Non-Bloch PT symmetry of non-Hermitan systems

Zhong Wang

Institute for Advanced Study, Tsinghua University

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Outline

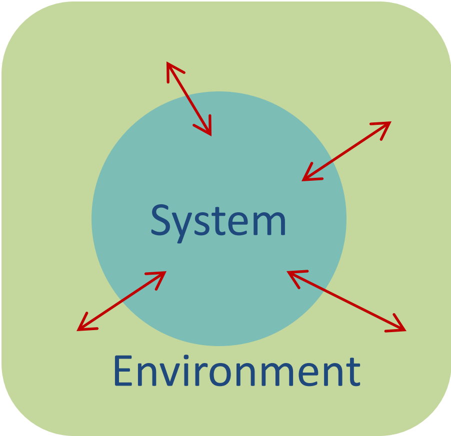
- Non-Hermitian physics and PT symmetry
- Non-Bloch band theory
- Non-Bloch parity-time symmetry: 1D and 2D

Hermiticity vs non-Hermiticity

$$i\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$$

$H^\dagger = H$ for closed systems

$H^\dagger \neq H$ for open systems



Many ways to non-Hermiticity

252 *Y. Ashida et al.*

Table 1. A wide variety of classical and quantum systems described by non-Hermitian matrices/operators, the physical origins of non-Hermiticity, and theoretical methods presented in order of appearance in the present review.

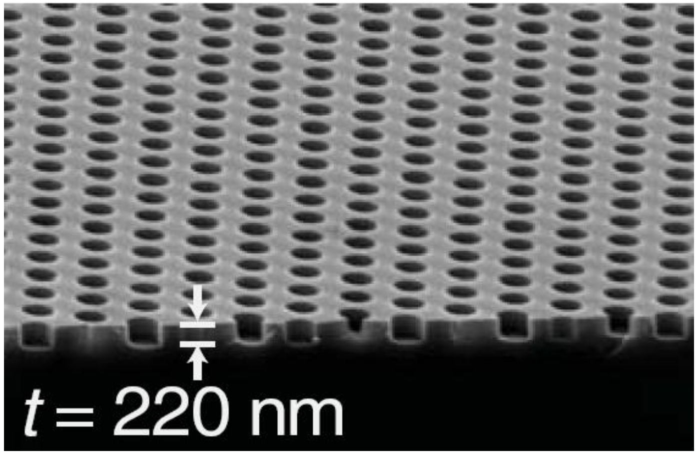
Systems/processes	Physical origin of non-Hermiticity	Theoretical methods
Photonics	Gain and loss of photons	Maxwell equations [12,13]
Mechanics	Friction	Newton equation [14–16]
Electrical circuits	Joule heating	Circuit equation [17]
Stochastic processes	Nonreciprocity of state transitions	Fokker-Planck equation [18,19]
Soft matter and fluid	Nonlinear instability	Linearized hydrodynamics [20–22]
Nuclear reactions	Radioactive decays	Projection methods [4–6]
Mesoscopic systems	Finite lifetimes of resonances	Scattering theory [23,24]
Open quantum systems	Dissipation	Master equation [25,26]
Quantum measurement	Measurement backaction	Quantum trajectory approach [27–32]

Y. Ashida, Z. Gong, M. Ueda, Advances in Phys. 2020

Open classical-wave systems

Photonic open systems
(Loss by radiation)

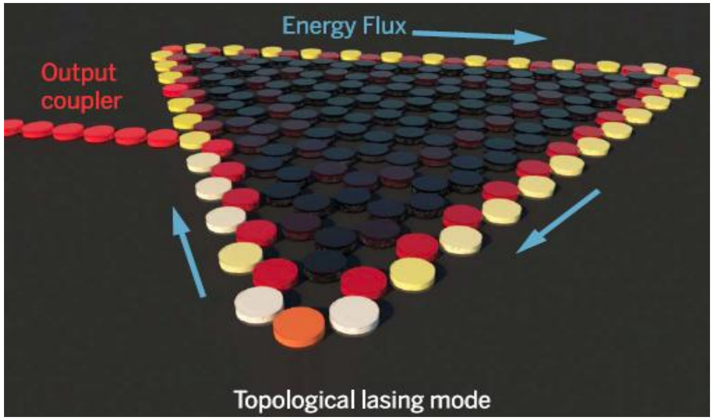
$$H_{\text{eff}} = \omega_{\text{D}} - i\gamma_0 + (v_g\delta k_x - i\gamma)\sigma_z + v_g\delta k_y\sigma_x$$



Zhou et al, Science 359, 1009 (2018)

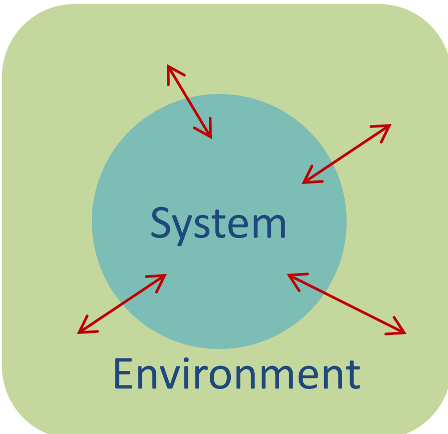
"Topological insulator laser"
Non-Hermitian Haldane model
(Gain by pumping)

$$i\frac{\partial\psi}{\partial t} = H_{\text{Haldane}}\psi - i\gamma\psi + \frac{ig\mathbb{P}}{1+|\psi|^2/I_{\text{sat}}}\psi + H_{\text{output}}\psi$$



Harari, et al, Science, 359, 1230(2018)

Open quantum systems: Density matrix



Quantum master equation:

$$\frac{d\rho(t)}{dt} = \mathcal{L}\rho = -i[H,\rho] + \sum_{\mu}\left(L_\mu \rho L_\mu^\dagger - \frac{1}{2}\{L_\mu^\dagger L_\mu, \rho\}\right)$$

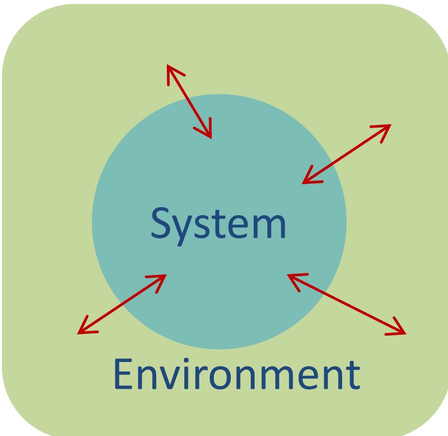
Environment effect

$$\rho = \sum_{m,n} \rho_{mn}|m\rangle\langle n| \Leftrightarrow \tilde{\rho} = \sum_{m,n} \rho_{mn}|m\rangle \otimes |n\rangle$$

$$\frac{d\tilde{\rho}(t)}{dt} = \tilde{\mathcal{L}}\tilde{\rho} \qquad \tilde{\mathcal{L}} = -iH \otimes I + iI \otimes H^T + \sum_{\mu}(L_\mu \otimes L_\mu^* - \frac{1}{2}L_\mu^\dagger L_\mu \otimes I - \frac{1}{2}I \otimes L_\mu^T L_\mu^*)$$

- Liouvillian is non-Hermitian: $i\tilde{\mathcal{L}} \neq (i\tilde{\mathcal{L}})^\dagger$
- Liouvillians as non-Hermitian "Hamiltonian"(density matrix as "wavefunction")
- Applications in quantum optics, cold atom systems…

Open quantum systems: Quantum trajectory



Quantum master equation:

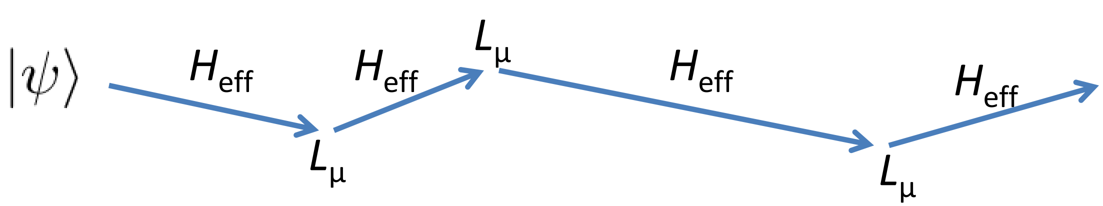
$$\frac{d\rho(t)}{dt} = \mathcal{L}\rho = -i[H,\rho] + \sum_{\mu}\left(L_{\mu}\rho L_{\mu}^{\dagger} - \frac{1}{2}\{L_{\mu}^{\dagger}L_{\mu},\rho\}\right)$$

$$H_{\text{eff}} = H - \frac{i}{2}\sum_{\mu} L_{\mu}^{\dagger}L_{\mu}$$

$$\frac{d\rho(t)}{dt} = -i(H_{\text{eff}}\rho - \rho H_{\text{eff}}^{\dagger}) + \sum_{\mu} L_{\mu}\rho L_{\mu}^{\dagger}$$

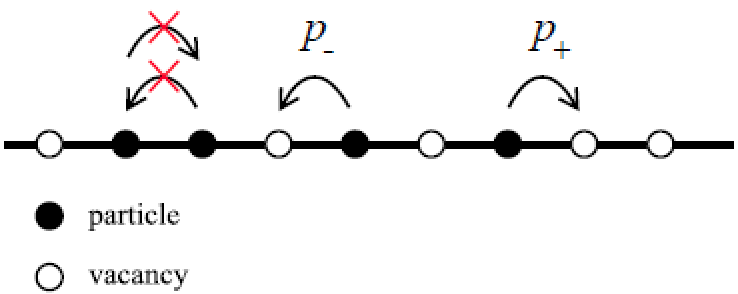
- H_{eff} + Quantum jumps
- Post-selection → H_{eff}

$$i\frac{d|\psi\rangle}{dt} = H_{\text{eff}}|\psi\rangle \Longleftrightarrow i\frac{d(|\psi\rangle\langle\psi|)}{dt} = H_{\text{eff}}|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|H_{\text{eff}}^{\dagger}$$



Non-Hermitian Hamiltonian from statistical mechanics

Asymmetric exclusion process (ASEP):



Probability distribution

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = W\mathbf{p}$$

$$W = \sum_{j=1}^{L} \left[p_+ \sigma_j^+ \sigma_{j+1}^- + p_- \sigma_j^- \sigma_{j+1}^+ + \frac{p_+ + p_-}{4} (\sigma_j^z \sigma_{j+1}^z - 1) \right]$$

For more examples, see Y. Ashida et al, Advances in Physics, 69, 249 (2020)

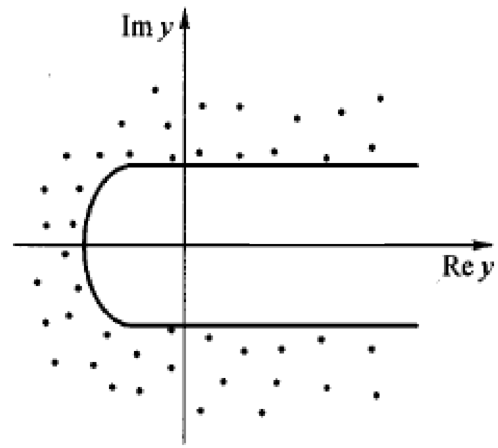
Non-Bloch PT symmetry of non-Hermitan systems

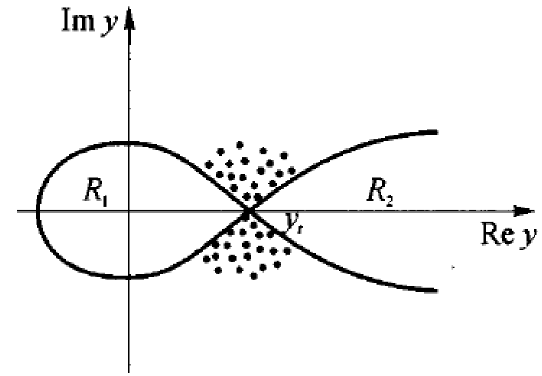
Recent concept

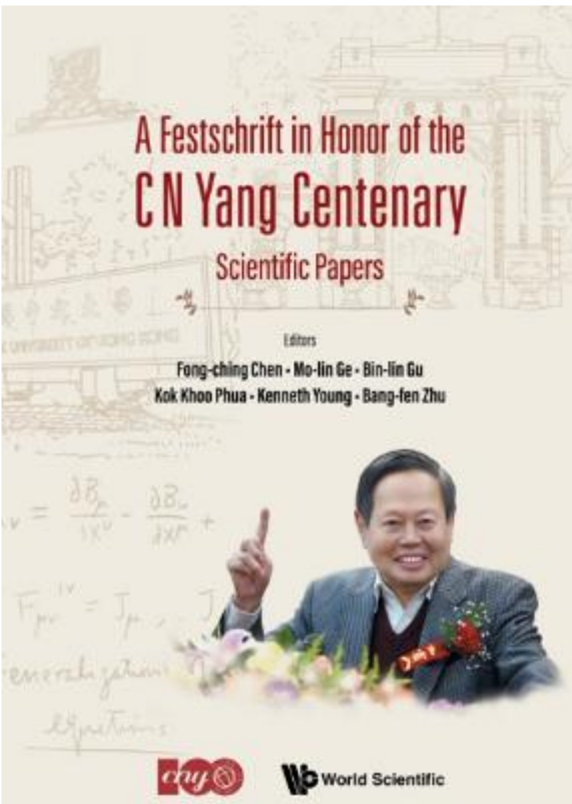
Bender-Boettcher 1998

Lee-Yang 1952

To understand phase transitions, Lee-Yang studied the behavior of partition function as a function of complex-valued chemical potential (or fugacity).







