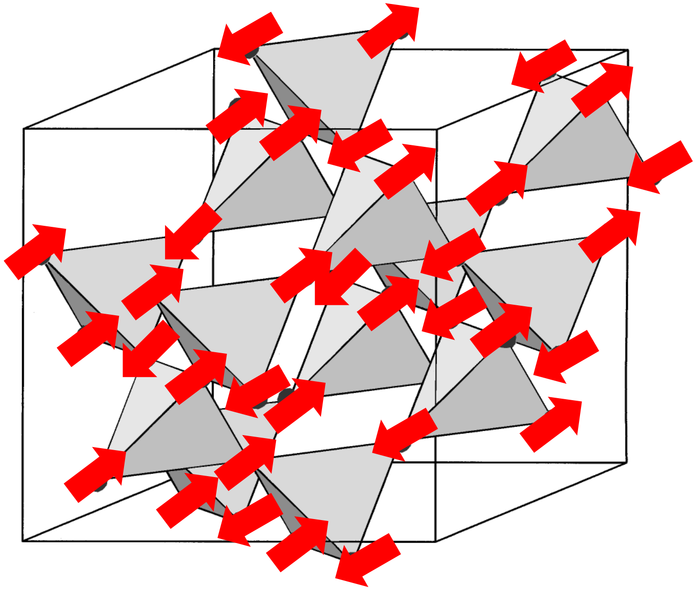


Disordered state

Increase temperature



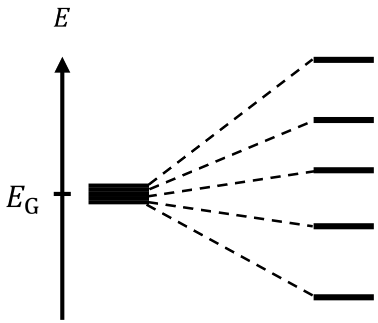
Finite temperature



Long-ranged order

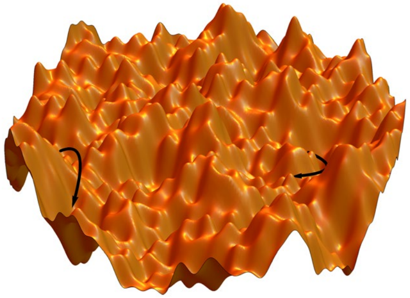
# Many-body physics in geometrical frustration

- Order-by-disorder



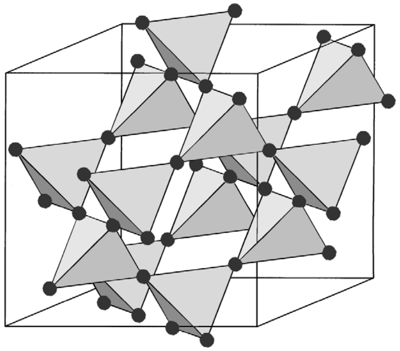
Villain, et al., J. Physique (1980)

- Spin glass

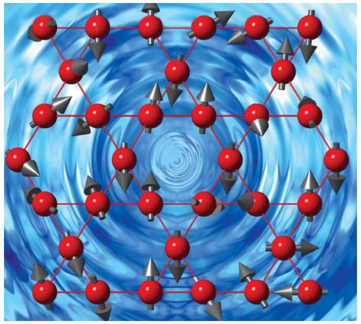


(Image from https://scglass.uchicago.edu/ )

- Quantum/Classical spin liquid



R. Mossener and J. T. Chalker, PRL 1998

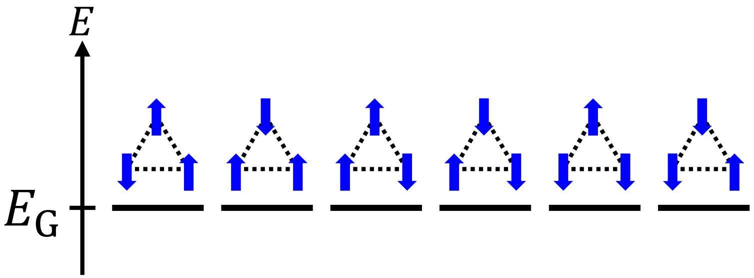


T. Imai and Y. Lee, Physics Today 2016

# Geometrical vs Non-reciprocal frustration

## Geometrical frustration

Accidental degeneracy of
ground state



## Non-reciprocal frustration

?

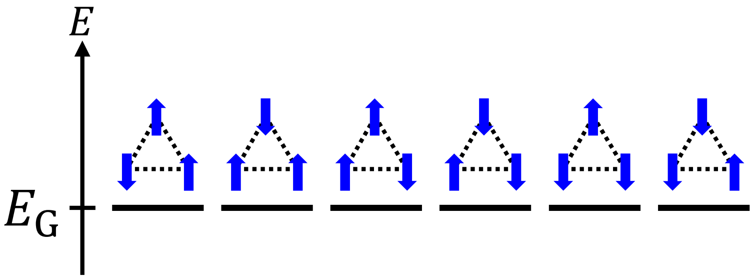
Energy cannot be defined …

May not even converge to
a static state…

# Geometrical vs Non-reciprocal frustration

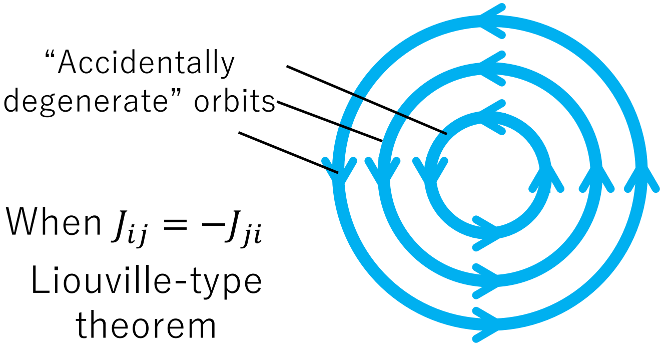
## Geometrical frustration

Accidental degeneracy of ground state



## Non-reciprocal frustration

"Accidental degeneracy" of orbits



When $J_{ij} = -J_{ji}$

Liouville-type theorem

- *Dynamical counterpart* of order-by-disorder and spin glass occurs!

RH, arXiv:2208.08577

# Dissipative XY spin dynamics

$$\dot{\theta}_i = \sum_j J_{ij} \sin(\theta_j - \theta_i)$$

Here, couplings are non-reciprocal in general $J_{ij} \neq J_{ji}$

➢ Reciprocal case $(J_{ij} = J_{ji})$ ：Potential energy minimization problem

$$\dot{\theta}_i = -\frac{\partial V(\theta)}{\partial \theta_i} \quad \text{with} \quad V(\theta) = -\sum_{i,j} \cos(\theta_j - \theta_i)$$

Potential with
geometrical frustration

→

Accidentally degenerate
ground states

# "Accidental degeneracy" of *orbits*

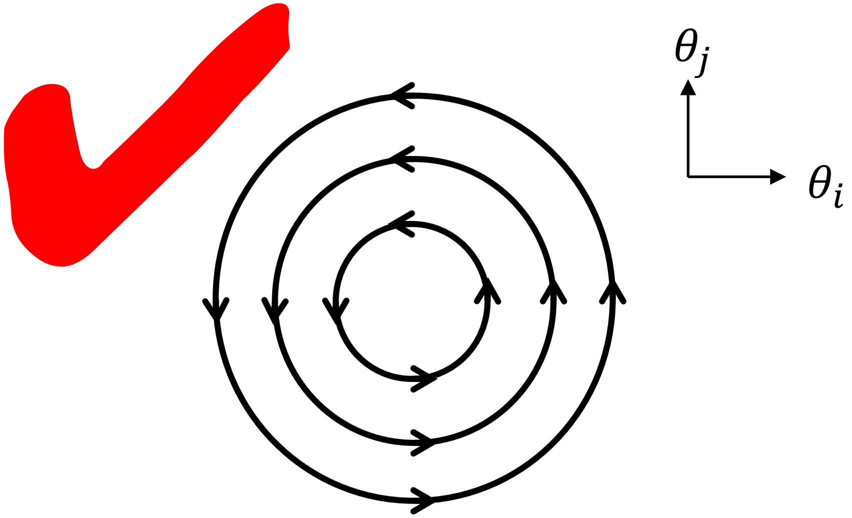
- Anti-symmetric case ($J_{ij} = -J_{ji}$)

Liouville-type theorem

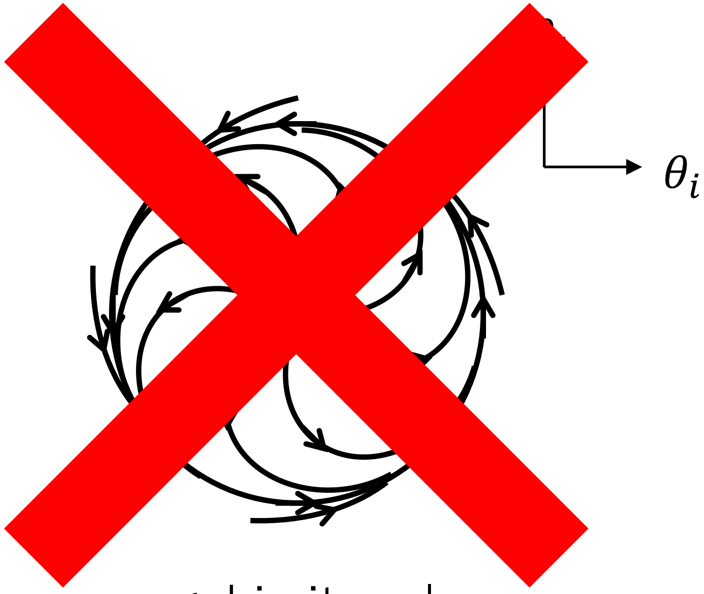
$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \sum_i \frac{\partial\rho}{\partial\theta_i}\dot{\theta}_i = 0$$

RH, arXiv:2208.08577

Conservation of phase volume = **Non-dissipative** dynamics



Marginal



e.g. Limit cycles

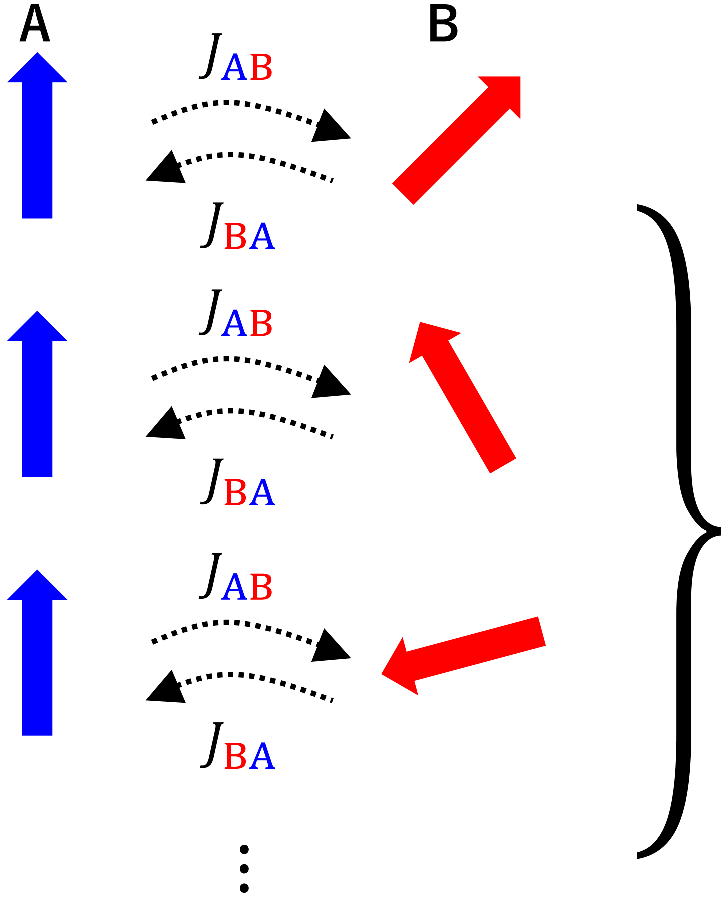
# "Accidental degeneracy” of *orbits*

(e.g., two XY spin system)

$$J_{\mathrm{AB}} = -J_{\mathrm{BA}}$$

$$\dot{\theta}_{\mathrm{A}} = J_{\mathrm{AB}} \sin(\theta_{\mathrm{B}} - \theta_{\mathrm{A}})$$

$$\dot{\theta}_{\mathrm{B}} = J_{\mathrm{BA}} \sin(\theta_{\mathrm{A}} - \theta_{\mathrm{B}})$$



Initial state 1 :

Initial state 2:

Initial state 3:

⋮

Non-reciprocal frustration induced

“accidental degeneracy” of orbits

RH, arXiv:2208.08577

# “Accidental degeneracy” of *orbits*

**[Proof]**

Continuity equation: $\frac{\partial \rho}{\partial t} = -\sum_i \frac{\partial\left(\rho \dot{\theta}_i\right)}{\partial \theta_i} = -\sum_i \left[\frac{\partial \rho}{\partial \theta_i} \dot{\theta}_i + \xcancel{\rho \frac{\partial \dot{\theta}_i}{\partial \theta_i}}\right]$

$$\sum_i \frac{\partial \dot{\theta}_i}{\partial \theta_i} \underset{\uparrow}{=} \sum_{ij}\left[J_{ij} \cos\left(\theta_j - \theta_i\right)\right] \underset{\uparrow}{=} 0$$

$$J_{ij} = -J_{ji}$$

$$\dot{\theta}_{\mathrm{i}} = \sum_j J_{ij} \sin(\theta_j - \theta_i)$$

Therefore, $\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial \theta_i} \dot{\theta}_i = 0.$

■

RH, arXiv:2208.08577

# “Accidental degeneracy” of *orbits*

**[Proof]**

Continuity

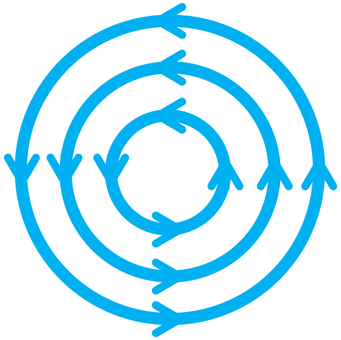
How does the “accidentally degenerate” orbits affect collective properties of many-body system?