PHYSICAL REVIEW VOLUME 87, NUMBER 3 AUGUST 1, 1952

Statistical Theory of Equations of State and Phase Transitions. I. Theory of Condensation

C. N. YANG AND T. D. LEE
Institute for Advanced Study, Princeton, New Jersey
(Received March 31, 1952)

PHYSICAL REVIEW VOLUME 87, NUMBER 3 AUGUST 1, 1952

Statistical Theory of Equations of State and Phase Transitions. II. Lattice Gas and Ising Model

T. D. LEE AND C. N. YANG
Institute for Advanced Study, Princeton, New Jersey
(Received March 31, 1952)

Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry

Carl M. Bender[1] and Stefan Boettcher[2,3]
[1]Department of Physics, Washington University, St. Louis, Missouri 63130
[2]Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
[3]CTSPS, Clark Atlanta University, Atlanta, Georgia 30314
(Received 1 December 1997; revised manuscript received 9 April 1998)

The condition of self-adjointness ensures that the eigenvalues of a Hamiltonian are real and bounded below. Replacing this condition by the weaker condition of $\mathcal{PT}$ symmetry, one obtains new infinite classes of complex Hamiltonians whose spectra are also real and positive. These $\mathcal{PT}$ symmetric theories may be viewed as analytic continuations of conventional theories from real to complex phase space. This paper describes the unusual classical and quantum properties of these theories. [S0031-9007(98)06371-6]

[1] D. Bessis (private communication). This problem originated from discussions between Bessis and J. Zinn-Justin, who was studying Lee-Yang singularities using renormalization group methods. An $i\phi^3$ field theory arises if one translates the field in a ϕ^4 theory by an imaginary term.

Several years ago, Bessis conjectured on the basis of numerical studies that the spectrum of the Hamiltonian $H = p^2 + x^2 + ix^3$ is *real and positive* [1]. To date there is no rigorous proof of this conjecture. We claim that the reality of the spectrum of H is due to $\mathcal{PT}$ symmetry. Note that H is invariant *neither* under parity

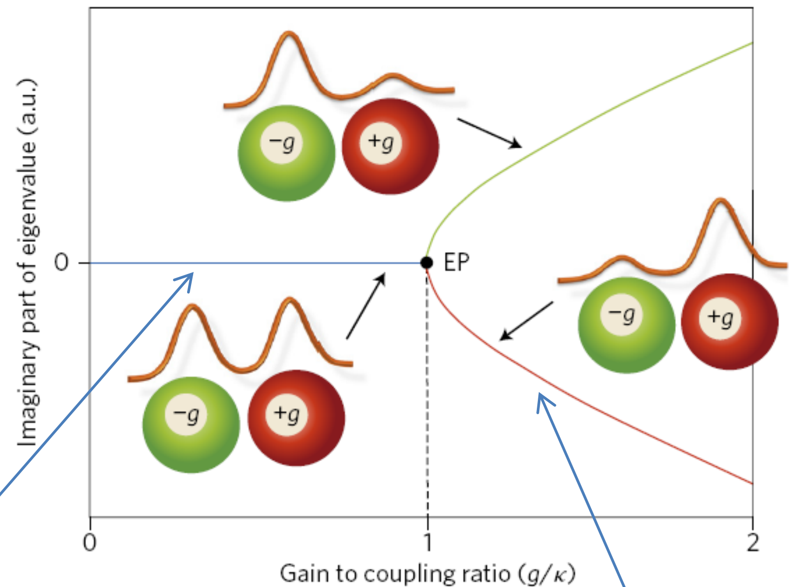
$H = p^2 + m^2x^2 - (ix)^N$ invariant under PT: $p \to -p$, $x \to -x$, and $i \to -i$.

PT symmetry and its breaking: a simplest model

$$H = \begin{pmatrix} -ig & \kappa \\ \kappa & ig \end{pmatrix}$$

PT symmetry: $\sigma_x H^* \sigma_x = H$

$$E_\pm = \pm\sqrt{\kappa^2 - g^2}$$



E real-valued (PT-exact phase)

$g < \kappa$

E complex-valued (PT-broken phase)

$g > \kappa$

A PT-symmetry Hamiltonian has two phases: PT-exact and PT-broken phases.

P (parity) is inessential; T (anti-unitary operation) is essential

Generalized PT symmetry and real spectra

Carl M Bender[1], **M V Berry**[2] **and Aikaterini Mandilara**[1]

[1] Department of Physics, Washington University, St Louis, MO 63130, USA
[2] H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK

E-mail: cmb@howdy.wustl.edu and amandula@hbar.wustl.edu

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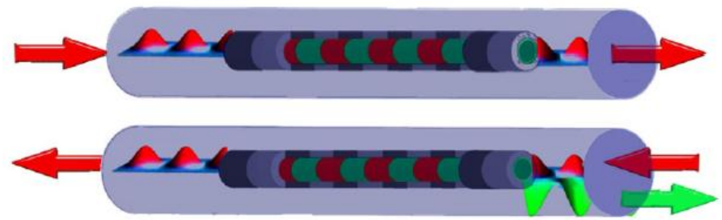
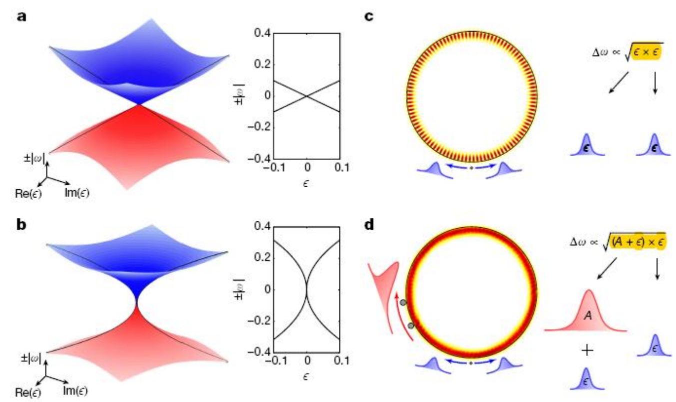
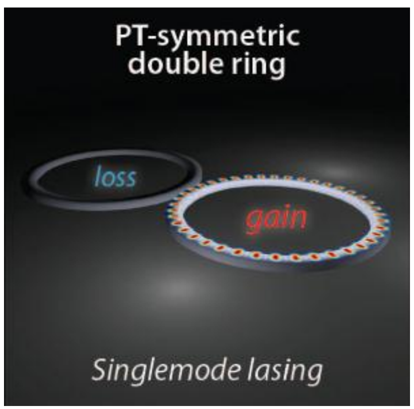
Abstract
The fact that eigenvalues of **PT**-symmetric Hamiltonians **H** can be real for some values of a parameter and complex for others is explained by showing that the matrix elements of **H**, and hence the secular equation, are real, not only for **PT** but also for any antiunitary operator **A** satisfying $\mathbf{A}^{2k} = 1$ with k odd. The argument is illustrated by a 2×2 matrix Hamiltonian, and two examples of the generalization are given.

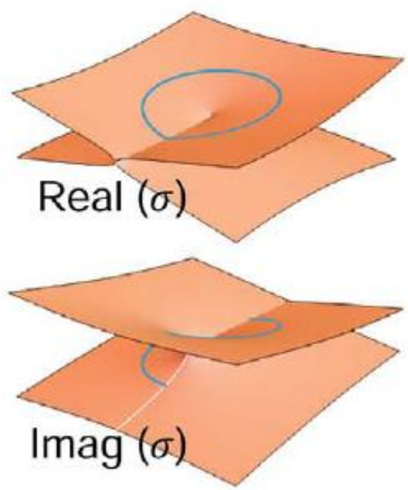
Significance of PT symmetry

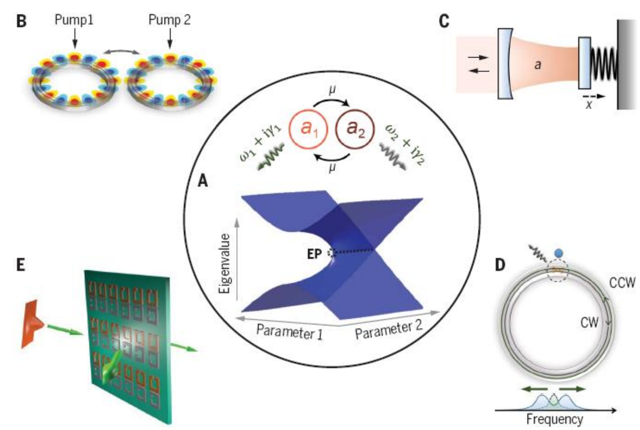
- PT-symmetric single-mode lasers
- Unidirectional invisibility
- Enhanced sensing
- Chiral mode conversion
- Robust wireless power transfer
- ……











R. El-Ganainy, et al, Nat. Phys. 14, 11(2018)
M. Miri, et al, Science 363, 42 (2019)

Non-Bloch PT symmetry of non-Hermitan systems

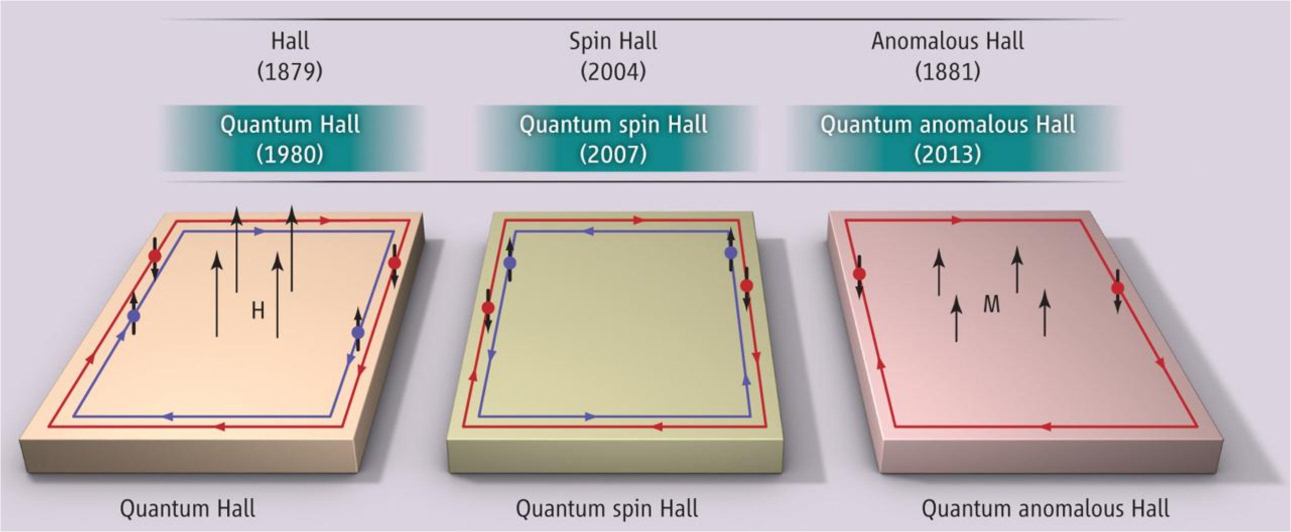
Recent concept

Outline

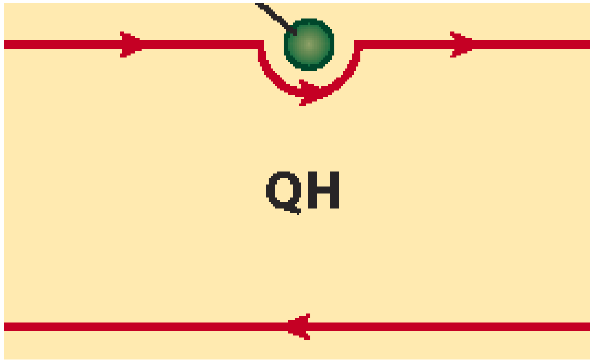
- Non-Hermitian physics and PT symmetry
- Non-Bloch band theory
- Non-Bloch parity-time symmetry