Order-by-disorder?

➡ **YES!**

Spin glass?

➡ **YES!**

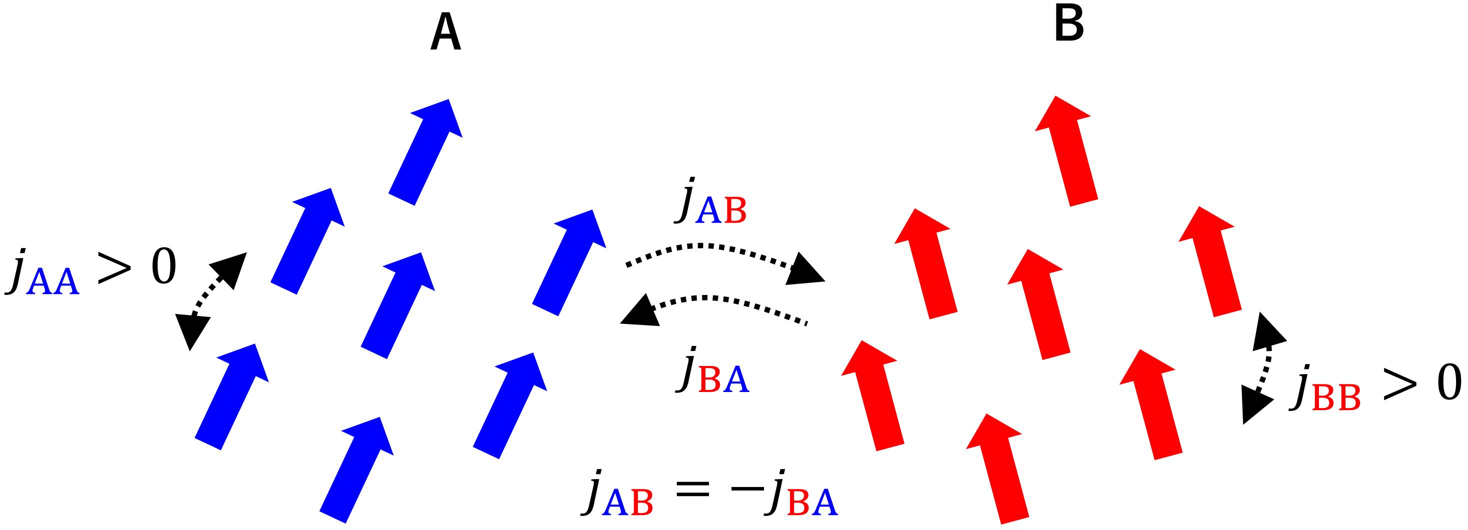


Therefore, $$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial \theta_i} \dot{\theta}_i = 0. \qquad \blacksquare$$

RH, arXiv:2208.08577

# Non-reciprocal two-group system

$$\dot{\theta}_i^a = \sum_b \sum_{i=1}^{N_b} \frac{j_{ab}}{N_b} \sin(\theta_j^b - \theta_i^a)$$



RH, arXiv:2208.08577

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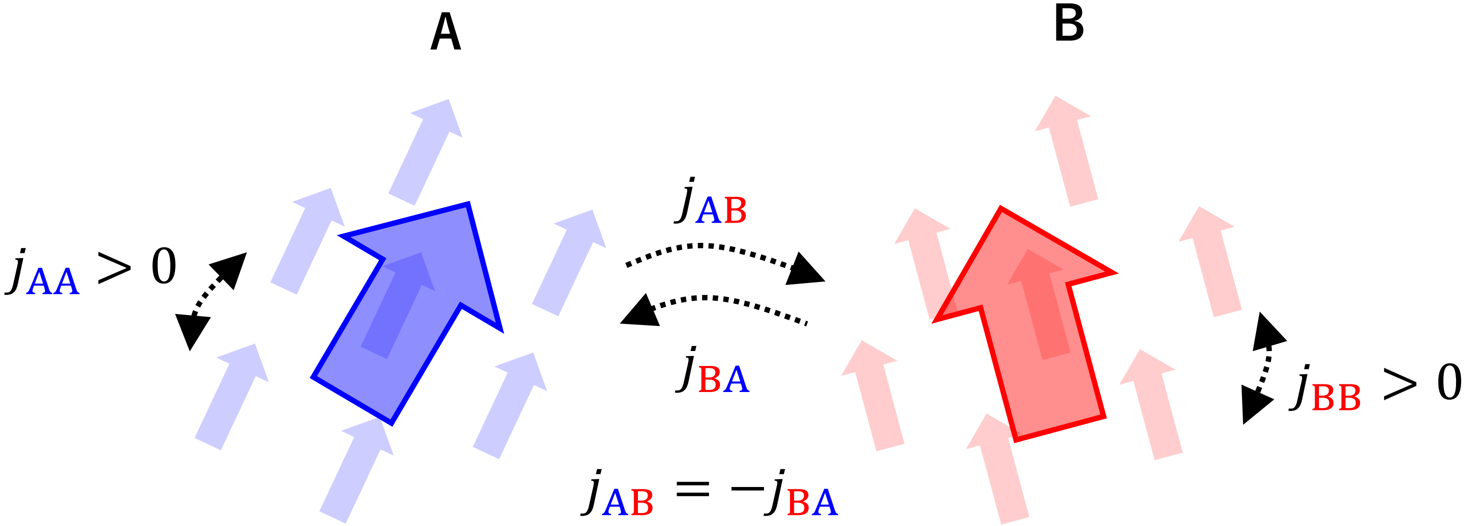

RH, arXiv:2208.08577

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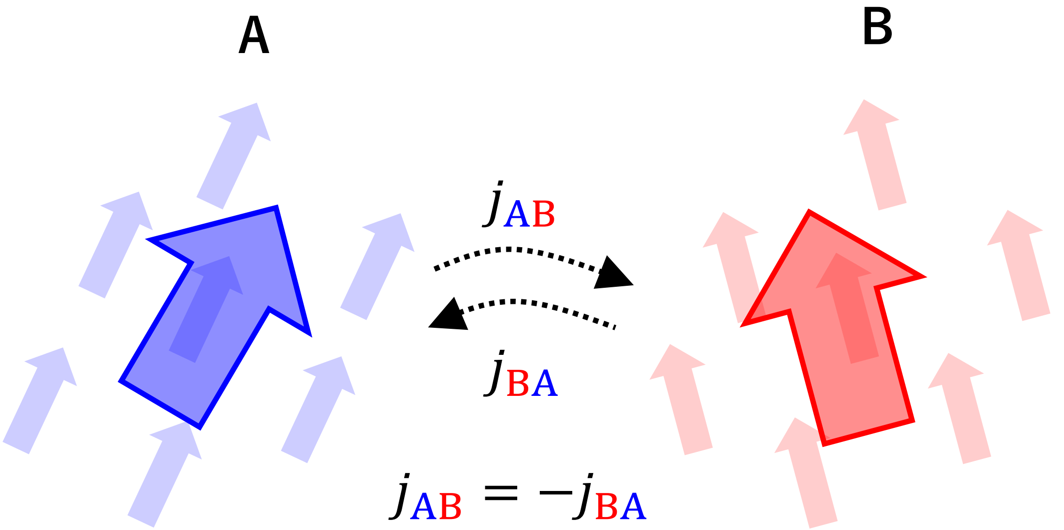

"Accidental degeneracy" parameterized by $\Delta\phi = \phi_{\mathrm{A}} - \phi_{\mathrm{B}}$

$$j_{\mathrm{AB}} = -j_{\mathrm{BA}}$$

Macroscopic spin (No noise)

$$\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a)$$

RH, arXiv:2208.08577

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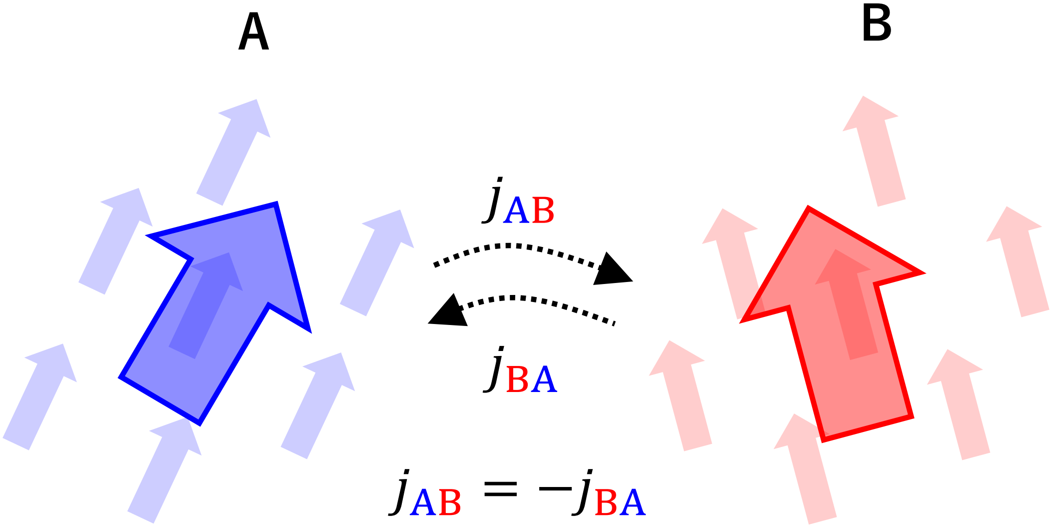

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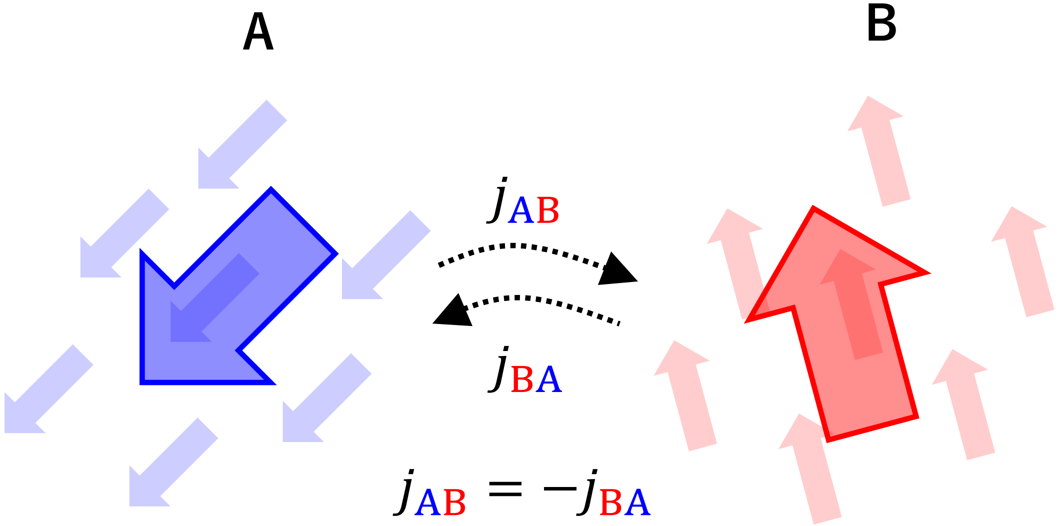

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Macroscopic spin (No noise)

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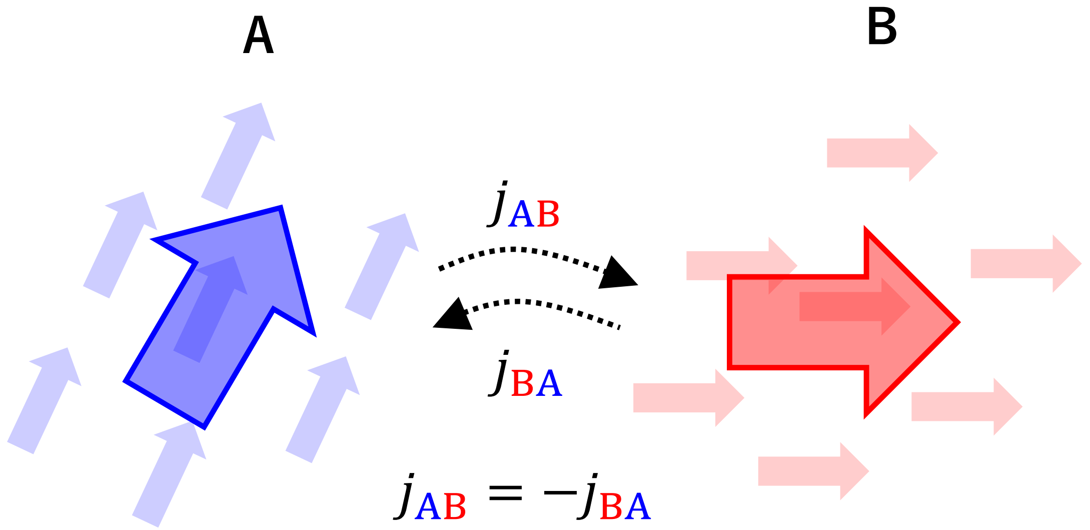
RH, arXiv:2208.08577

# Non-reciprocal two-group system

Noise

$$\dot{\theta}_i^a = \sum_b \sum_{i=1}^{N_b} \frac{j_{ab}}{N_b} \sin(\theta_j^b - \theta_i^a) + \boldsymbol{\eta_i^a}$$

“Accidental degeneracy” parameterized by $\Delta\phi = \phi_{\mathrm{A}} - \phi_{\mathrm{B}}$