Topological states of matter

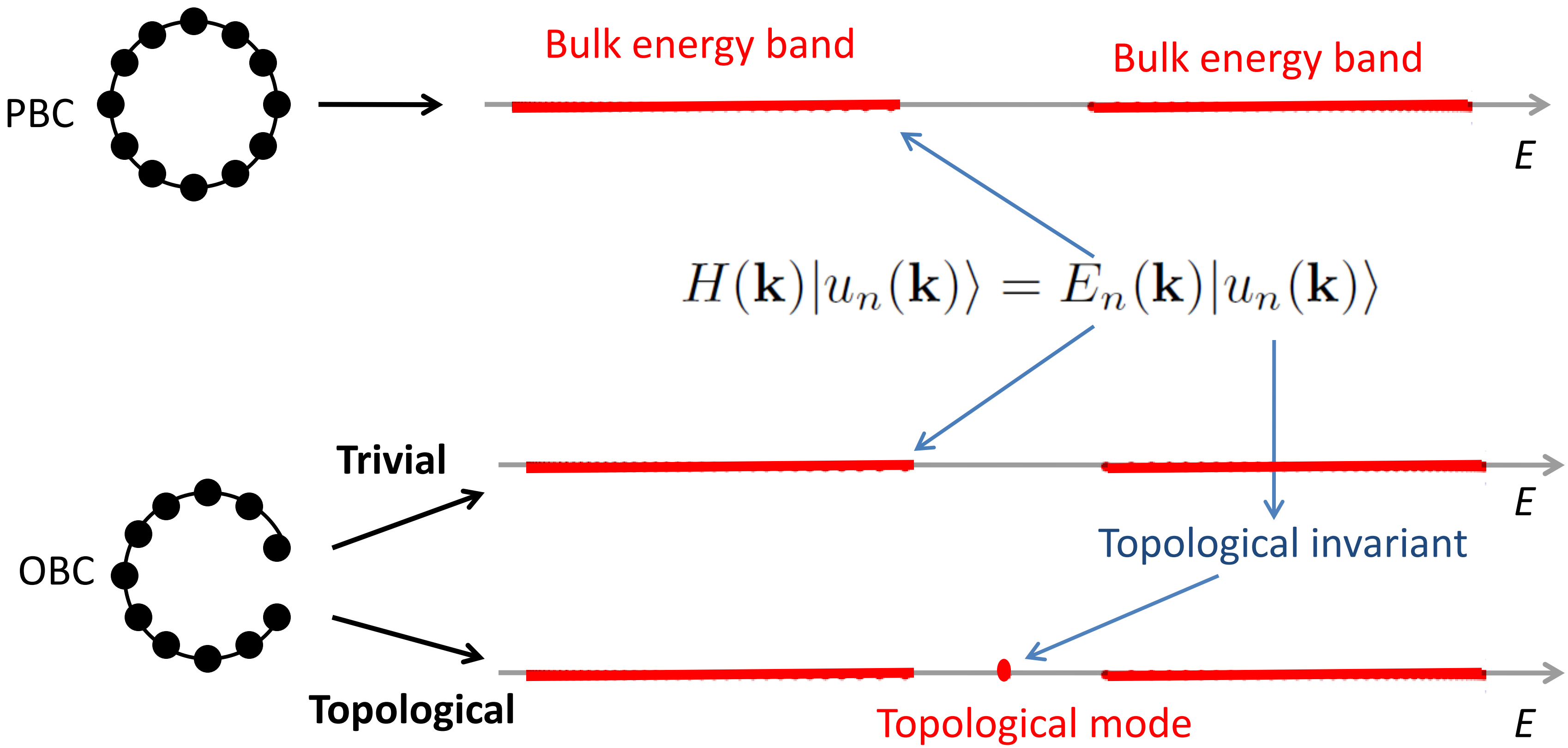
Robust boundary states; bulk-boundary correspondence



impurity



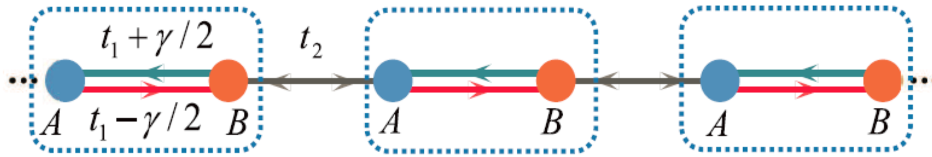
Topological band theory: Simplicity in reciprocal space



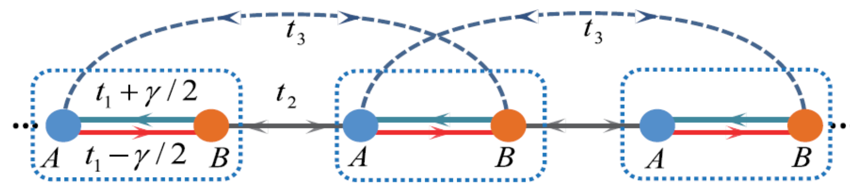
Generalize it to non-Hermitian systems?

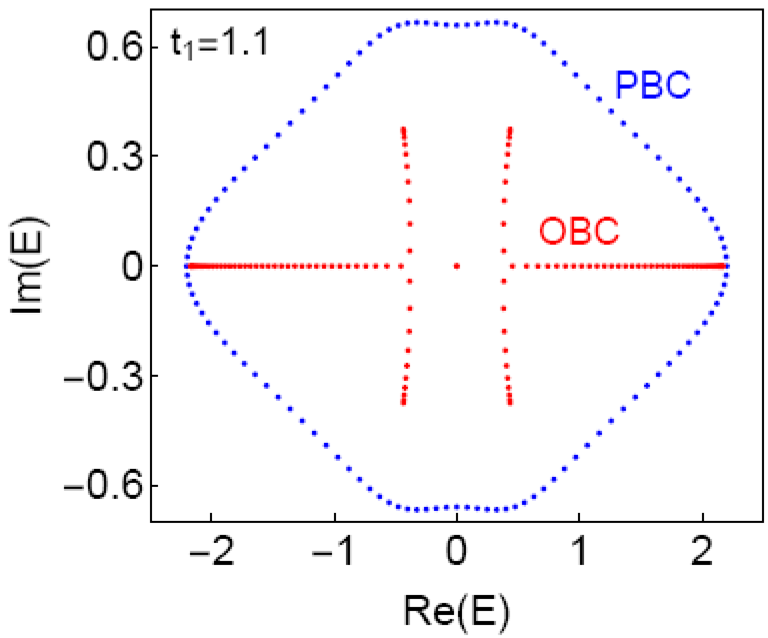
Difficulty: High sensitivity of spectrum to boundary condition

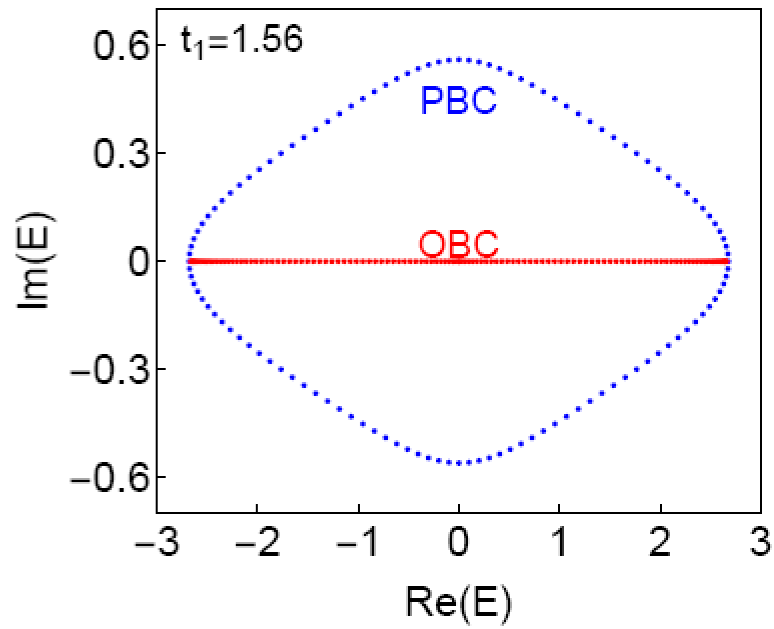
Non-Hermitian SSH model



Extended model

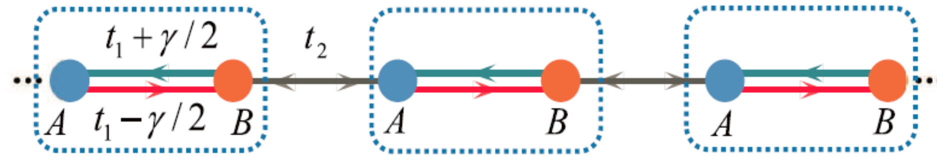




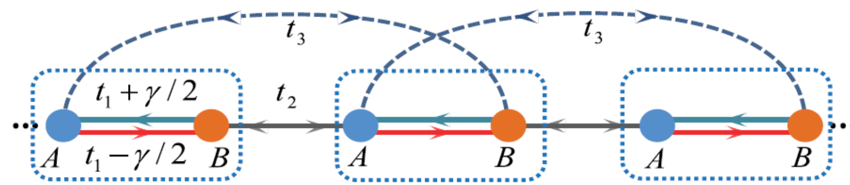


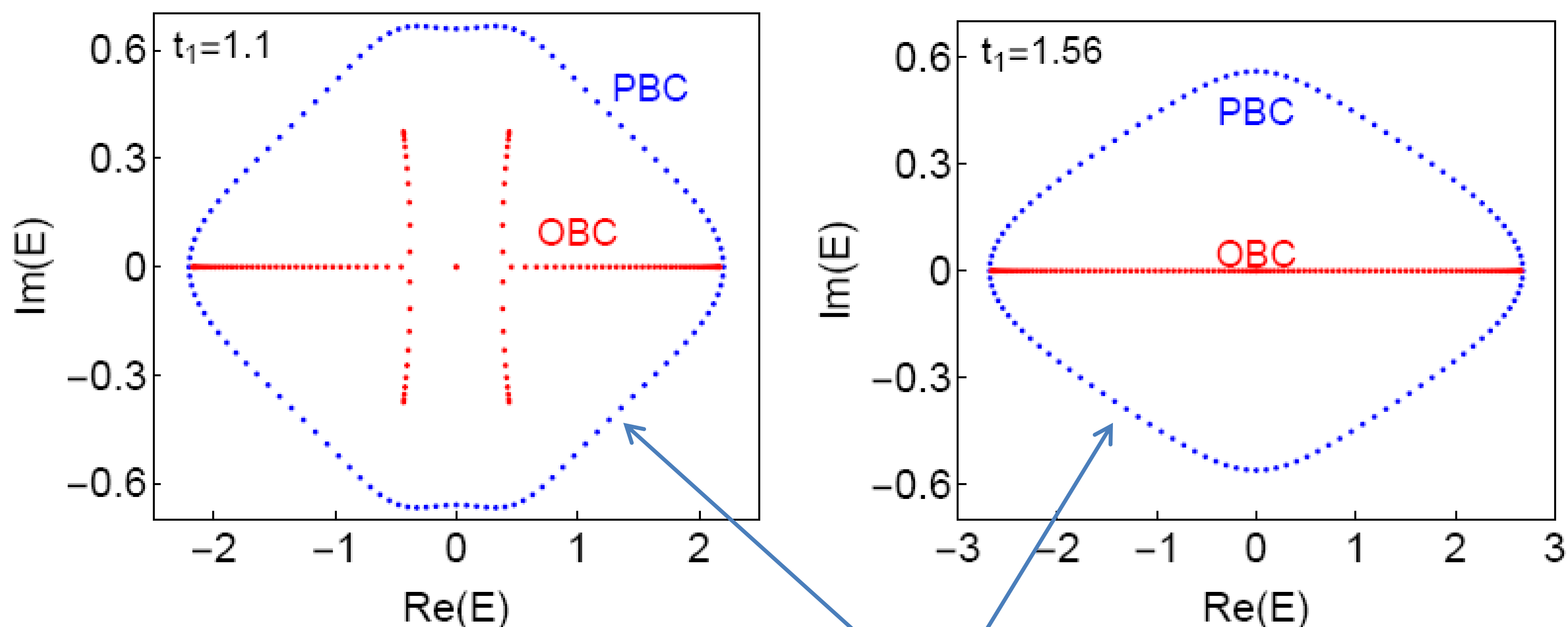
Difficulty: High sensitivity of spectrum to boundary condition

Non-Hermitian SSH model



Extended model

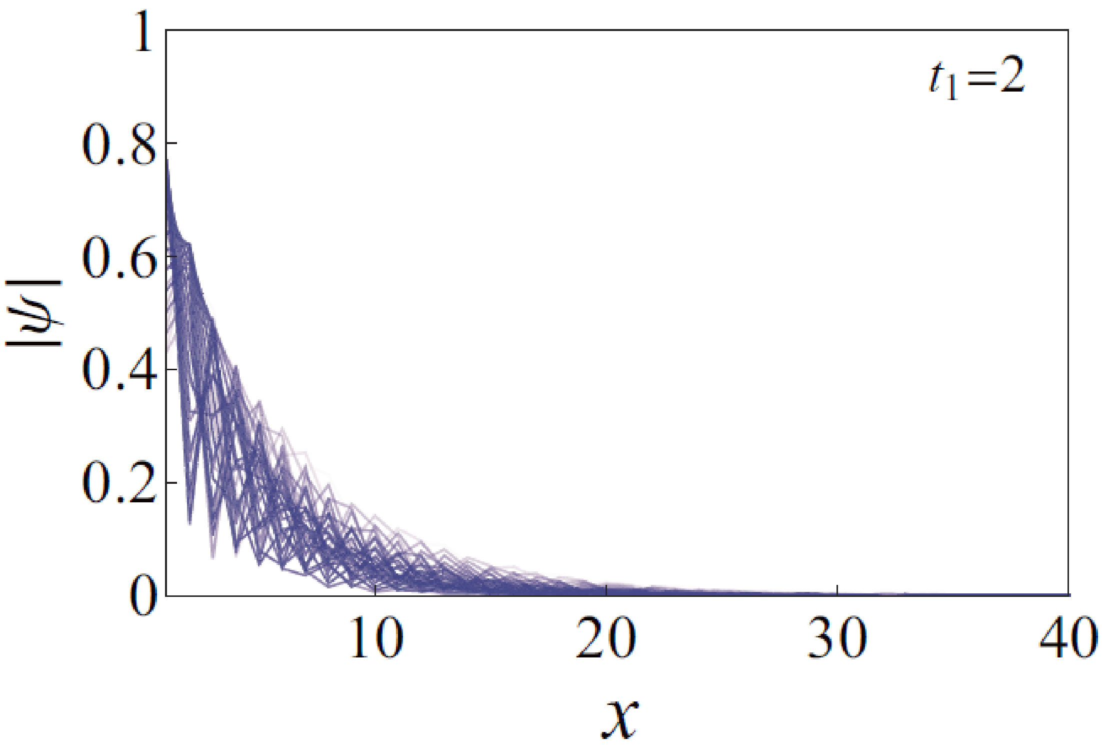




$$H(\mathbf{k})|u_n(\mathbf{k})\rangle = E_n(\mathbf{k})|u_n(\mathbf{k})\rangle$$