Macroscopic spin (No noise)

$$\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a)$$

RH, arXiv:2208.08577

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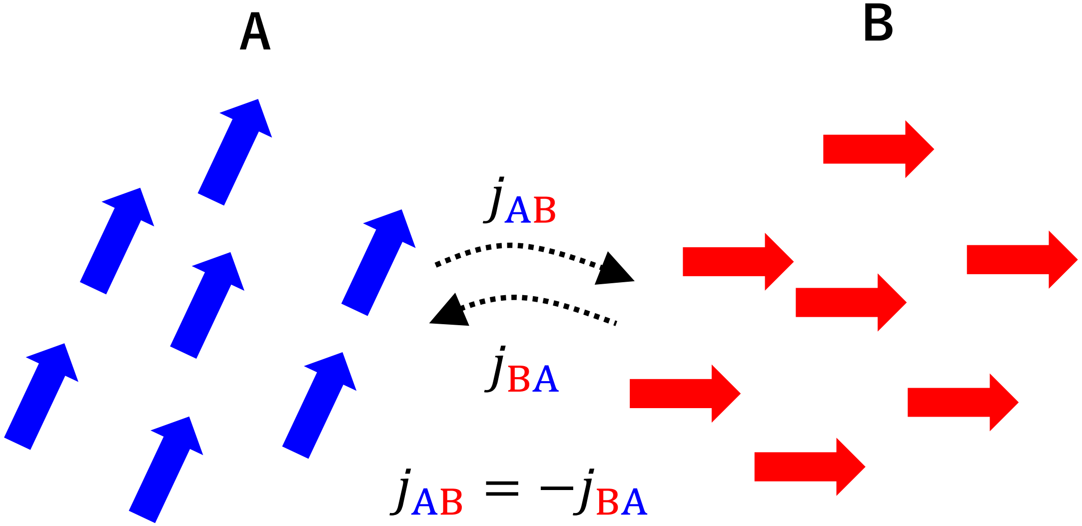

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# Non-reciprocal two-group system

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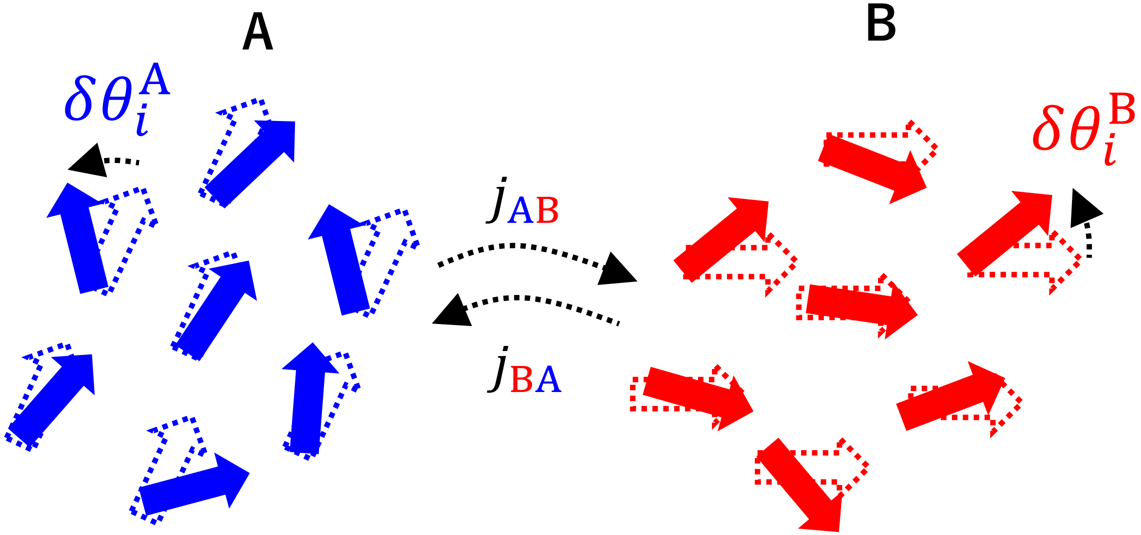
Noise

Probability distribution of $\delta\theta_i^a$ is $\Delta\phi$-dependent



"Accidental degeneracy" parameterized by $\Delta\phi = \phi_{\mathrm{A}} - \phi_{\mathrm{B}}$

Macroscopic spin (No noise)

$$\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a)$$

Macroscopic spin (With noise)

$$\dot{\phi}_a \approx \sum_b \underline{\boldsymbol{j_{ab}^{\star}(\Delta\phi)}} \sin(\phi_b - \phi_a)$$

$\Delta\phi$-dependent renormalized coupling

RH, arXiv:2208.08577

# Order-by-time-crystalline order

$$\Delta\dot{\phi} = -\left(j^{\star}_{\mathbf{AB}}(\boldsymbol{\phi}) + j^{\star}_{\mathbf{BA}}(\boldsymbol{\phi})\right)\sin\Delta\phi$$

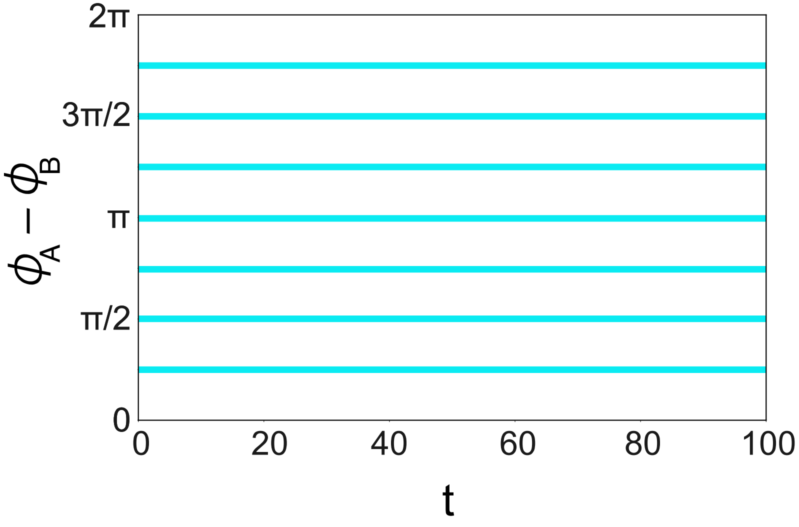
$j_{AB} = -j_{BA} = j_{-}, j_{AA} = j_{BB} = j_0$

$$\approx \frac{j_0 j_{-}^2 \sigma^2}{2} \frac{\cos\Delta\phi}{(j_0^2 - j_{-}^2\cos^2\Delta\phi)^2} \sin\Delta\phi$$

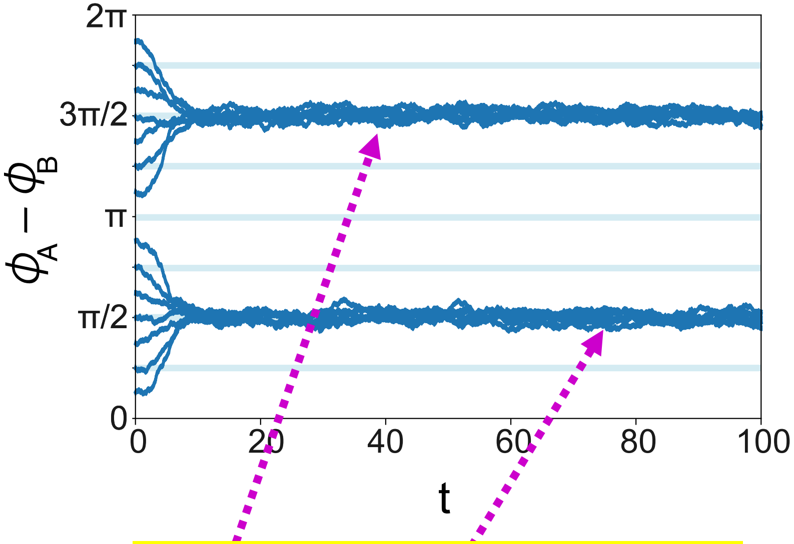
**Stable** fixed point $\Delta\phi_* = \pm\frac{\pi}{2}$

$\sigma$: Noise strength

No noise



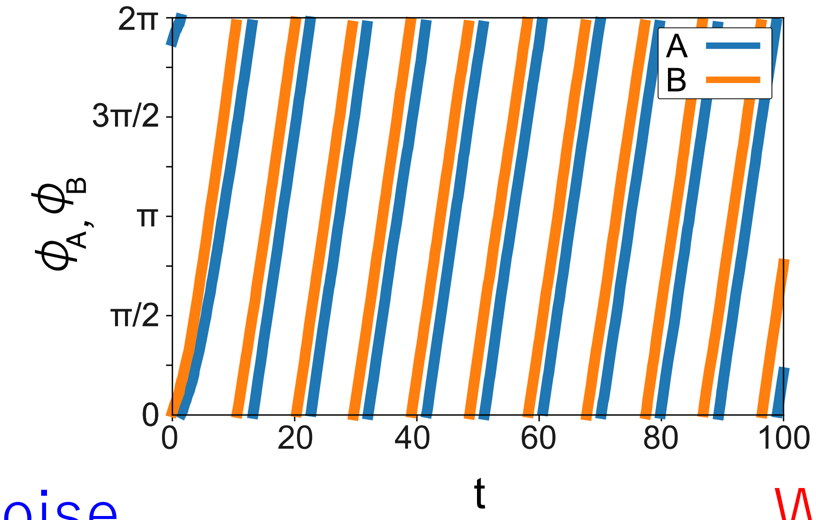
With noise



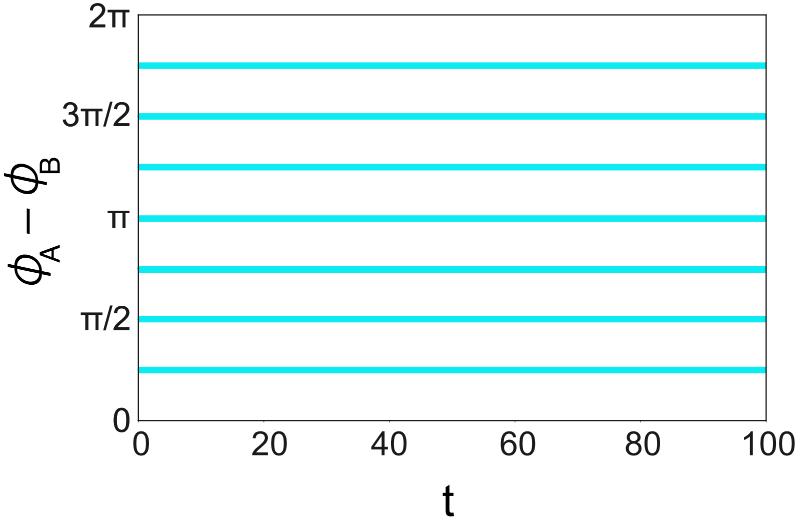
Order-by-disorder!

RH, arXiv:2208.08577

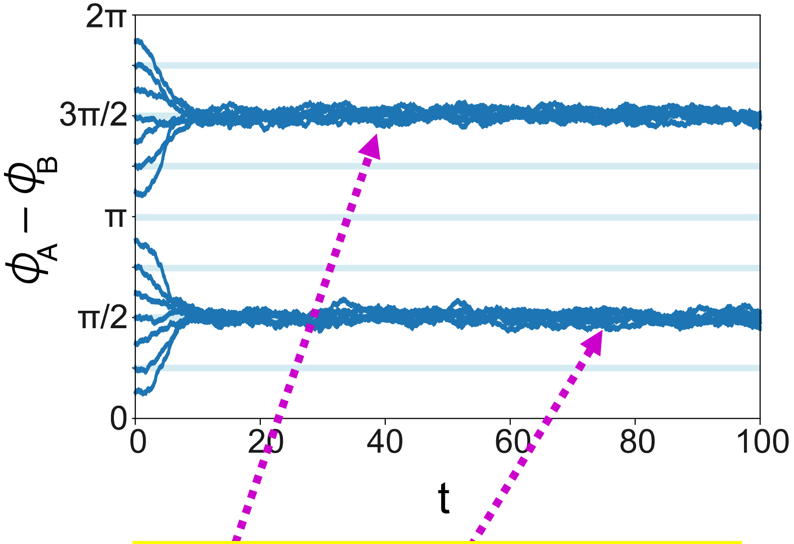
# Order-by-time-crystalline order



No noise



With noise



Order-by-disorder!