The mechanism is Non-Hermitian Skin Effect (NHSE)

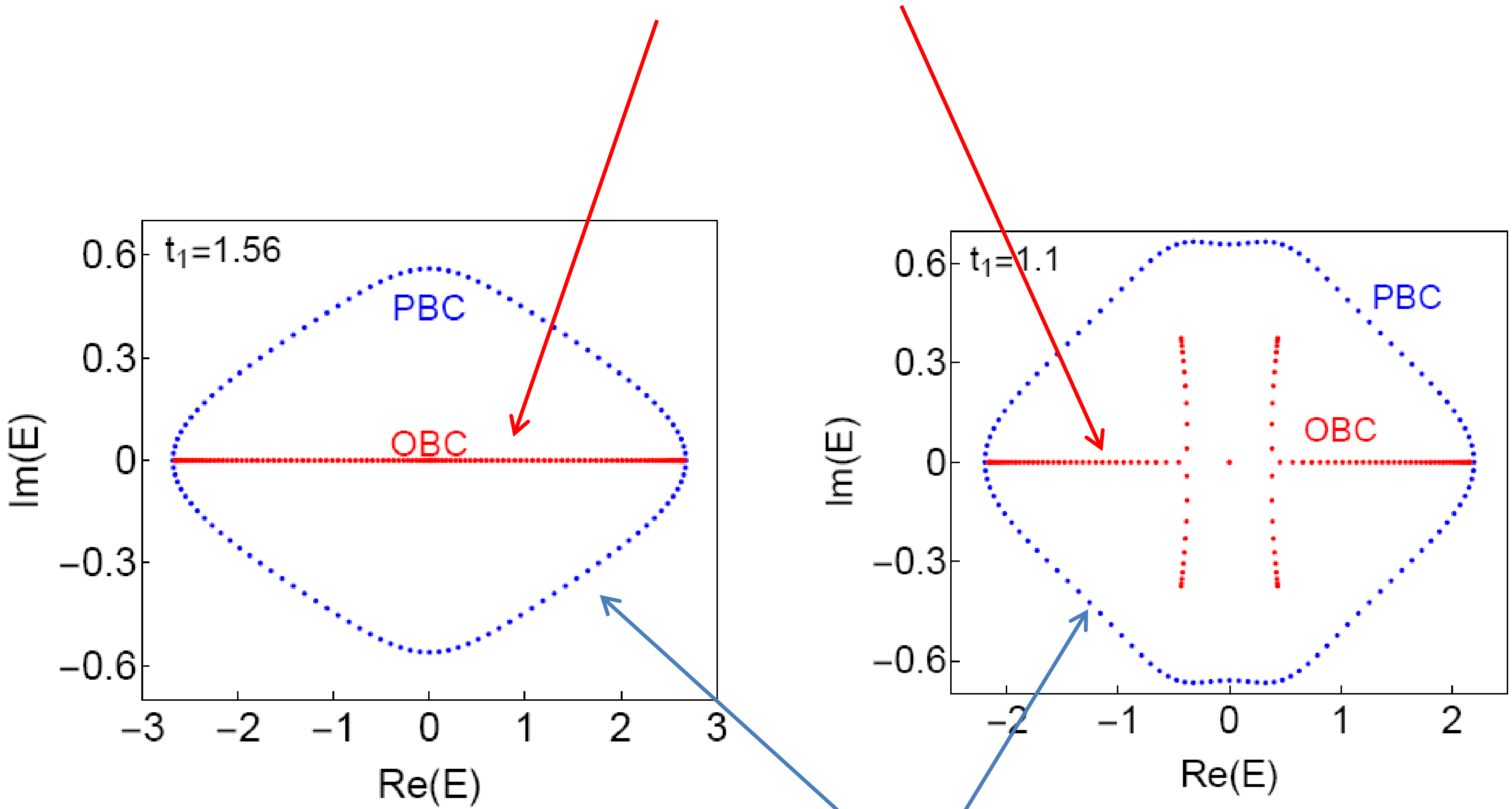
All eigenstates are localized at the boundary
→ Bloch wave picture fails!



NHSE can happen because eigenstates are not orthogonal.

S. Yao, Z. Wang PRL 121, 086803 (2018);
F. Kunst, et al PRL 121, 026808 (2018);
C.H.Lee, R. Thomale PRB 99, 201103 (2019)

Calculate energy spectrum from a general band theory?



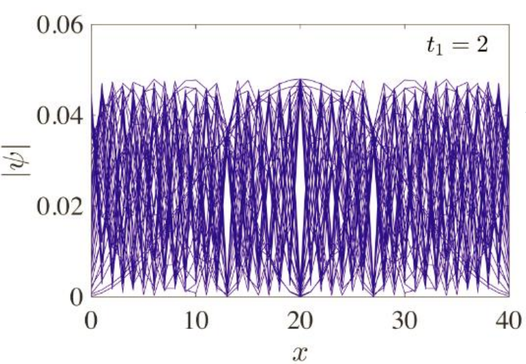
$$H(\mathbf{k})|u_n(\mathbf{k})\rangle = E_n(\mathbf{k})|u_n(\mathbf{k})\rangle$$

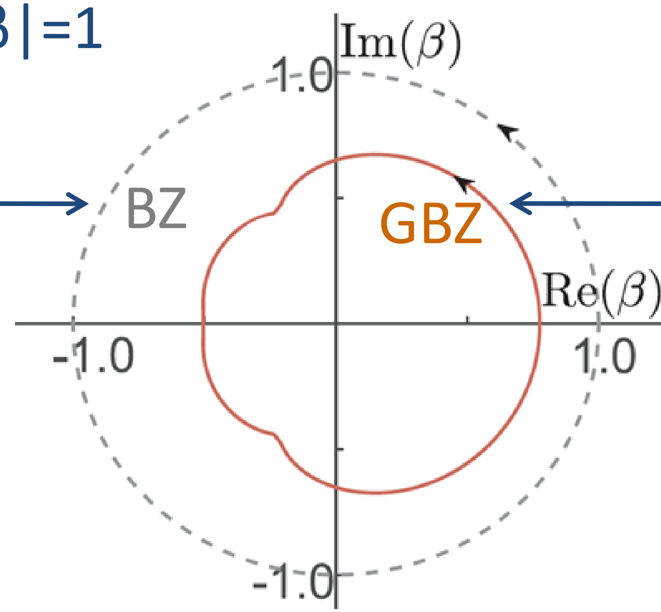
Generalized Brillouin Zone (GBZ)

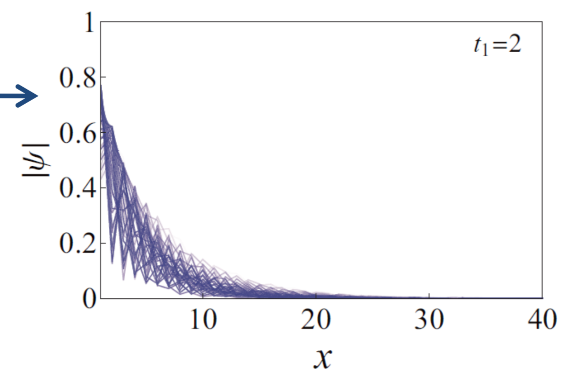
$$\psi(x) \sim e^{ikx} \xrightarrow{\beta = e^{ik}} \psi(x) \sim \beta^{x}$$

Bloch waves ↔ BZ:|β|=1

Non-Hermitian skin effect ↔ GBZ:|β|≠1







S. Yao, Z. Wang [PRL 121, 086803 (2018)]

Equation of generalized Brillouin zone (GBZ)

$$\det(H(\beta) - E\mathbb{I}) = 0$$

$$|\beta_1(E)| \leq |\beta_2(E)| \leq \cdots \leq |\beta_{2M}(E)|$$

Input boundary condition:

$$A_1[\beta_M(E)]^L + A_2[\beta_{M+1}(E)]^L = 0$$

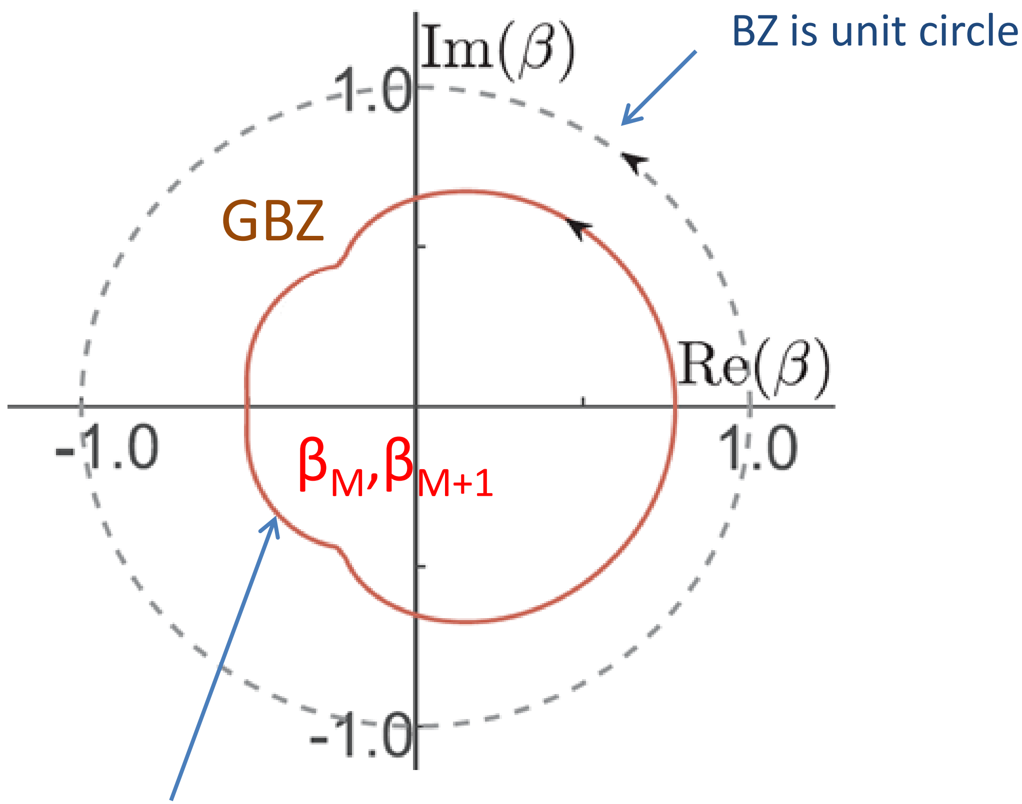
A_1 and A_2 are very complicated

Great simplification in the large-L limit!

$$|\beta_M(E)| = |\beta_{M+1}(E)|$$

General derivation: K. Yokomizo, S. Murakami, PRL 123, 066404 (2019)
The simplest M=1 case: S. Yao, Z. Wang, PRL 121, 086803 (2018)
Further extended in: Z. Yang, K. Zhang, C. Fang, J.P. Hu, PRL, 125, 226402 (2020)

GBZ = Trajectory of β



$$|\beta_M(E)| = |\beta_{M+1}(E)|$$

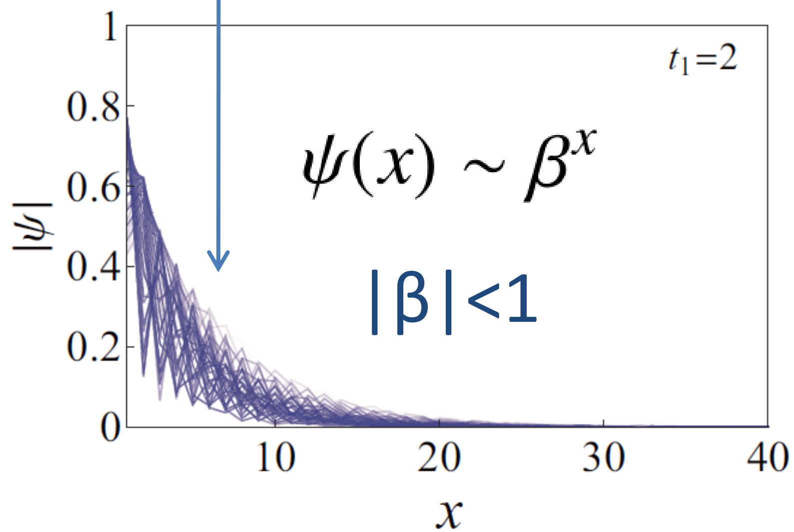
$$\det[H(\beta) - E] = 0$$

$$|\beta_1| \leq |\beta_2| \leq \cdots \leq |\beta_{2M-1}| \leq |\beta_{2M}|$$

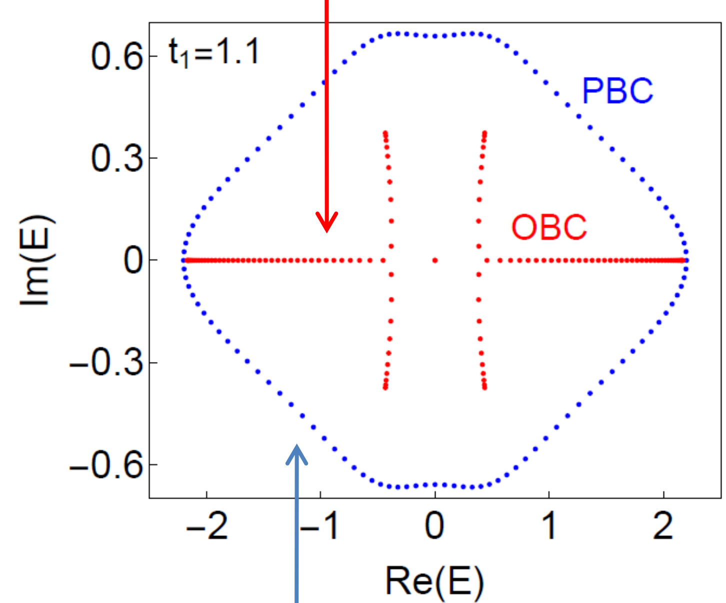
S. Yao, ZW [PRL 121, 086803 (2018)]; K. Yokomizo, S. Murakami, [PRL 123, 066404 (2019)]