RH, arXiv:2208.08577

# Three communities with natural frequency disorder

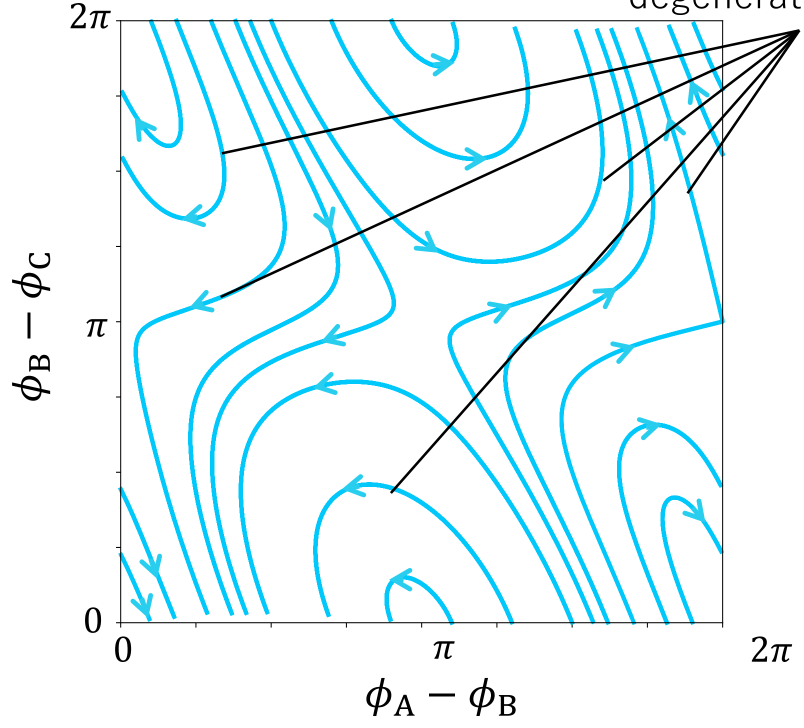
**Kuramoto model** $(\alpha = \mathrm{A}, \mathrm{B}, \mathrm{C})$

$$\dot{\theta}_i^\alpha = \omega_i^\alpha + \sum_{\beta=\mathrm{A,B,C}} \sum_{j=1}^{N_\beta} J_{\alpha\beta} \sin\left(\theta_j^\beta - \theta_i^\alpha\right)$$

Order parameter

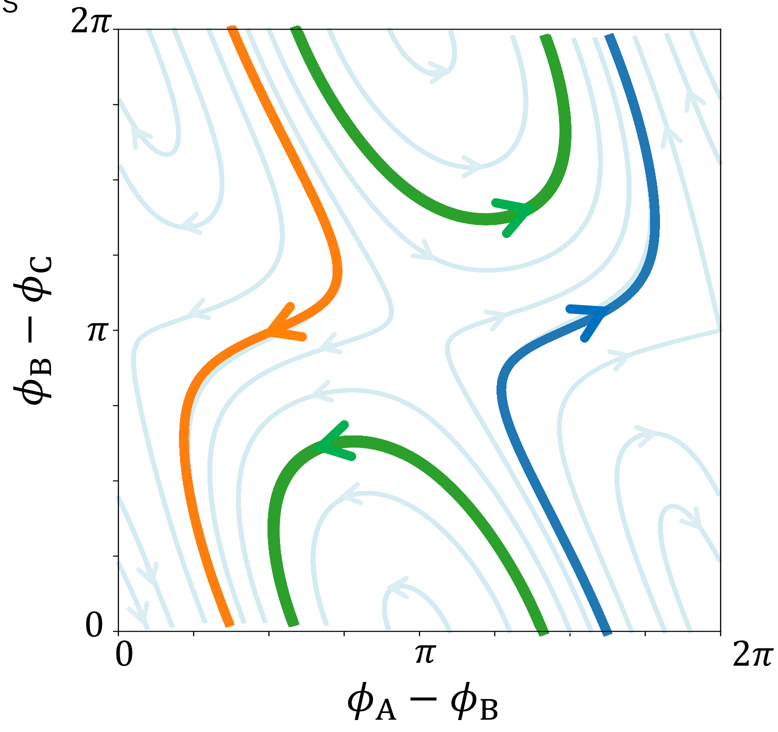
$$z_\alpha = r_\alpha e^{i\phi_\alpha} = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} e^{i\theta_\alpha}$$

Clean system



"Accidentally degenerate" orbits

Random natural frequency



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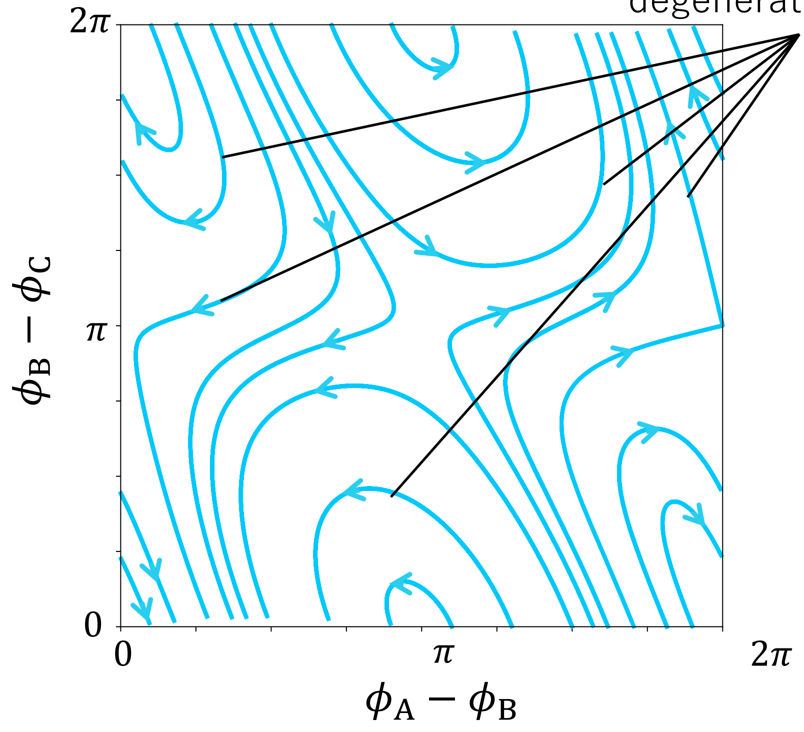

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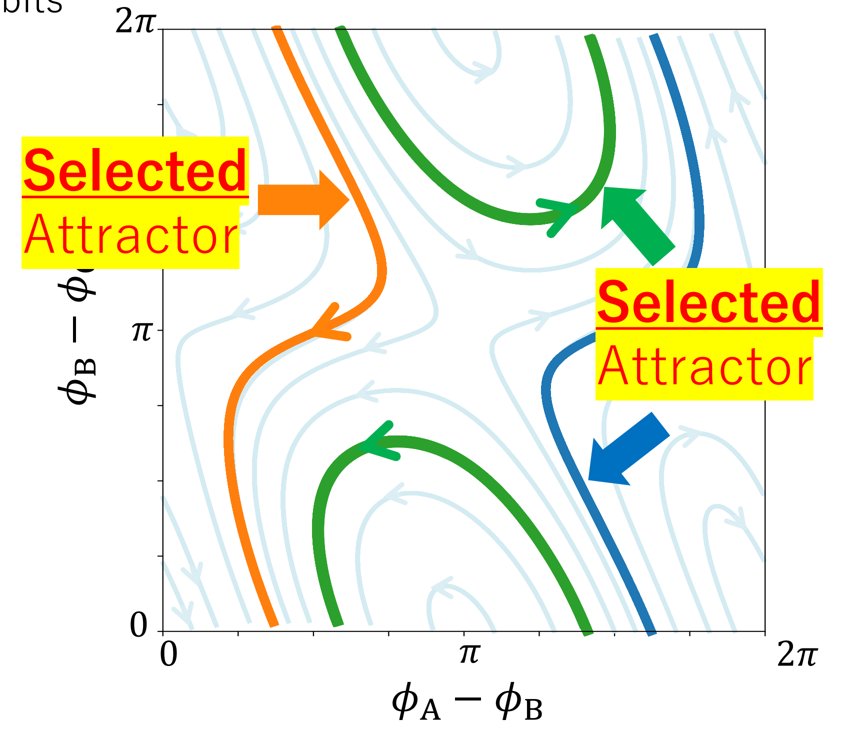
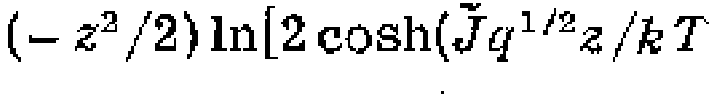
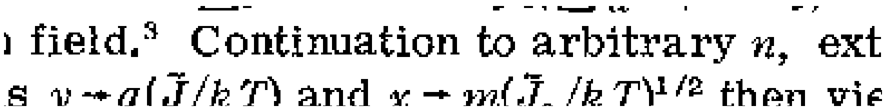


RH, arXiv:2208.08577

# Spin glass



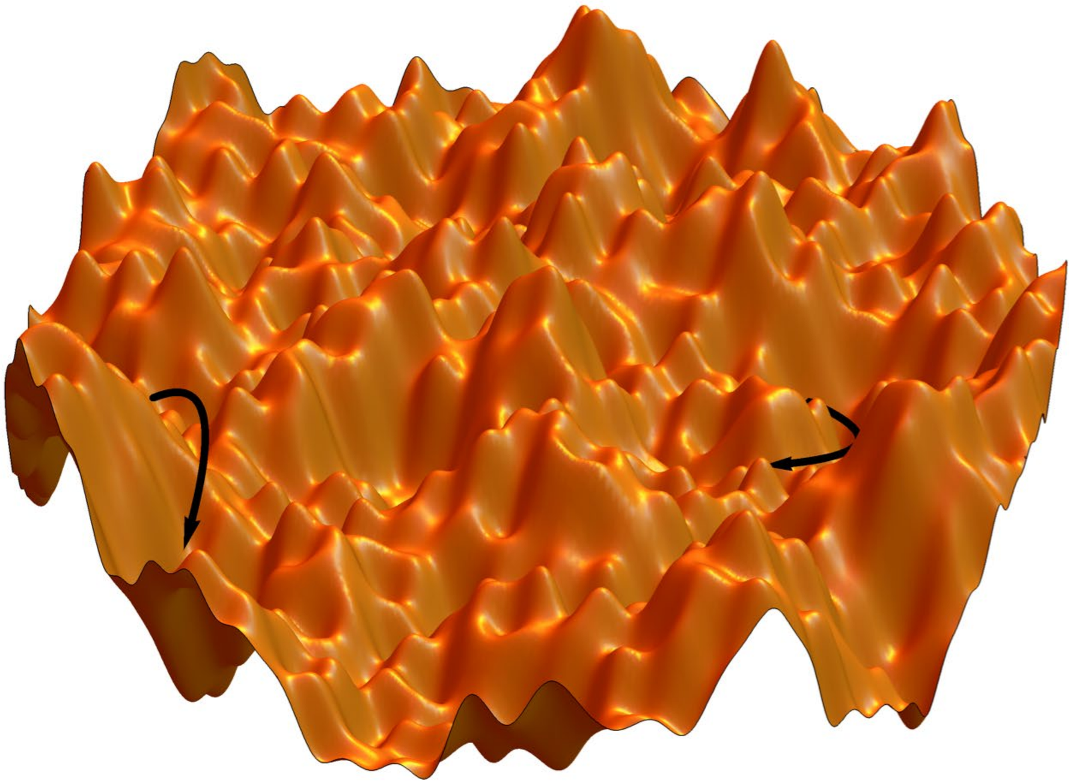
$(-z^2/2)\ln[2\cosh(\tilde{J}q^{1/2}z/kT$



field.$^3$ Continuation to arbitrary $n$, ext

s $y \to q(\tilde{J}/kT)$ and $x \to m(\tilde{J}_0/kT)^{1/2}$ then vie

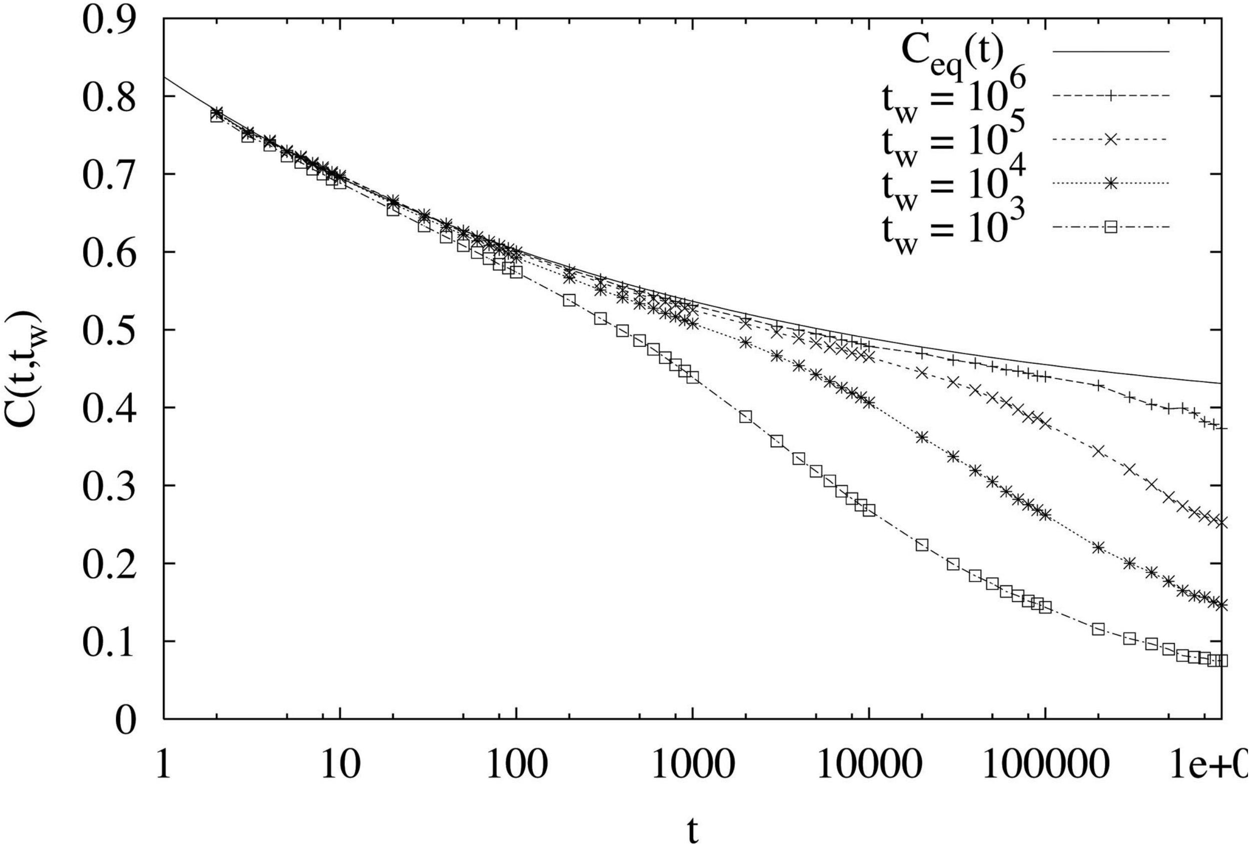
Sherrington and Kirkpatrick PRL1975



Extremely slow dynamics with no long ranged order

# Ageing phenomena

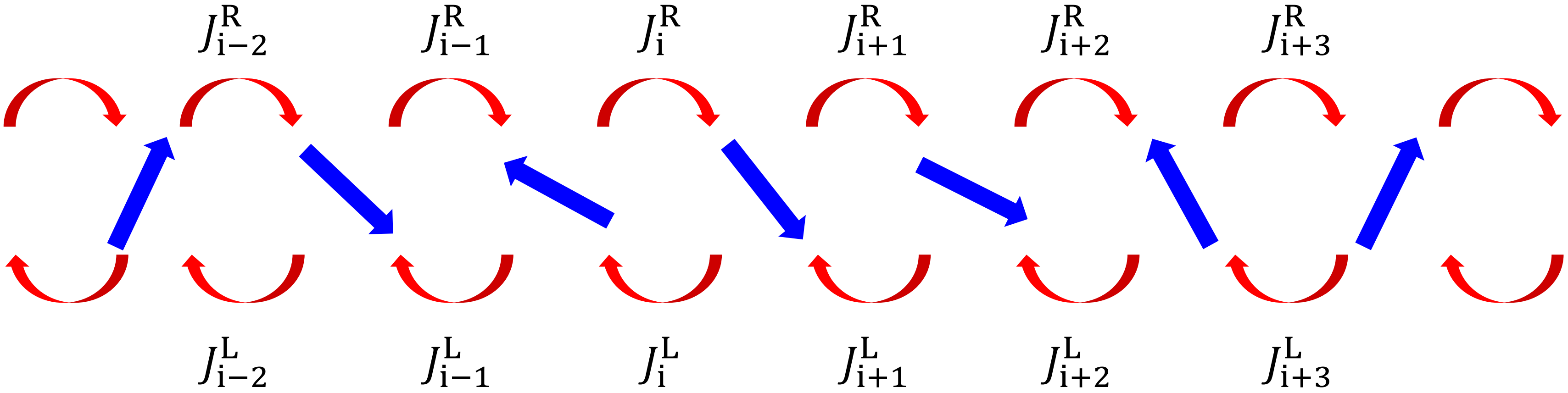
$$C(t, t_{\mathrm{w}}) = \frac{1}{N}\sum_{i=1}^{N}\langle \sigma_i(t_{\mathrm{w}})\sigma_i(t_{\mathrm{w}}+t)\rangle = q(t_{\mathrm{w}}, t_{\mathrm{w}}+t)$$