Spectrums and wavefunction obtained from GBZ

GBZ



Eigenvalues of $H(\beta)$; $\beta \in$ GBZ



Eigenvalues of $H(\beta)$; $|\beta|=1$

Non-Bloch band theory

Bloch band theory:

$$\beta = e^{ik}$$

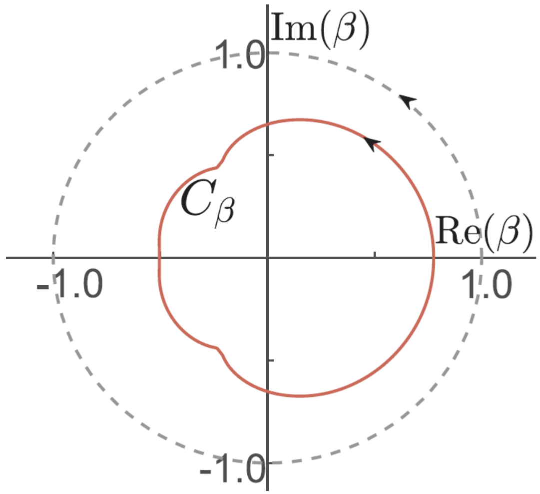
$$H(\beta)|u_n(\beta)\rangle = E_n(\beta)|u_n(\beta)\rangle \quad (\,|\beta|=1)$$

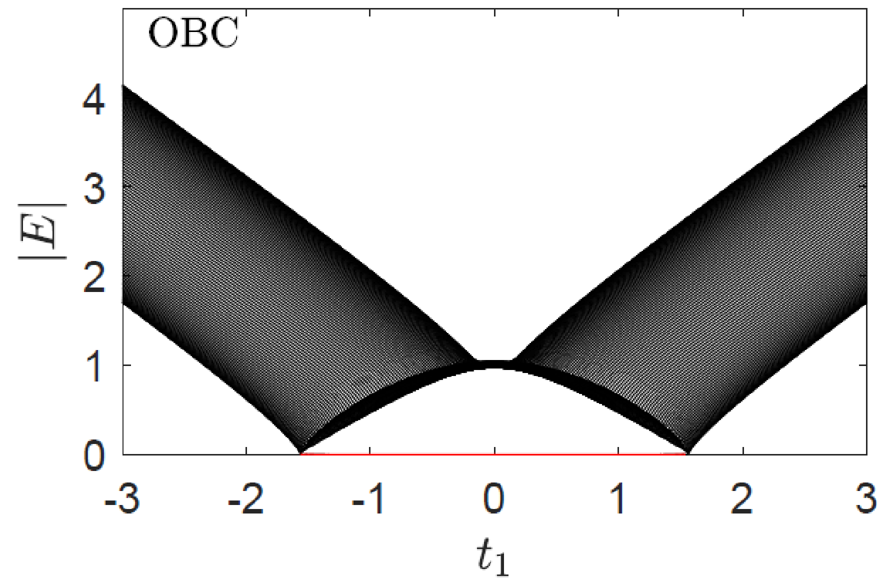
Non-Bloch band theory:

$$H(\beta)|u_n(\beta)\rangle = E_n(\beta)|u_n(\beta)\rangle \quad (\,\beta \in \mathrm{GBZ}\,)$$

(Generalized Brillouin Zone)

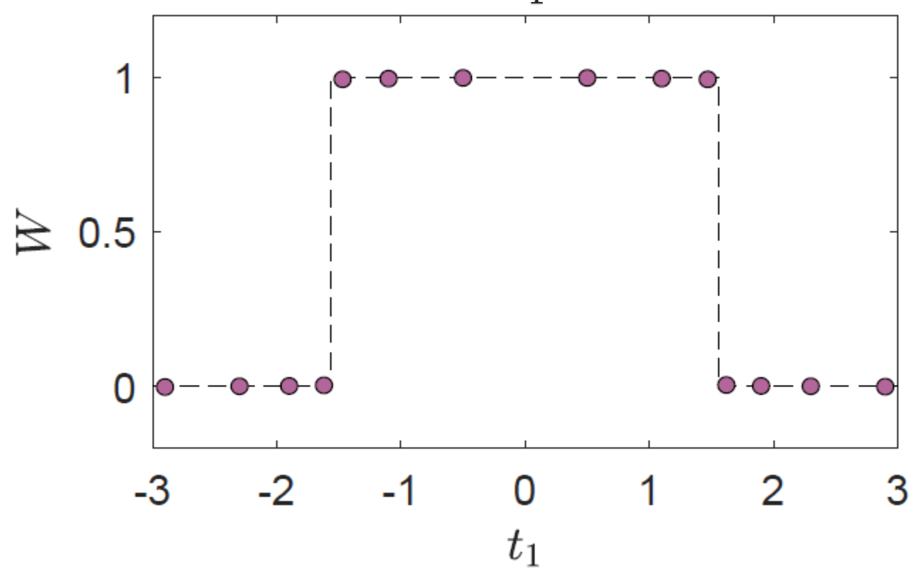
Topological invariants defined in GBZ:
Non-Bloch bulk-boundary correspondence





$$H(\beta)|u_R\rangle = E(\beta)|u_R\rangle$$

$$Q(\beta) = |u_{R1}(\beta)\rangle\langle u_{L1}(\beta)| - |u_{R2}(\beta)\rangle\langle u_{L2}(\beta)|$$

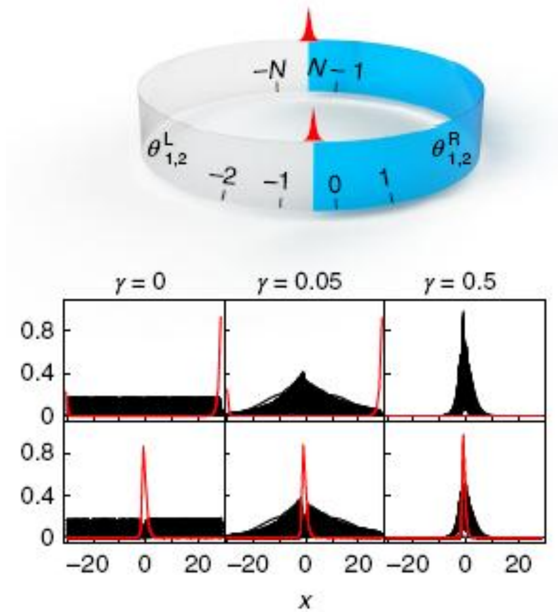


Non-Bloch winding number:

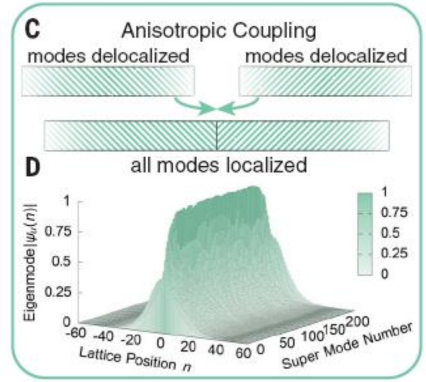
$$W = \frac{1}{4\pi i}\int_{\mathrm{GBZ}} \mathrm{Tr}[\sigma_z Q(\beta) dQ(\beta)],$$

S. Yao, Z. Wang [PRL 121, 086803 (2018)]

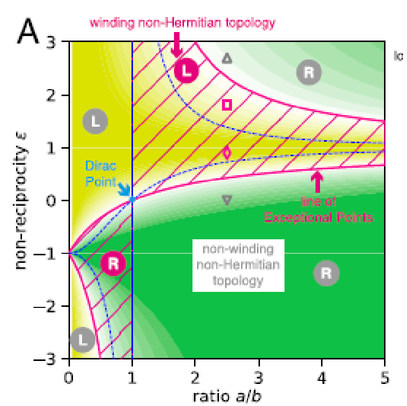
Many experimental confirmations



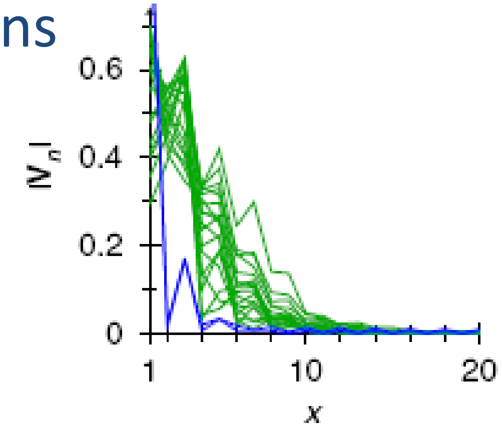
Beijing
Nat. Phys. 2020
(Peng Xue group
+ Wei Yi + Z.W.)



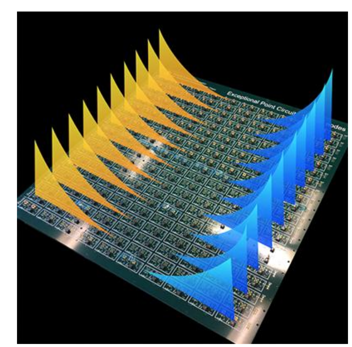
Rostock University
Science 2020



Amsterdam University
PNAS 2020



Wurzburg University
Nat. Phys. 2020
(R. Thomale +
L. W. Molenkamp +…)



University of Zurich
PR Research 2020
(T. Neupert group)

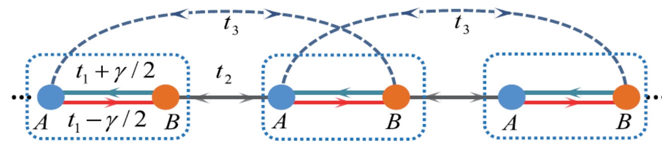
Some of the subsequent progresses

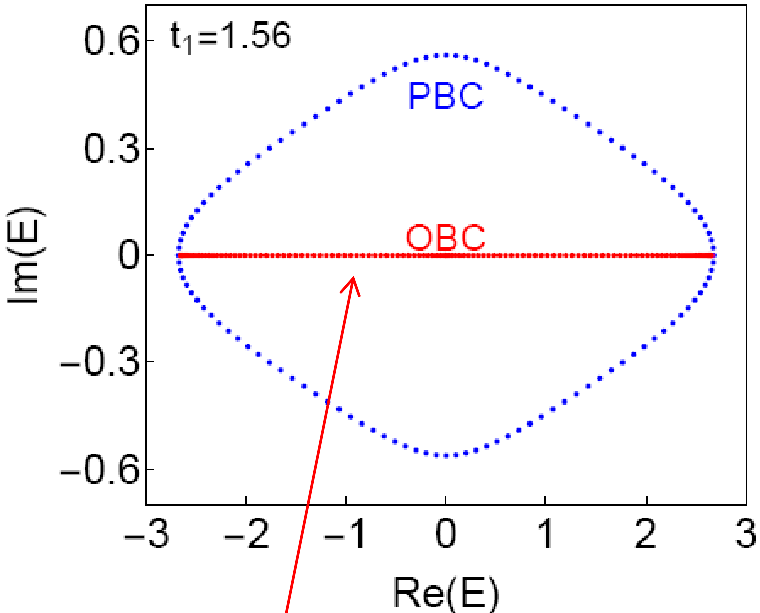
- Correspondence between winding number and skin modes
  [PRL, 125, 126402 (2020); PRL 124, 086801 (2020)]
- NHSE in correlated systems [PRL, 126, 176601 (2021)]
- Critical non-Hermitian skin effect [Nat. Comm. 11, 5491 (2021)
- Interplay between NHSE and disorders
- New relaxation pattern in open quantum systems [PRL, 123, 170401 (2019)]
- GBZ and aGBZ, and GBZ algorithm based on resultant [PRL, 125.226402 (2020)]
- Non-Bloch band collapse [PRL, 124, 066602 (2020)]
- Non-Bloch band theory for symplectic class [PRB, 101, 195147 (2020)
- Dislocation NHSE [PRL, 127, 066401 (2020)]
- Higher-order NHSE
  [PRB, 102, 205118 (2020); Nat. Comm. 12, 4691(2021); Nat. Comm. 12, 5377(2021)]
- NHSE in elastic media [PRL, 125, 118001 (2020)
- Topological switch of NHSE in lossy cold atom systems [PRL, 124, 250402 (2020)]
- NHSE-induced morphing of topological modes [Nature, 608, 50-55 (2022)]
- ....

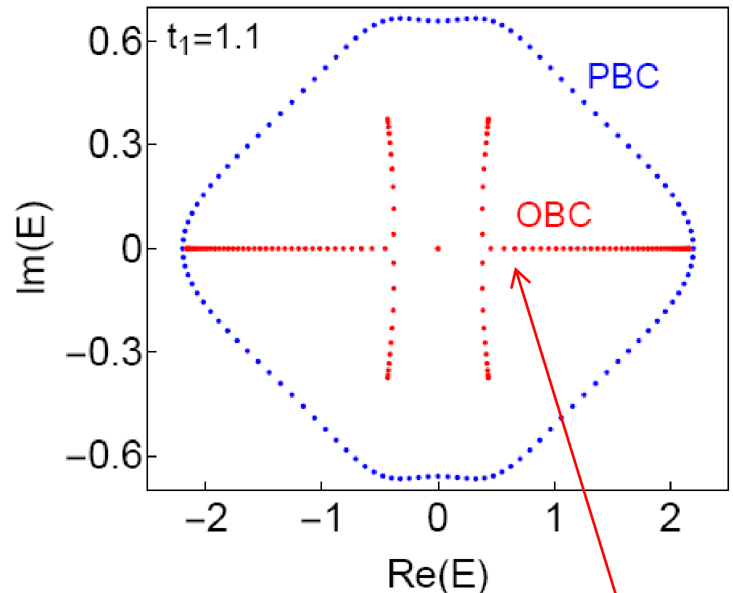
Outline

- Non-Hermitian physics and PT symmetry
- Non-Bloch band theory
- Non-Bloch parity-time symmetry: 1D and 2D

NHSE + PT symmetry = Non-Bloch PT symmetry







E real-valued (PT-exact phase)

E complex-valued (PT-broken phase)

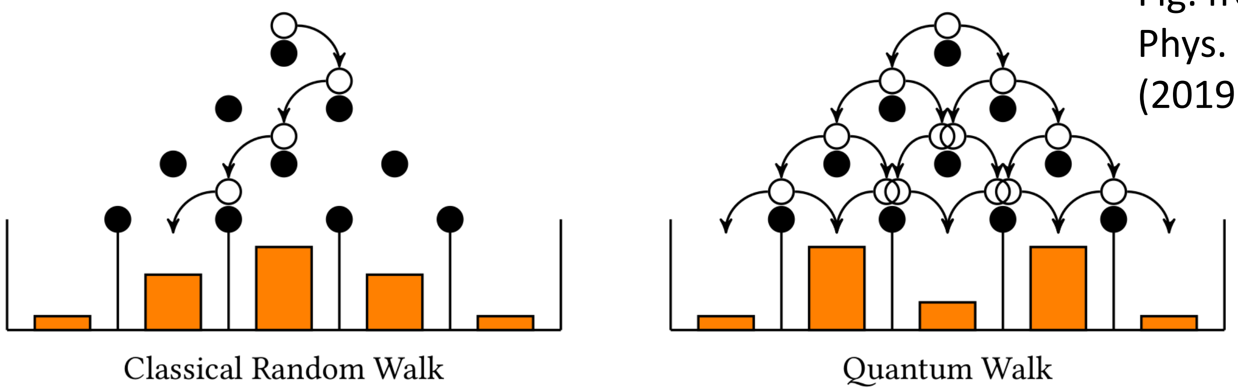
- Non-Bloch band theory → "Non-Bloch PT symmetry"
- It is a symmetry on the generalized Brillouin zone

$$\eta H(\beta)\eta^{-1} = H^{\dagger}(\beta)|_{\beta\in\mathrm{GBZ}}$$

S. Longhi, PR Research (2019); S. Longhi, Opt. Lett. (2019)

Quantum walk

Fig. from S. Dadras et al, Phys. Rev. A 99, 043617 (2019)



Non-Hermitian quantum walk of photons：

$$|\psi(t)\rangle = U^t|\psi(0)\rangle = \exp(-iH_{\text{eff}}t)|\psi(0)\rangle \qquad t = 0, 1, 2, \ldots$$

$$U := e^{-iH_{\text{eff}}}$$

$$U = R(\frac{\theta_1}{2})S_2R(\frac{\theta_2}{2})MR(\frac{\theta_2}{2})S_1R(\frac{\theta_1}{2})$$

$$R(\theta) = \mathbb{1}_w \otimes e^{-i\theta\sigma_y}$$

State rotation

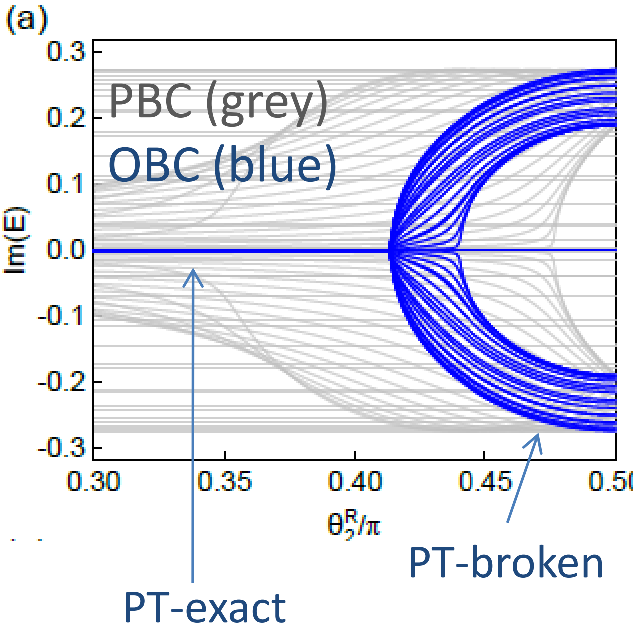
Spatial shift

$$M = \mathbb{1}_w \otimes (e^{\gamma}|0\rangle\langle 0| + e^{-\gamma}|1\rangle\langle 1|)$$

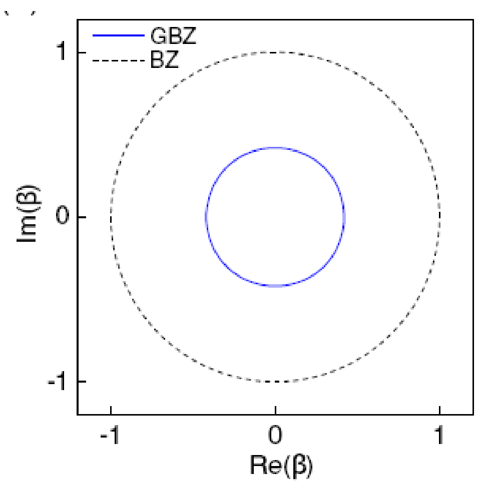
Onsite loss

Xiao et al (Peng Xue group), Phys. Rev. Lett. 126, 230402 (2021)

Non-Bloch PT symmetry

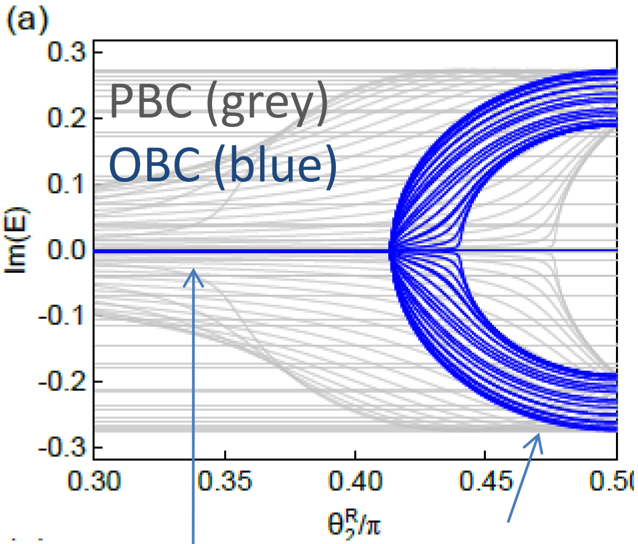


$$\eta H_{\text{eff}}(\beta)\eta^{-1} = H_{\text{eff}}^{\dagger}(\beta)|_{\beta\in\text{GBZ}}$$



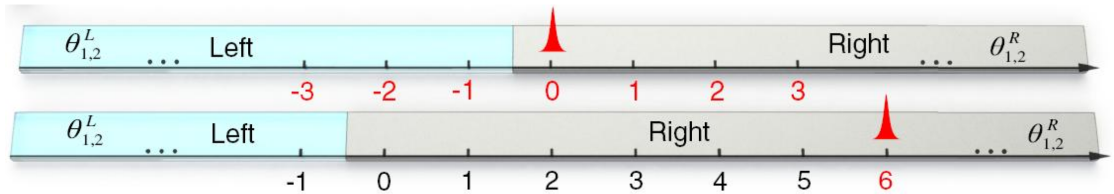
Xiao et al (Peng Xue group), [Phys. Rev. Lett. 126, 230402 (2021)]

Non-Bloch PT symmetry



PT-broken

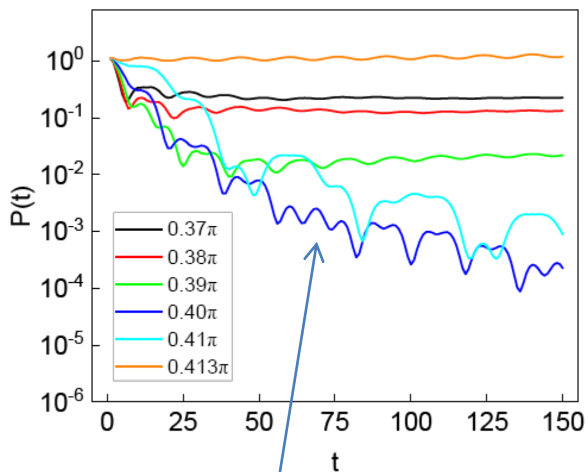
PT-exact

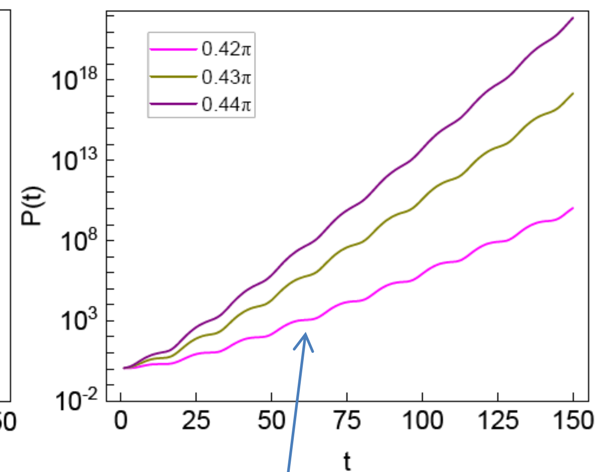


$$|\psi(t)\rangle = U^t|\psi(0)\rangle = \exp(-iH_{\text{eff}}t)|\psi(0)\rangle$$

$$P(t) = \langle\psi(t)|\,\psi(t)\rangle \sim e^{2\max[\text{Im}(E)]t}$$

$$\eta H_{\text{eff}}(\beta)\eta^{-1} = H^{\dagger}_{\text{eff}}(\beta)|_{\beta\in\text{GBZ}}$$

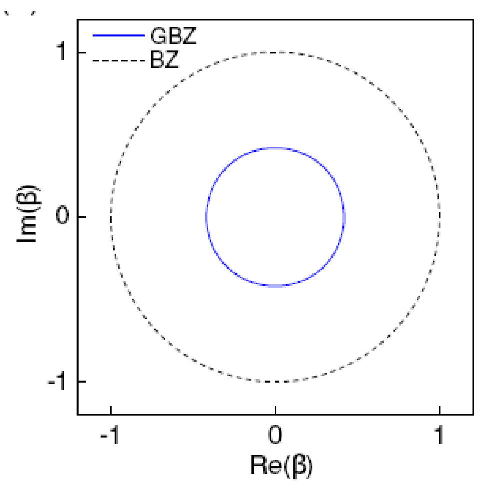




Numerical Simulation

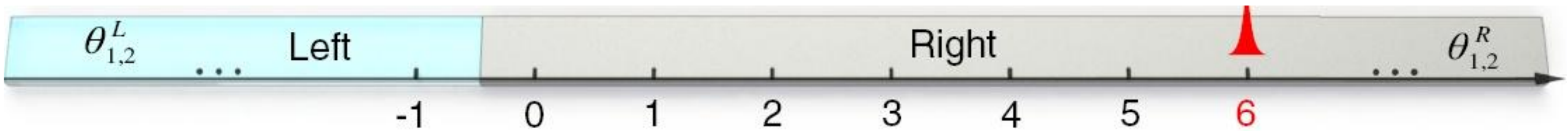
PT-exact

PT-broken

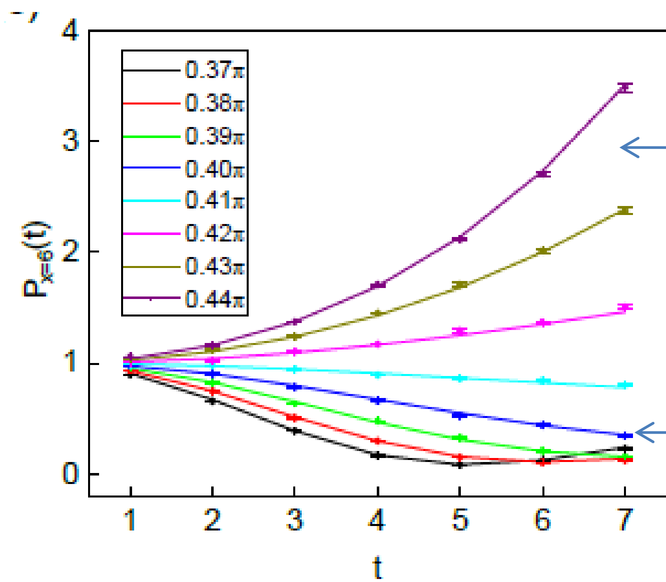


Xiao et al (Peng Xue group), Phys. Rev. Lett. 126, 230402 (2021)