Parisi PNAS 2006 (Review)

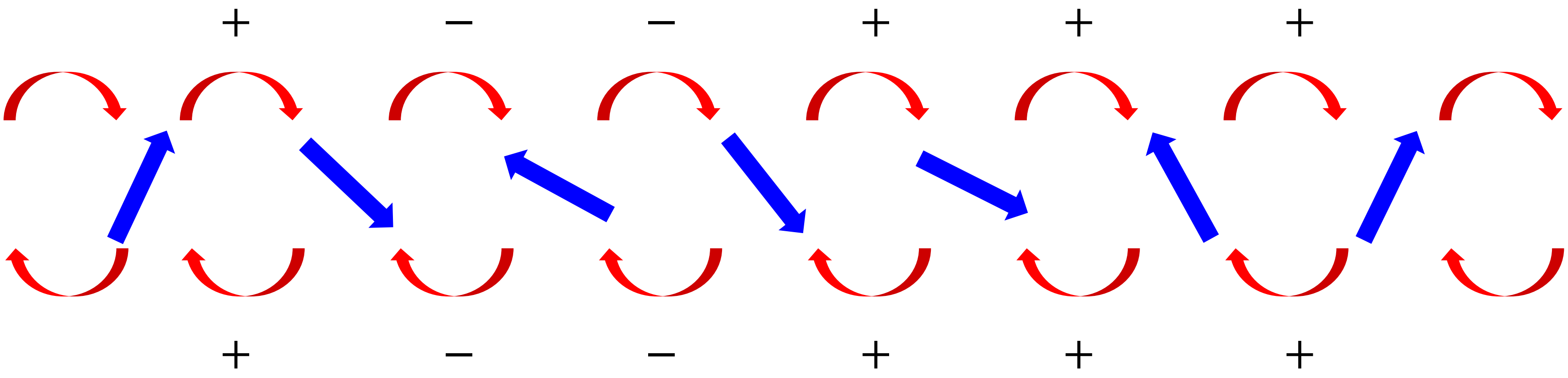
Q: Can **non-reciprocal frustrations** also give rise to these **glassy** dynamics?

# One dimensional random spin chain



$$p(J_i^{\mathrm{L/R}}) \propto \begin{cases} e^{-(J_i^{\mathrm{L/R}})^2/(2\sigma_J^2)} & |J_i^{\mathrm{L/R}}| \geq J_c \\ 0 & |J_i^{\mathrm{L/R}}| < J_c \end{cases}$$

# Reciprocal case $J_{ij} = J_{ji}$



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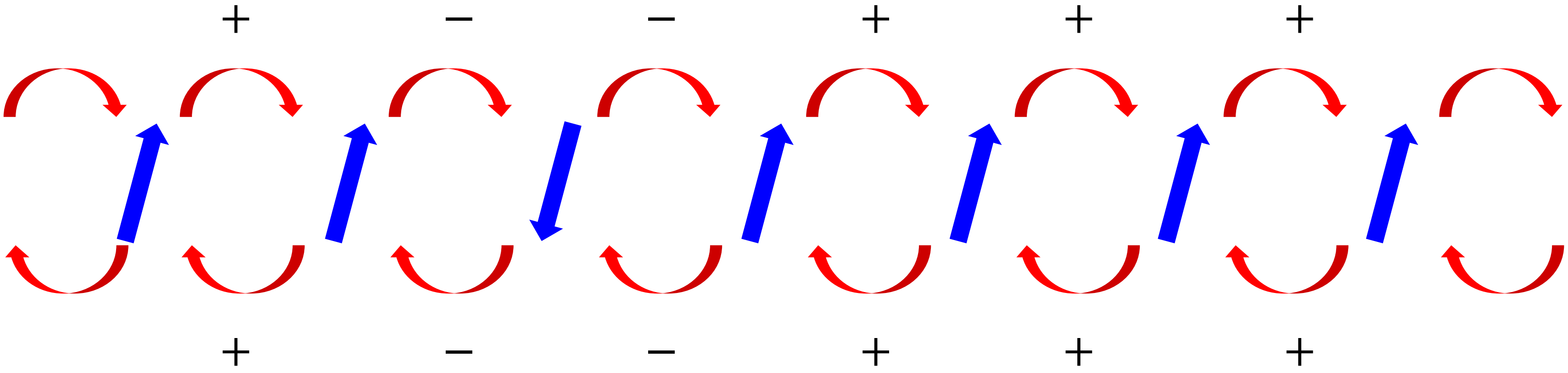


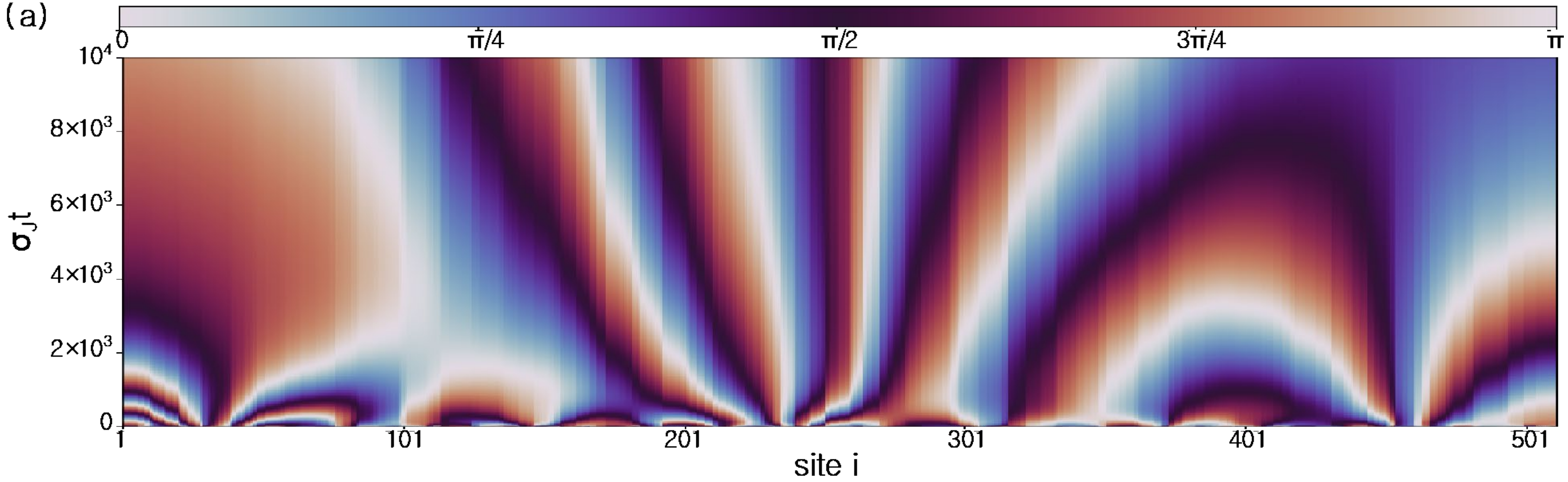
**Nematic order develops**

$$p(J_i^{\mathrm{L/R}}) \propto \begin{cases} e^{-(J_i^{\mathrm{L/R}})^2/(2\sigma_J^2)} & |J_i^{\mathrm{L/R}}| \geq J_c \\ 0 & |J_i^{\mathrm{L/R}}| < J_c \end{cases}$$

Reciprocal limit of this model = *no* geometrical frustrations

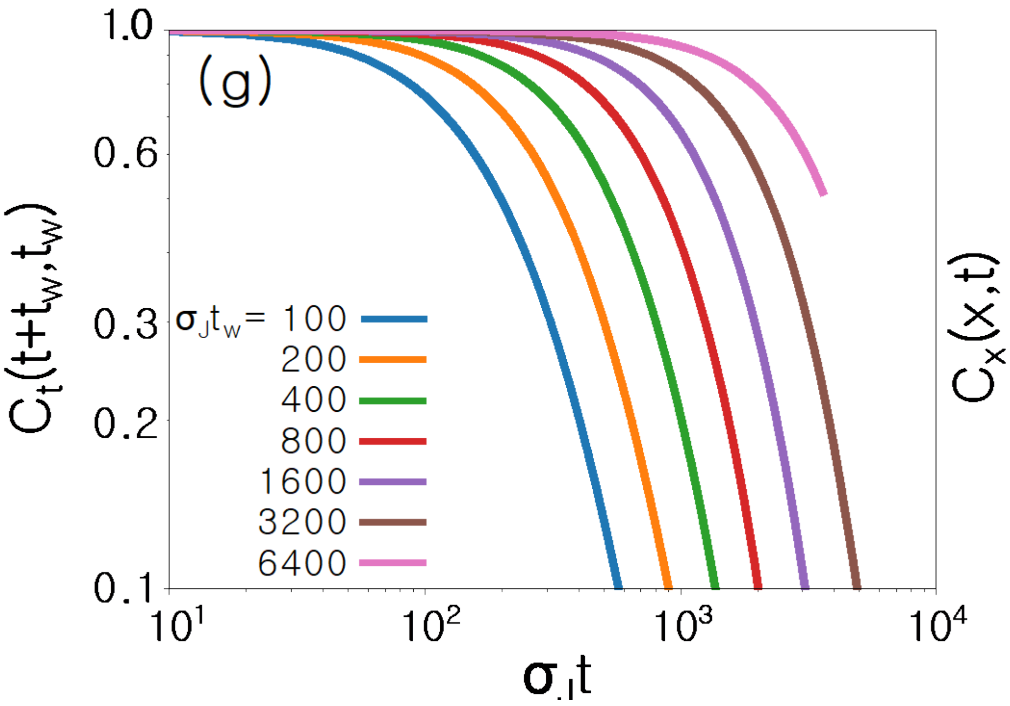
# Reciprocal case $J_{ij} = J_{ji}$
# =Domain wall annihilation dynamics

$\varphi_i = \theta_i \pmod{\pi}$



# Reciprocal case $J_{ij} = J_{ji}$
# =Domain wall annihilation dynamics

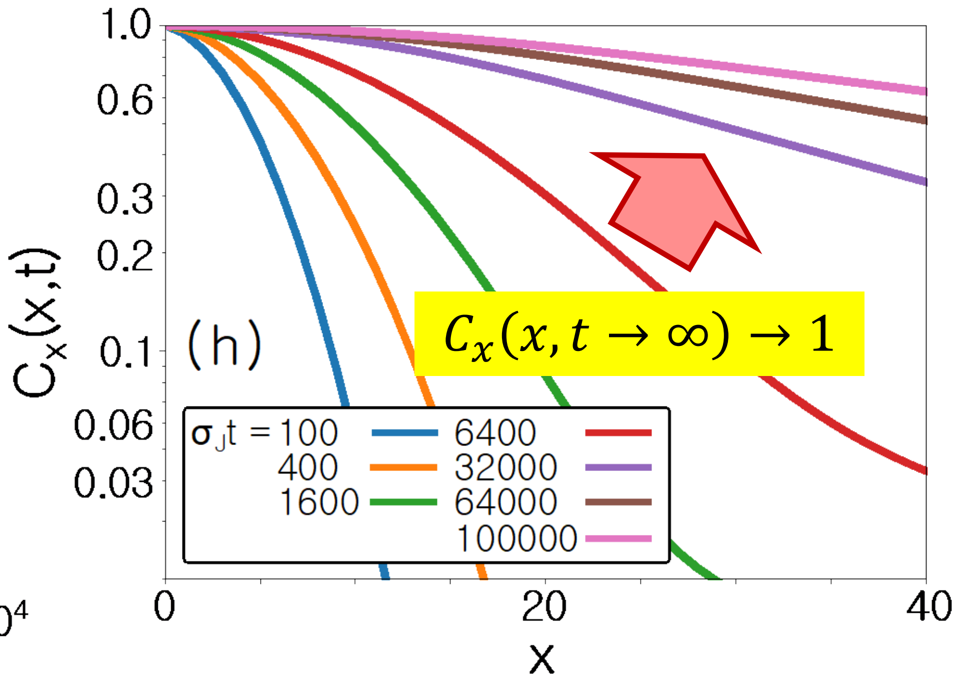
**Time** correlation function



$C_t(t_w + t, t_w) = \left|(1/N)\sum_{i=1}^{N} \overline{\delta\psi_{2,i}(t_w + t)\delta\psi^*_{2,i}(t_w)}\right|$,

where $e^{2i\theta_i(t)} - \psi_2(t)$ and $_2 = (1/N)' \ ^{N}_{i=1} e^{2i\checkmark_i}$

**Spatial** correlation function



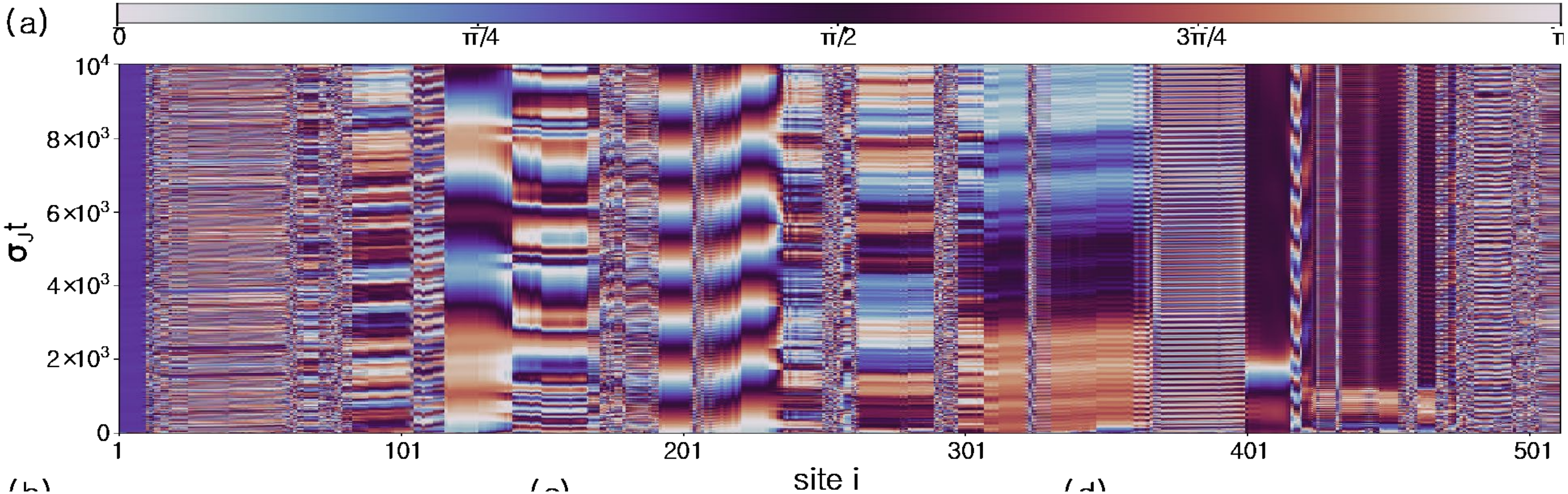
$C_x(x, t \to \infty) \to 1$

$C_x(x,t) = \left|(1/(N-x))\sum_{j=1}^{N-x} \overline{\psi_{2,j+x}(t)\psi^*_{2,j}(t)}\right|$

**Slow** dynamics towards **long-ranged** nematically ordered phase

# Non-reciprocal case ($J_{ij}$and $J_{ji}$ independent) = Periodic and chaotic domain dynamics

$\varphi_i = \theta_i \ (\mathrm{mod}\ \pi)$



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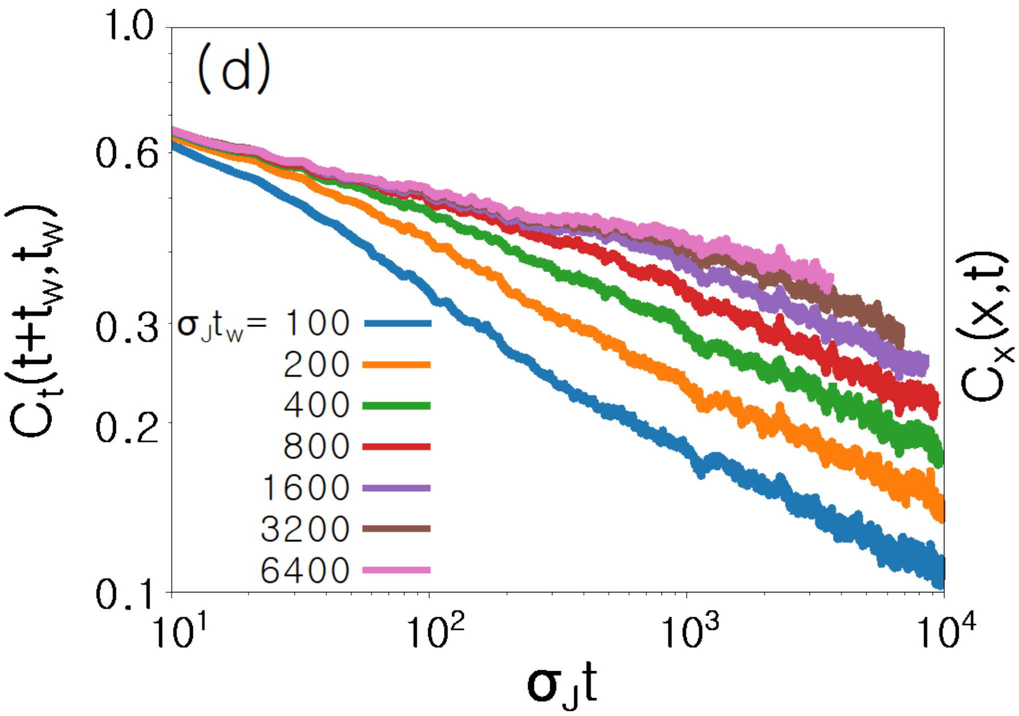
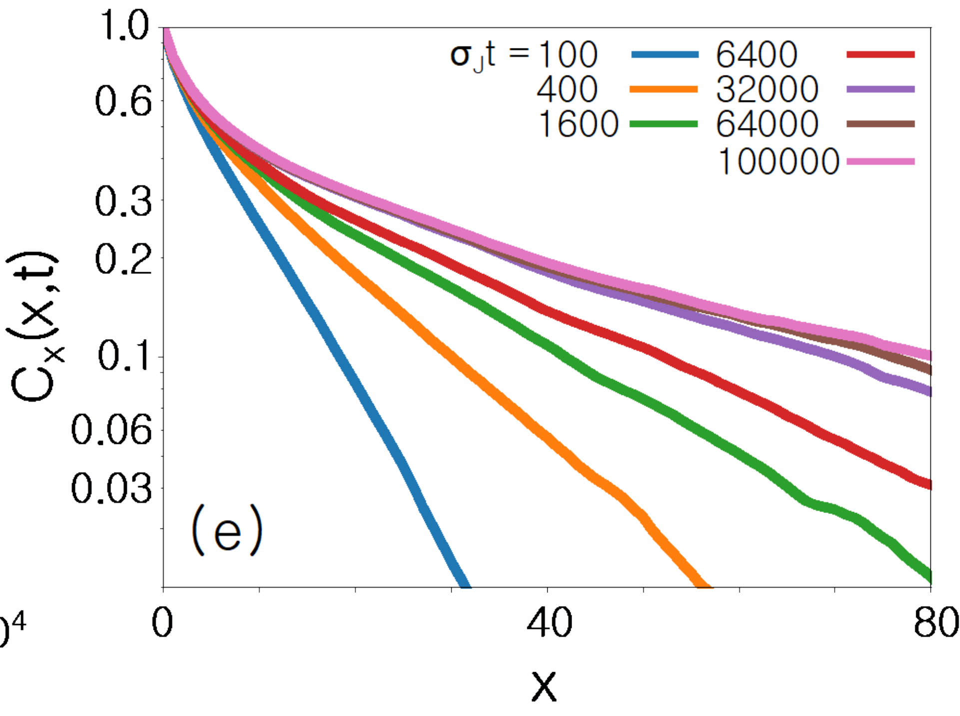


**Spatial** correlation function



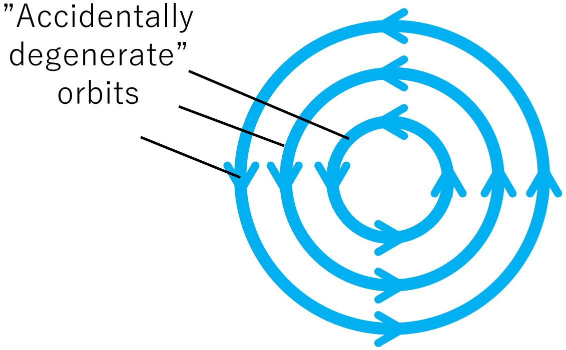
**Power law** time correlation with **ageing** + ***short*-range** spatial correlations

Reminiscent of spin glasses!

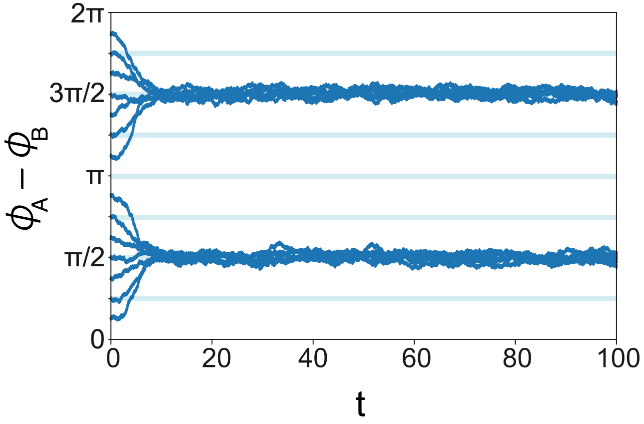
# Summary

- Pointed out a direct analogy between geometrical and non-reciprocal frustration

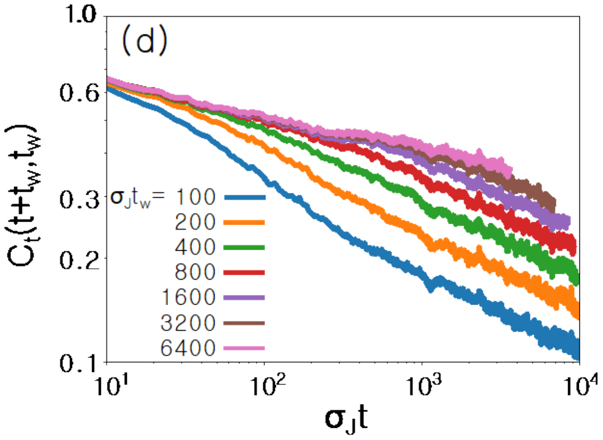
”Accidental degeneracy” of orbits



Order-by-disorder



Spin-glass-like state



RH, arXiv:2208.08577

# “Accidental degeneracy” of *orbits*

**[Proof]**

Continuity equation: $\frac{\partial\rho}{\partial t} = -\sum_i \frac{\partial\left(\rho\dot{\theta}_i\right)}{\partial\theta_i} = -\sum_i \left[\frac{\partial\rho}{\partial\theta_i}\dot{\theta}_i + \xcancel{\rho\frac{\partial\dot{\theta}_i}{\partial\theta_i}}\right]$

$$\sum_i \frac{\partial\dot{\theta}_i}{\partial\theta_i} \underset{\uparrow}{=} \sum_{ij}\left[J_{ij}\cos\left(\theta_j - \theta_i\right)\right] \underset{\uparrow}{=} 0$$

$$J_{ij} = -J_{ji}$$

$$\dot{\theta}_\mathrm{i} = \sum_j J_{ij}\sin(\theta_j - \theta_i)$$

Therefore, $\frac{\partial\rho}{\partial t} + \sum_i \frac{\partial\rho}{\partial\theta_i}\dot{\theta}_i = 0.$

■

RH, arXiv:2208.08577

# Orbit-dependent fluctuations

**Renormalized macroscopic spin dynamics**

$$\dot{\phi}_a = \sum_b \boldsymbol{j^\star_{ab}(\phi)} \sin(\phi_b - \phi_a) + \bar{\eta}_a$$