Non-Bloch PT symmetry



$$|\psi(t)\rangle = U^t|\psi(0)\rangle = \exp(-iH_{\rm eff}t)|\psi(0)\rangle$$



PT-broken phase

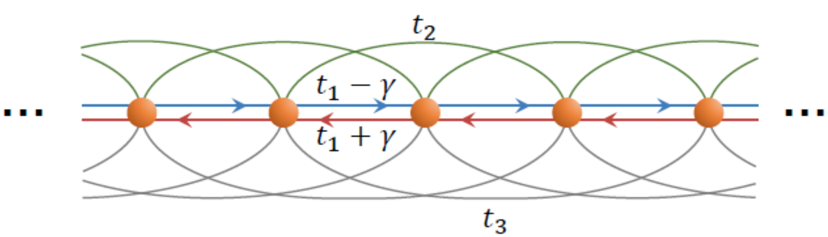
PT-exact phase

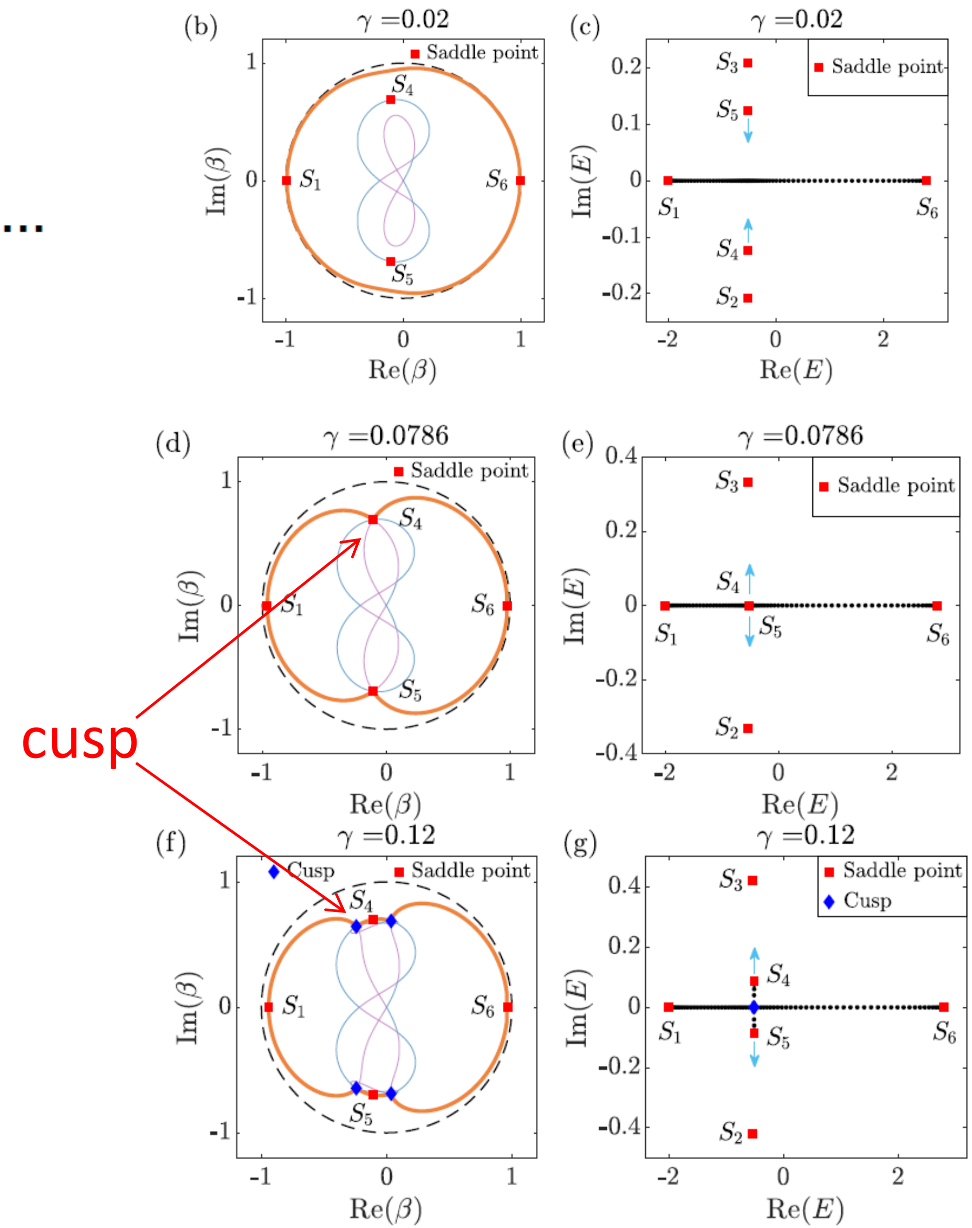
Experimentally observed

$$P(t) = \langle\psi(t)|\,\psi(t)\rangle \sim e^{2\max[\mathrm{Im}(E)]t}$$

Xiao et al (Peng Xue group), [Phys. Rev. Lett. 126, 230402 (2021)]

Geometric origin of non-Bloch PT symmetry breaking in 1D





- ➢ See PT symmetry from the shape of GBZ
- ➢ GBZ cusps means PT symmetry breaking

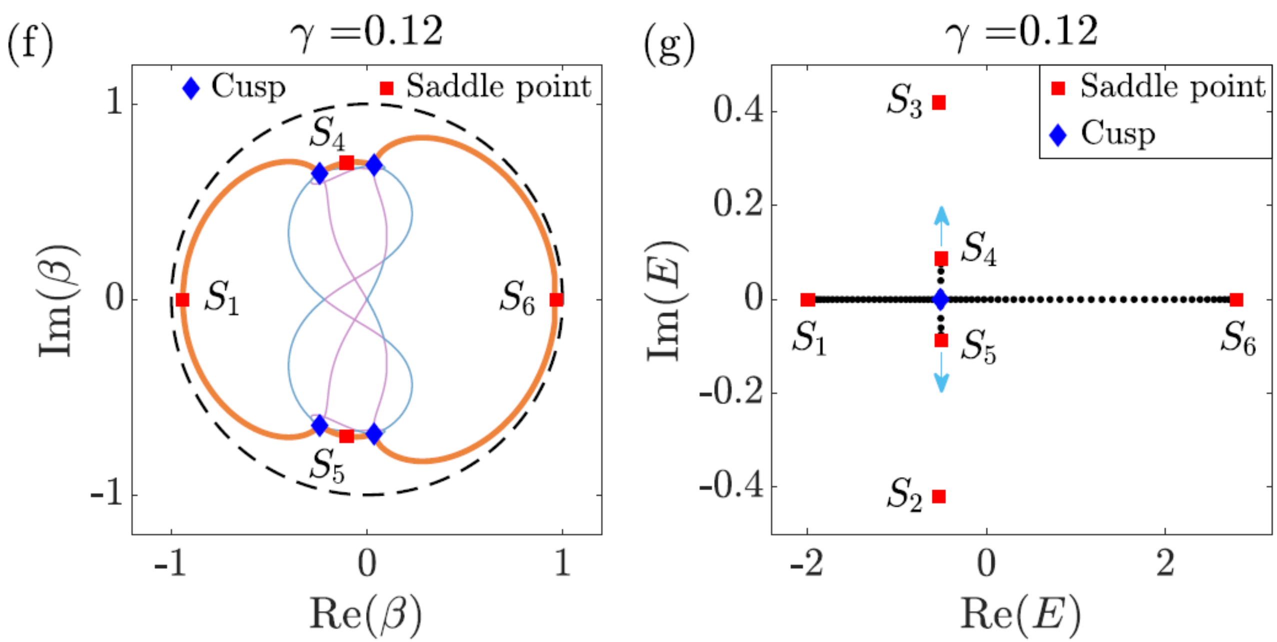
H.M. Hu, H.Y. Wang, ZW, F. Song, to appear

Geometric origin of non-Bloch PT symmetry breaking in 1D

$$E(\theta) = H(\beta(\theta)) = H(|\beta(\theta)|e^{i\theta})$$

Discontinuous at GBZ cusp ⟶

$$\frac{dE(\theta)}{d\theta} = \frac{\partial H(\beta)}{\partial \beta}\left(\boxed{\frac{\partial|\beta(\theta)|}{\partial\theta}}e^{i\theta} + i\beta\right)$$

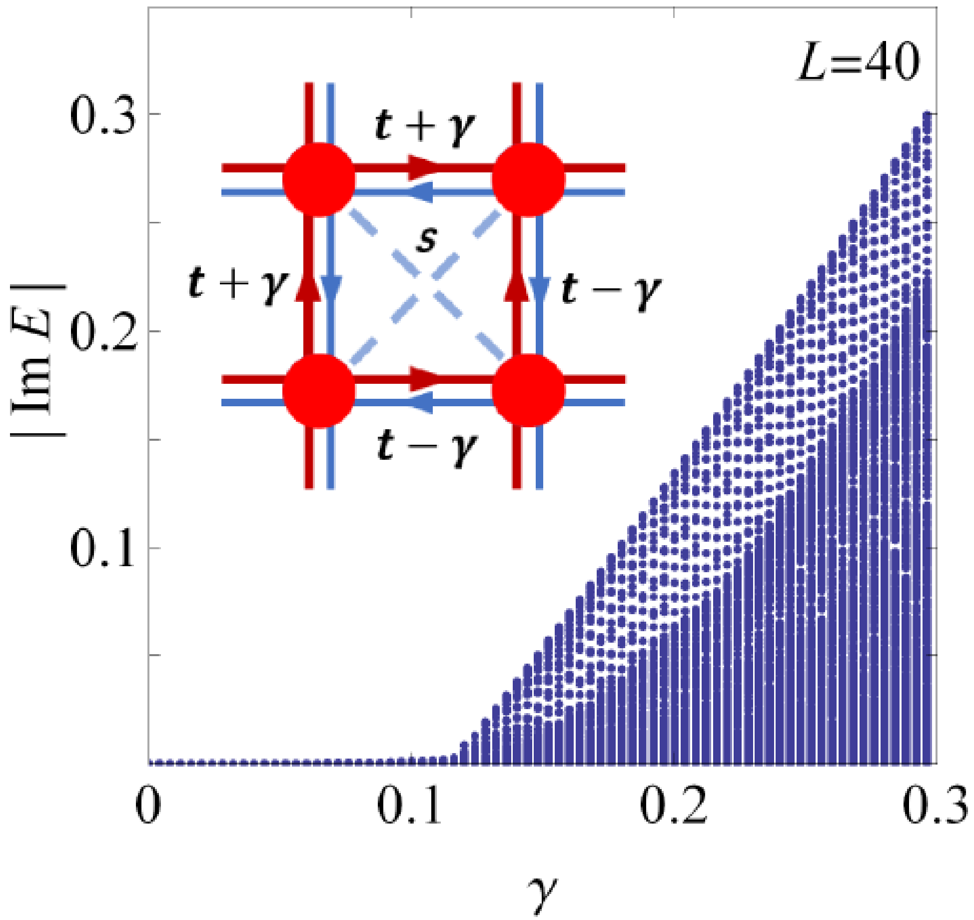


See PT symmetry breaking from the shape of GBZ!