Noise strength $\sim\sigma/N_a$

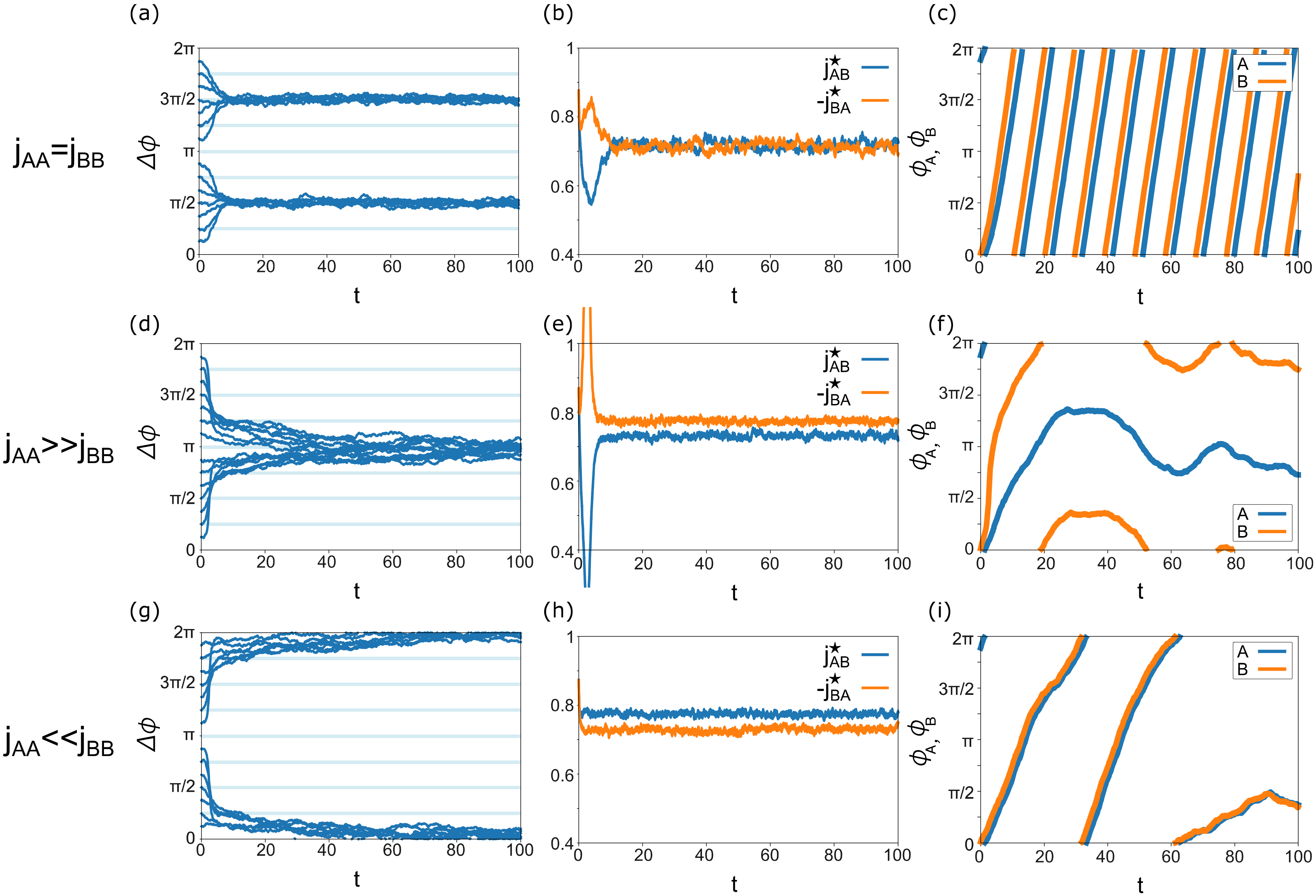
**Orbit-dependent** renormalized coupling

$$\boldsymbol{j^\star_{ab}(\phi)} = j_{ab}\frac{r_b(\phi)}{r_a(\phi)}\langle\cos^2\delta\theta_i^a\rangle_\phi$$

with $\psi_a = r_a e^{i\phi_a} = \frac{1}{N_a}\sum_{i=1}^{N_a} e^{i\theta_i^a}$ and $\langle\cdots\rangle_\phi = \int d\delta\theta_i^a \rho_i^a(\delta\theta_i^a;\phi(t))$

Stabilizes certain orbits=**"Orbit selection"** takes place!

RH, arXiv:2208.08577



RH, arXiv:2208.08577

# Three communities with natural frequency disorder

**Kuramoto model** $(\alpha = \mathrm{A}, \mathrm{B}, \mathrm{C})$

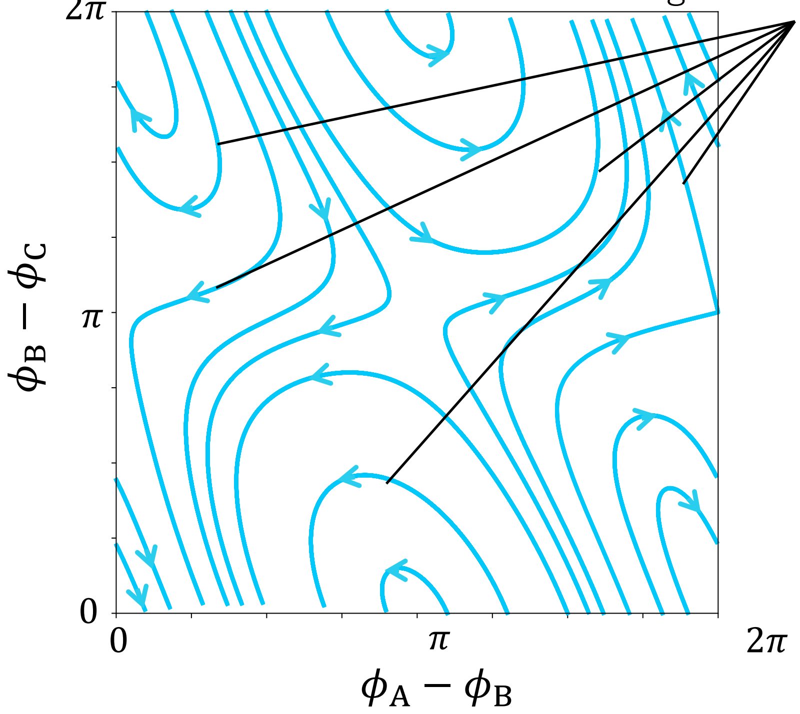
$$\dot{\theta}_i^\alpha = \omega_i^\alpha + \sum_{\beta=\mathrm{A,B,C}} \sum_{j=1}^{N_\beta} J_{\alpha\beta} \sin\left(\theta_j^\beta - \theta_i^\alpha\right)$$

Order parameter

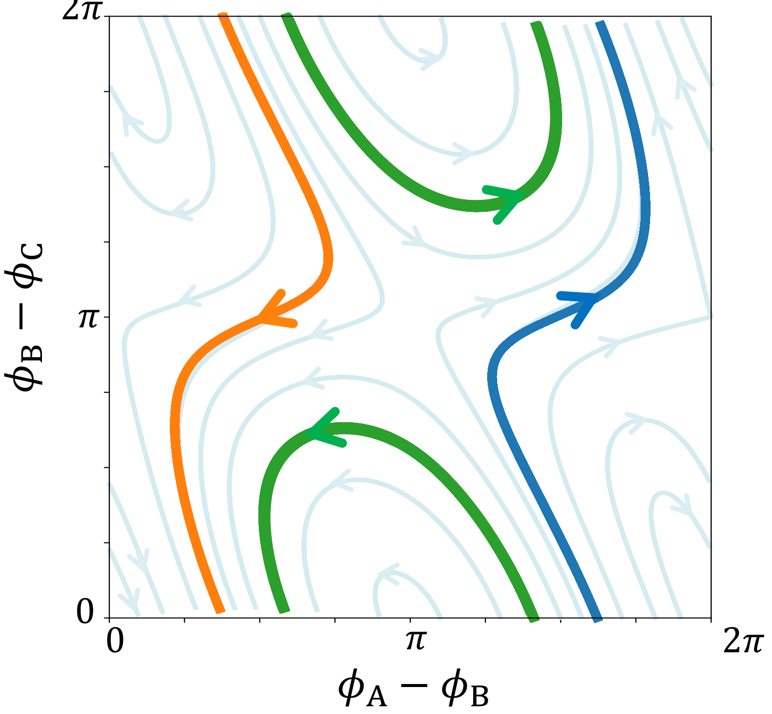
$$z_\alpha = R_\alpha e^{i\phi_\alpha} = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} e^{i\theta_\alpha}$$

Clean system

"Accidentally degenerate" orbits



Random natural frequency



RH, arXiv:2208.08577

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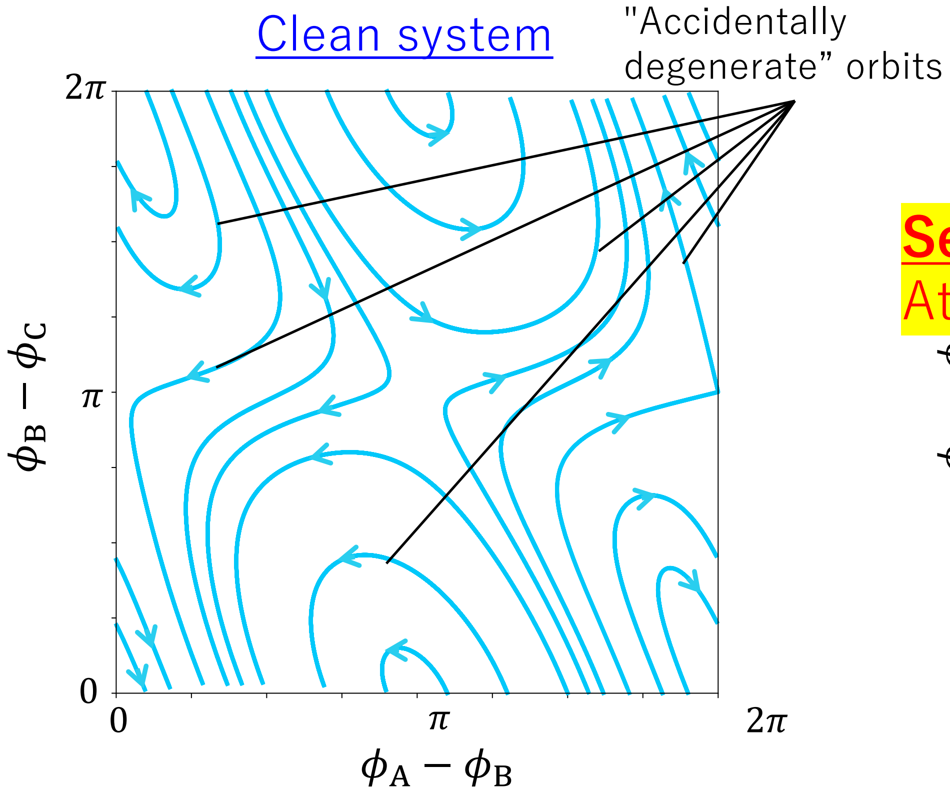
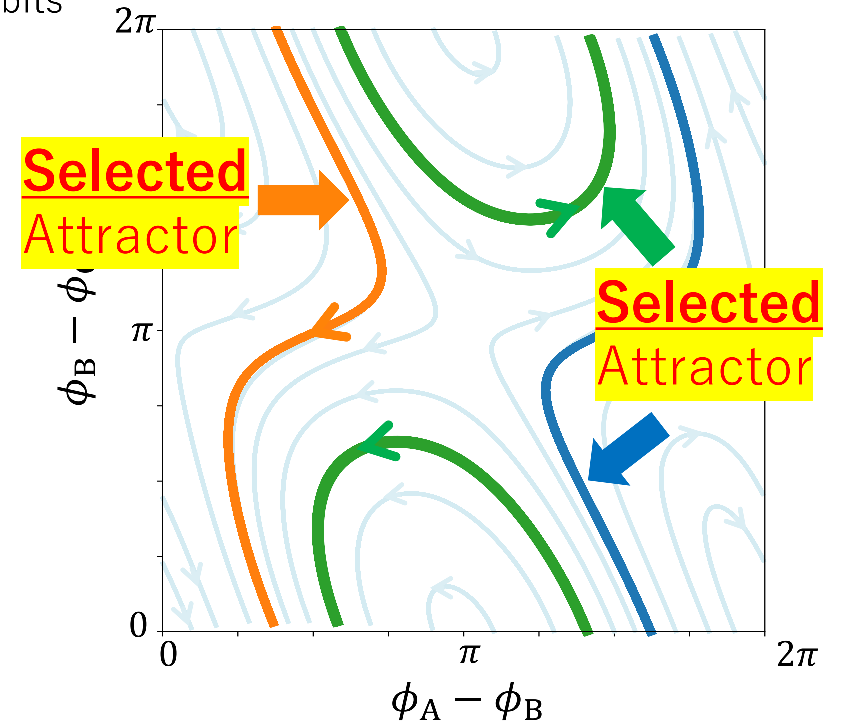
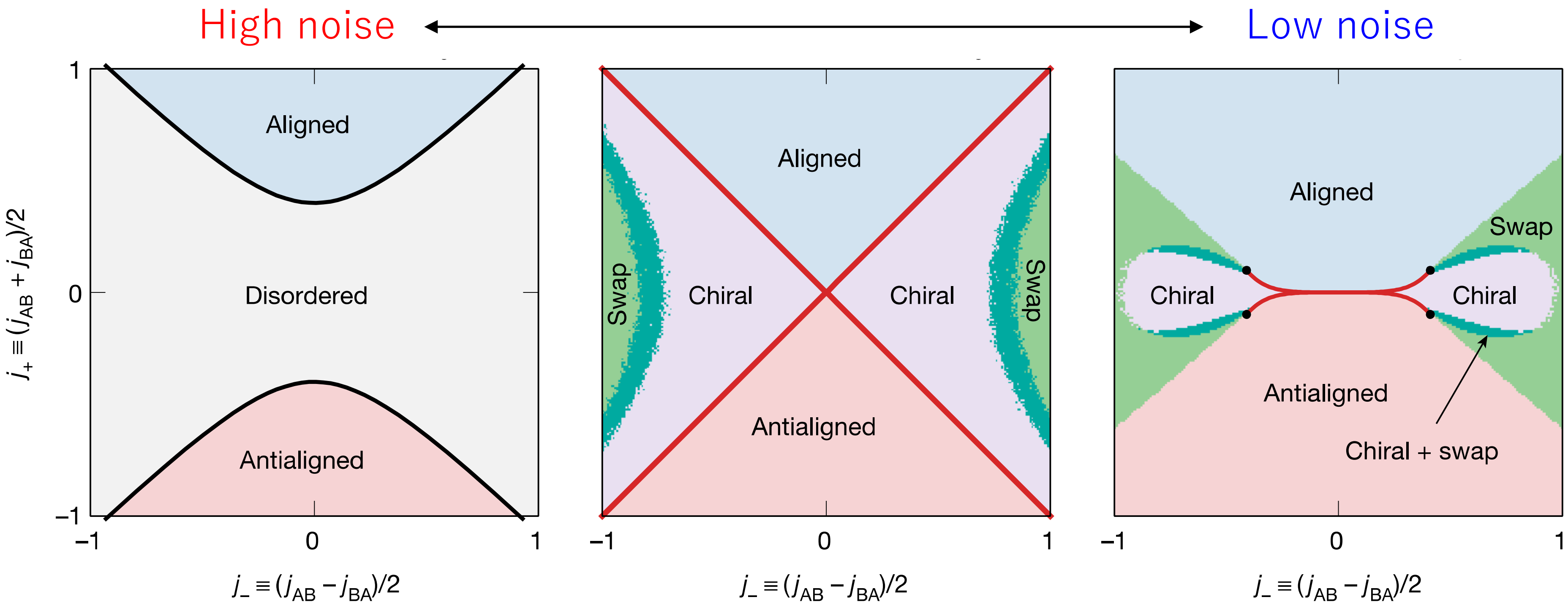