H.M. Hu, H.Y. Wang, ZW, F. Song, to appear

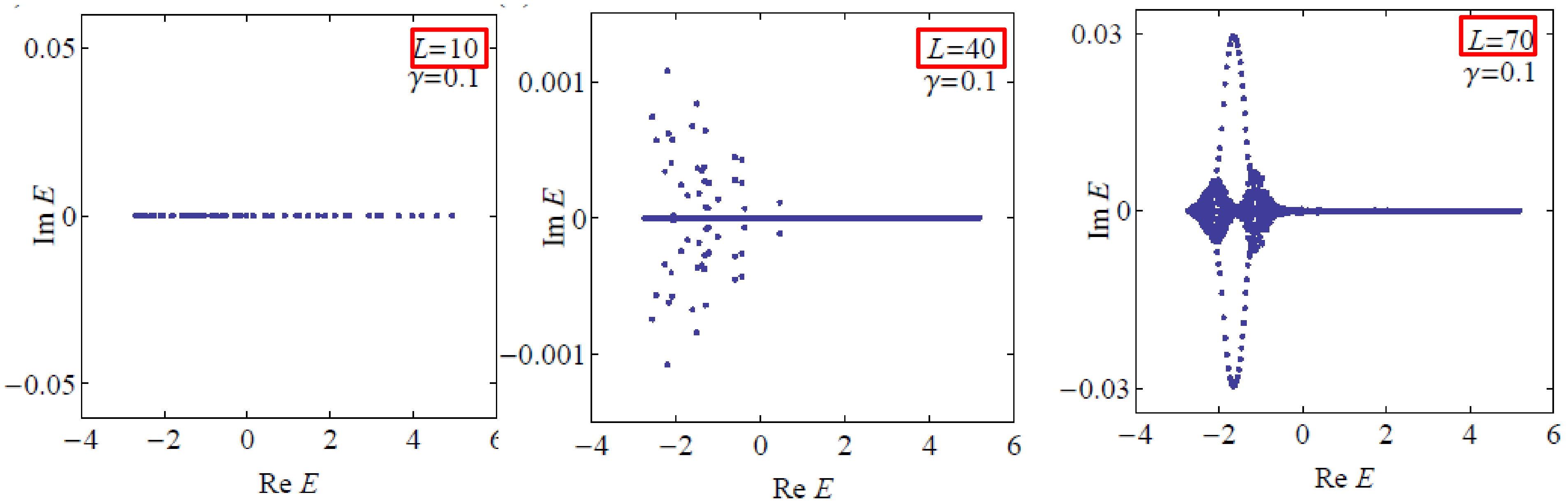
2D non-Bloch PT symmetry breaking

Seems normal



2D non-Bloch PT symmetry breaking

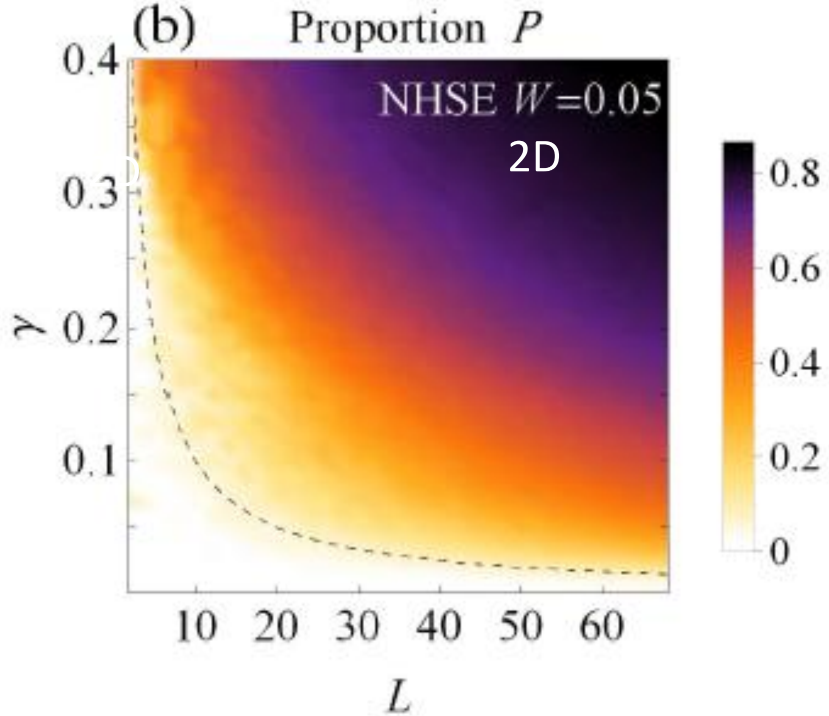
But the size dependence is strange



PT breaking induced by simply increasing size

2D non-Bloch PT symmetry breaking

P = Proportion of complex-valued eigenenergies

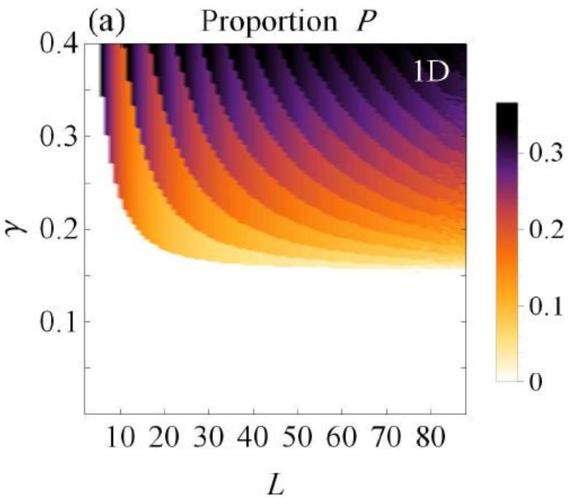


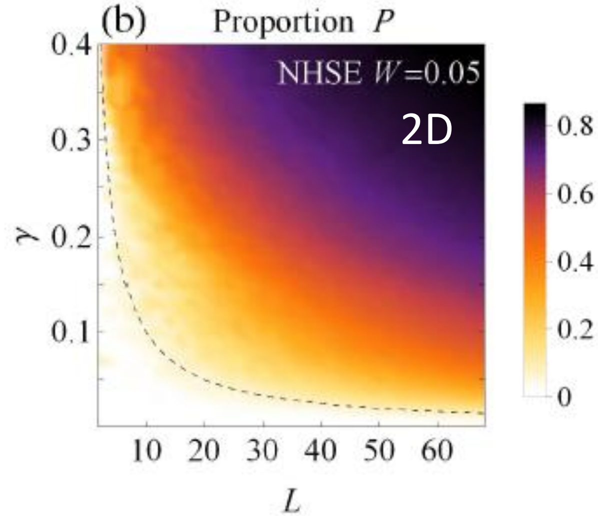
Threshold universally approaches 0 as size increases

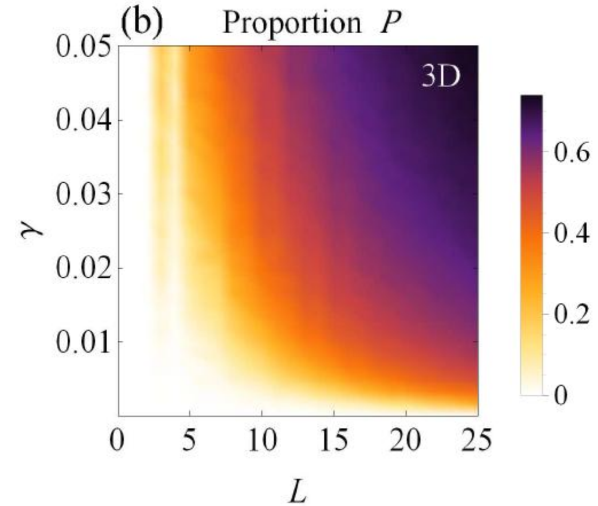
F. Song, H. Y. Wang, Z. Wang , [A Festschrift in Honor of the C N Yang Centenary, pp. 299-311 (2022)]

2D non-Bloch PT symmetry breaking

P = Proportion of complex-valued eigenenergies







For Bloch or 1D non-Bloch PT breaking, *P* quickly saturates as size grows.

Threshold universally approaches 0 as size increases

3D is similar to 2D; both different from 1D

Open question: Universal functions characterizing the PT-breaking transitions beyond 1D?

F. Song, H. Y. Wang, Z. Wang , [A Festschrift in Honor of the C N Yang Centenary, pp. 299-311 (2022)]

2D non-Bloch PT symmetry breaking

Wave-function overlap

$$\eta(n,m) = \frac{|\langle\psi_n|\psi_m\rangle|}{\sqrt{\langle\psi_n|\psi_n\rangle\langle\psi_m|\psi_m\rangle}}$$ →1 at PT breaking point

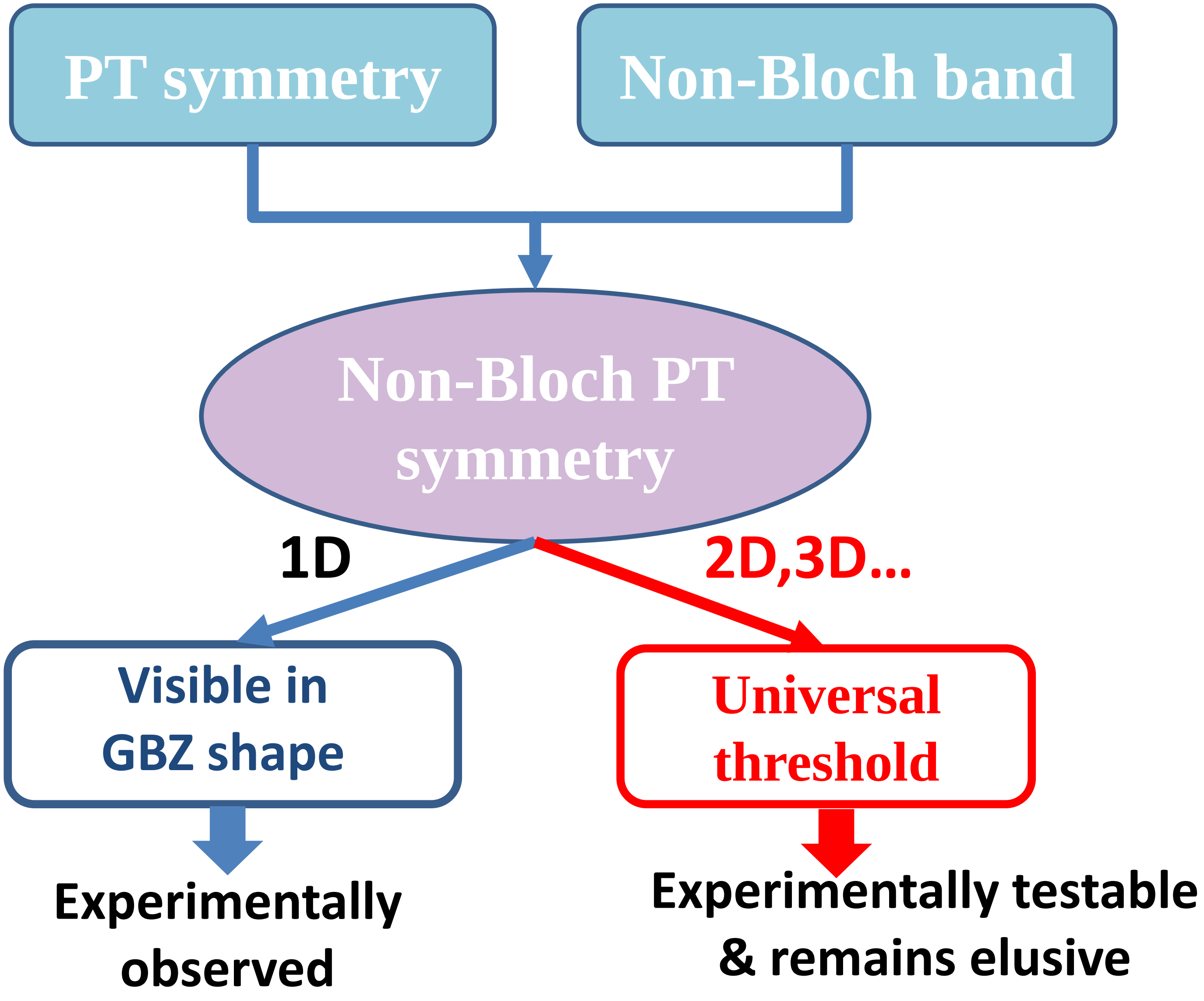
Strong size sensitivity of non-Bloch bands:

$$\eta(n,m) \sim \gamma L.$$ Increases with size!

Question: This explains the role of NHSE, but it seems independent of spatial dimensions. Why are 2D/3D drastically differ from 1D?
Partial answer: Because 1D is special; it is constrained by spectral theorems:

Kai Zhang, Zhesen Yang, and Chen Fang, "Correspondence between winding numbers and skin modes in non-hermitian systems," Phys. Rev. Lett. **125**, 126402 (2020).
Nobuyuki Okuma, Kohei Kawabata, Ken Shiozaki, and Masatoshi Sato, "Topological origin of non-hermitian skin effects," Phys. Rev. Lett. **124**, 086801 (2020).

F. Song, H. Y. Wang, Z. Wang , A Festschrift in Honor of the C N Yang Centenary, pp. 299-311 (2022)

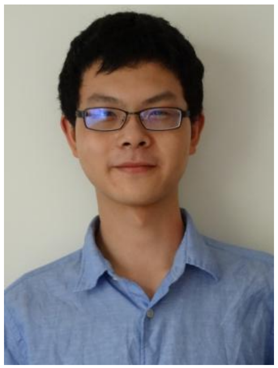


H.-M. Hu, H.-Y. Wang, ZW, F. Song, to appear

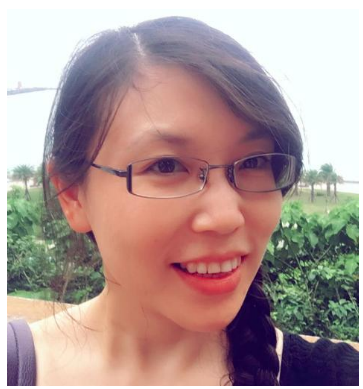
F. Song, H.-Y. Wang, ZW , A Festschrift in Honor of the C N Yang Centenary, pp. 299-311 (2022)

Outlook

- Other symmetries on the GBZ (non-Bloch symmetries)?
- Theory and experiments of higher-dimensional non-Bloch bands?
- Transport theory based on non-Bloch band theory?
- Many-body non-Hermitian physics?
- Many other open questions…




Students: Shunyu Yao (now at Stanford) Fei Song Hong-Yi Wang Yu-Min Hu

Prof. Wei Yi Prof. Peng Xue

Thank you!