Random natural frequency



RH, arXiv:2208.08577

# Hint: non-reciprocal flocking model

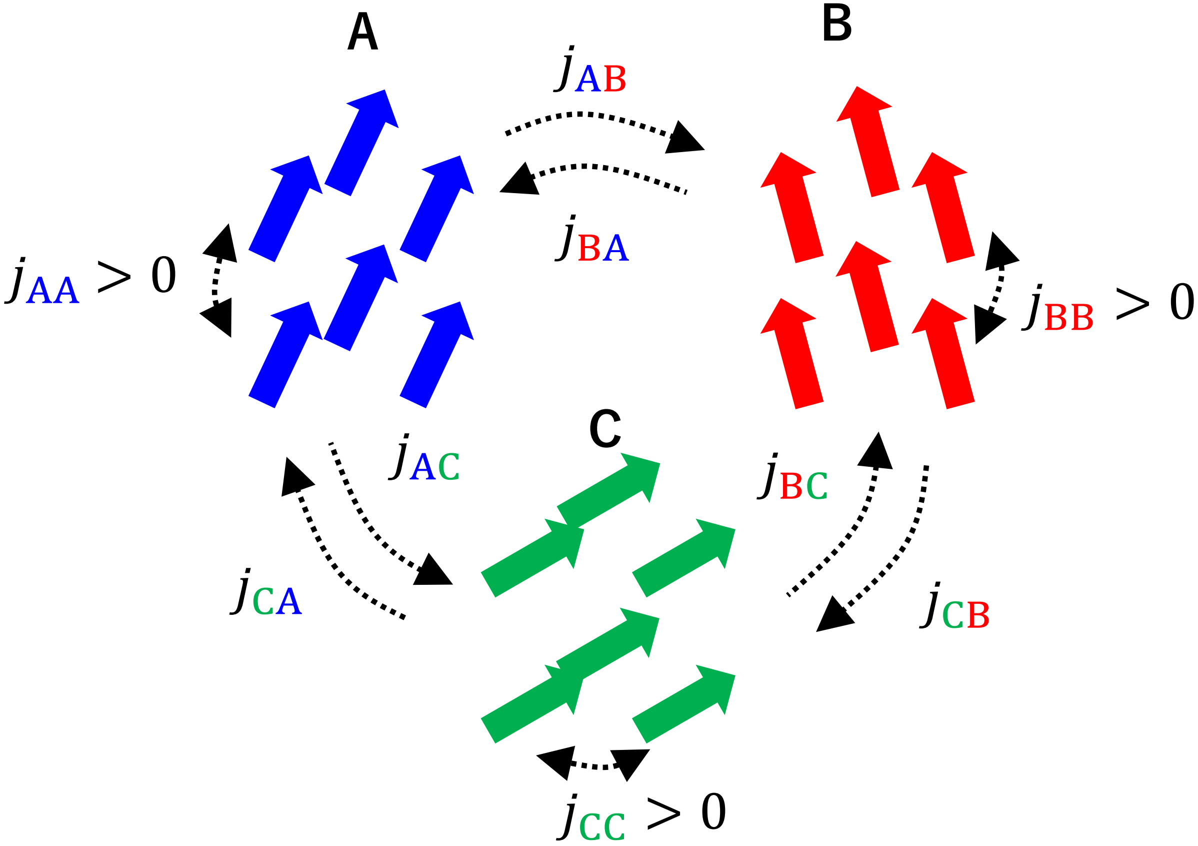


Chiral phase being *enhanced* by noise!

M. Fruchart*, RH*****, P. B. Littlewood, and V. Vitelli, Nature **592**, 363 (2021).

# 無秩序による時間結晶秩序

$$\dot{\theta}_i^a = \sum_b \sum_{i=1}^{N_b} \frac{j_{ab}}{N_b} \sin\left(\theta_j^b - \theta_i^a\right)$$



RH, arXiv:2208.08577

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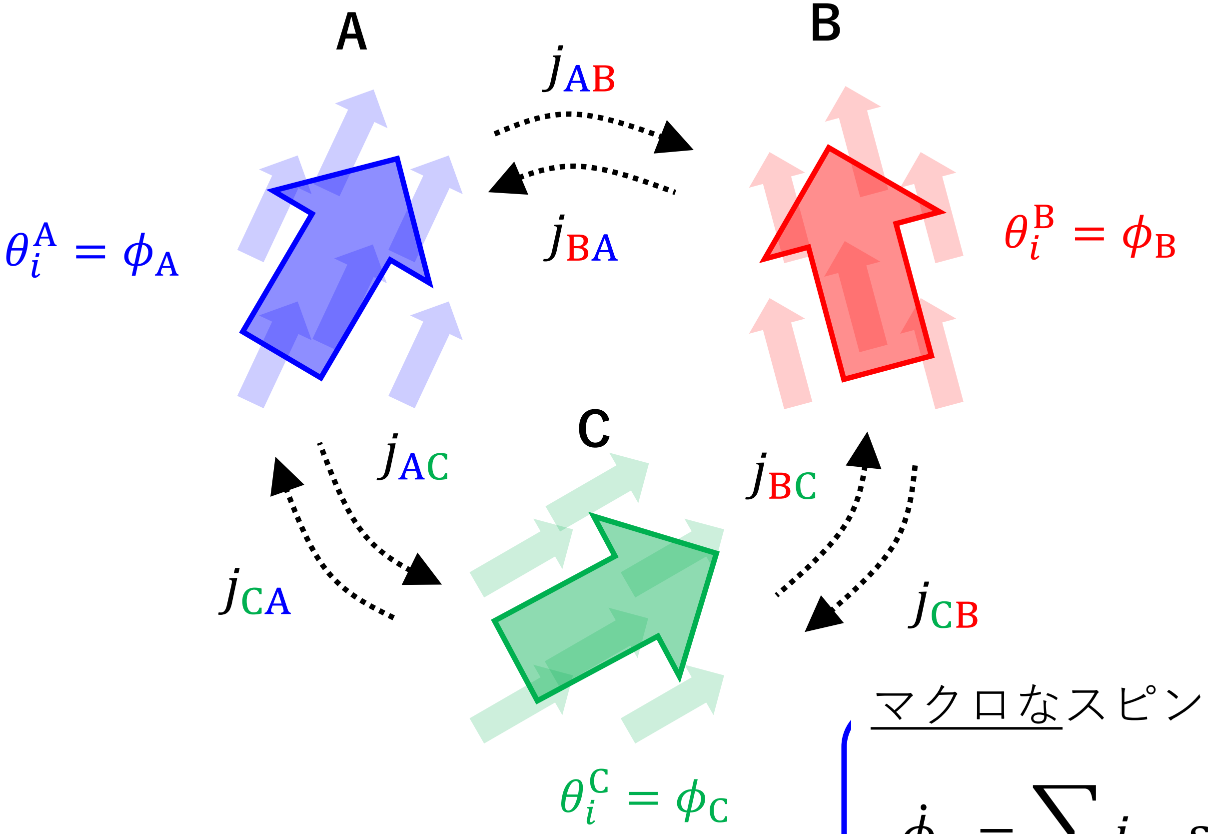


マクロなスピンのダイナミクス

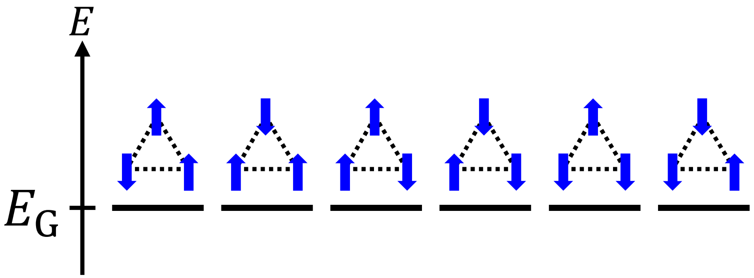
$$\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a)$$

RH, arXiv:2208.08577

# 幾何学的 vs 非相反
# フラストレーション

## 幾何学的
## フラストレーション

基底状態の偶然縮退



## 非相反
## フラストレーション

?

静止した状態に収束
するとは限らない…

そもそもエネルギーを
定義できない…

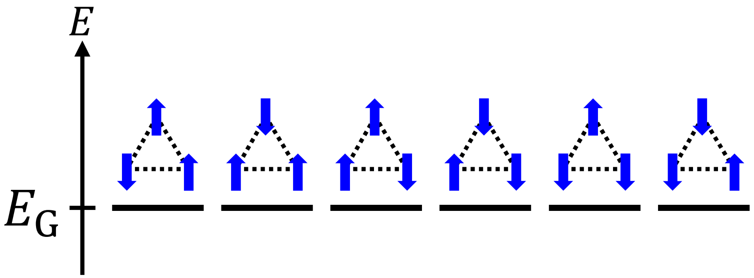
# 幾何学的 vs 非相反 フラストレーション

## 幾何学的 フラストレーション

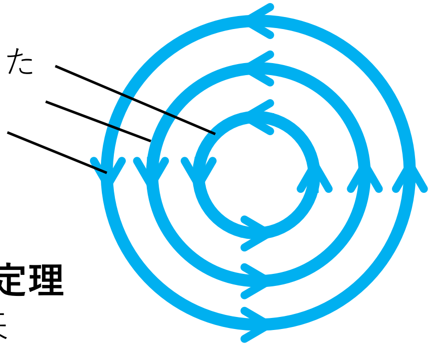
## 非相反 フラストレーション

基底状態の偶然縮退

（力学系の意味での）軌道の「偶然縮退」



「偶然縮退」した 軌道



$J_{ij} = -J_{ji}$のとき

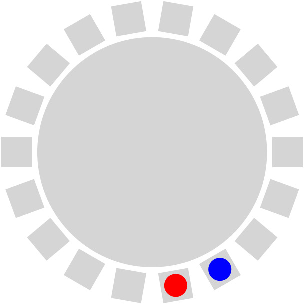
**リウビル型の定理**
成立に由来

- ➢ 無秩序による秩序や、スピンガラスの動的対応物が出現

RH, arXiv:2208.08577

# Non-reciprocity induced frustration
# + noise + many-body interaction
# = *Time-periodic ordered phase*

No noise, two agents

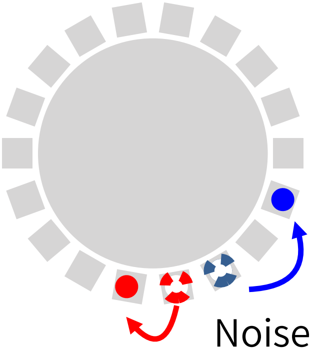


$\Delta\theta \to 0$

M. Fruchart*, RH*, P. B. Littlewood, and V. Vitelli, Nature **592**, 363 (2021).

# Non-reciprocity induced frustration + **noise** + **many-body interaction** = *Time-periodic ordered phase*

With noise, two agents

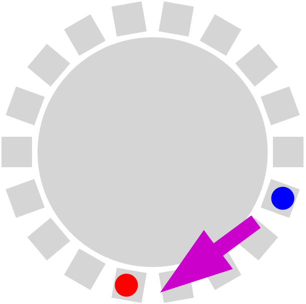


Noise perturbs the state

M. Fruchart*, RH*, P. B. Littlewood, and V. Vitelli, Nature **592**, 363 (2021).

# Non-reciprocity induced frustration + noise + many-body interaction = *Time-periodic ordered phase*

With noise, two agents



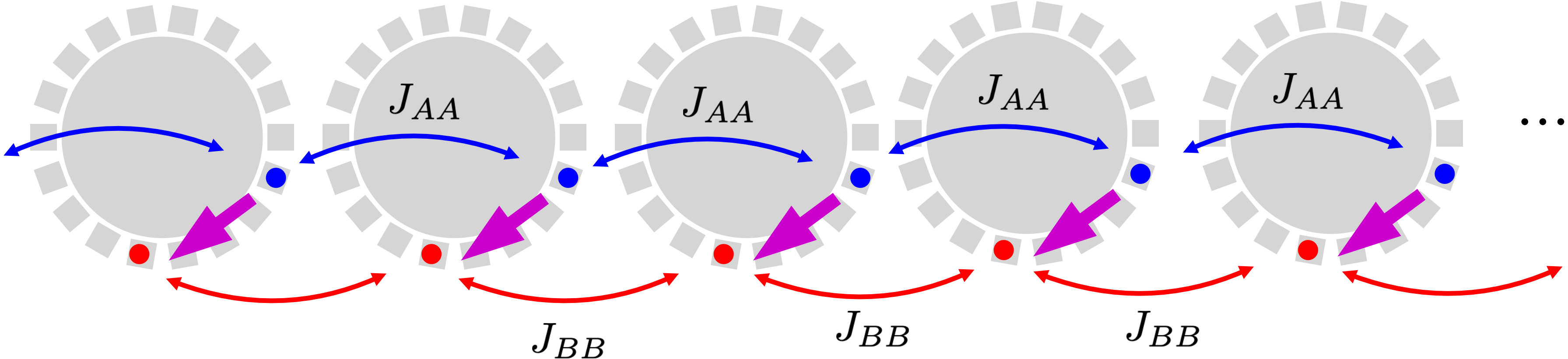
Chase-and-runaway

Noise restarts the chase-and-runaway motion

M. Fruchart*, RH*, P. B. Littlewood, and V. Vitelli, Nature **592**, 363 (2021).

# Non-reciprocity induced frustration + noise + many-body interaction = *Time-periodic ordered phase*

With noise, many agents



Noise constantly (randomly) kicks the state out of the fixed point and restarts the chase-and-runaway motion

Many-body interaction gives rise to macroscopic correlation to **stabilize the collective motion** = chiral phase

Noise-activated symmetry breaking reminiscent of order-by-disorder transition known in frustrated systems

J. Villian, et al., J. de Phys. 41, 1263 (1980).

M. Fruchart*, RH*, P. B. Littlewood, and V. Vitelli, Nature **592**, 363 (2021).