Tkachenko wave: From modern field theory viewpoint

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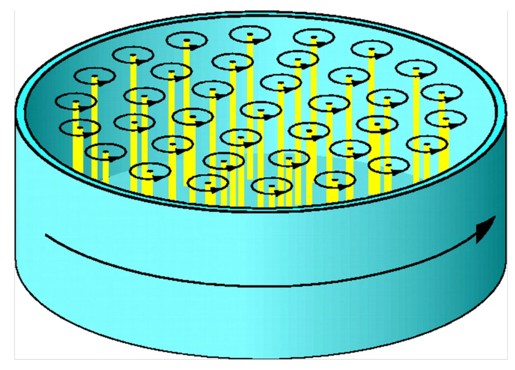
Motivation

- *Higher-rank symmetries* play an important role in high-energy and condensed matter physics. One example, the *dipole symmetry*, induces UV/IR mixing and restricts the mobility of fractonic excitations.
- Vortex lattice in a rotating superfluid is a quantum crystal that exhibits a *noncommutative* version of *dipole symmetry*.
- The collective oscillations of the vortex lattice, known as the Tkachenko mode, play the role of a Goldstone boson shared by the spontaneously broken dipole and charge symmetries.
- We developed a noncommutative effective field theory of the Tkachenko mode that predicts a *cubic dependence* of the decay width of the Tkachenko mode on its energy.

The Tkachenko wave

- Quantum vortices in superfluids have quantized circulation in units of the Planck constant.
- Vortex lattice is formed in a rotating superfluid where the vortices are distributed in a triangular lattice.
- Tkachenko wav is the vibration of the vortex lattice with strange behavior:
  - Quadratic dispersion $\omega = ck^2$.
  - Only one polarization: transverse.
  - One Nambu-Goldstone boson for breaking 2-translation symmetries and $U(1)$



Circulation around a vortex line

$$\kappa = \oint d\mathbf{l}\cdot \mathbf{v}_s = \frac{\hbar}{m_{He}}\oint d\mathbf{l}\cdot\nabla\phi = \frac{h}{m_{He}}$$

Lifshitz model

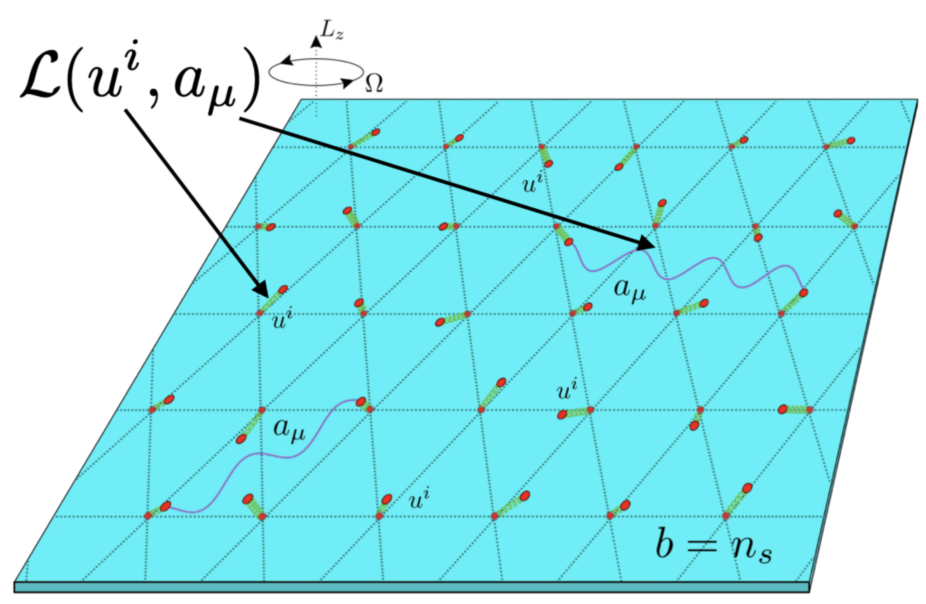
- Effective theory of vortex lattice in dual description

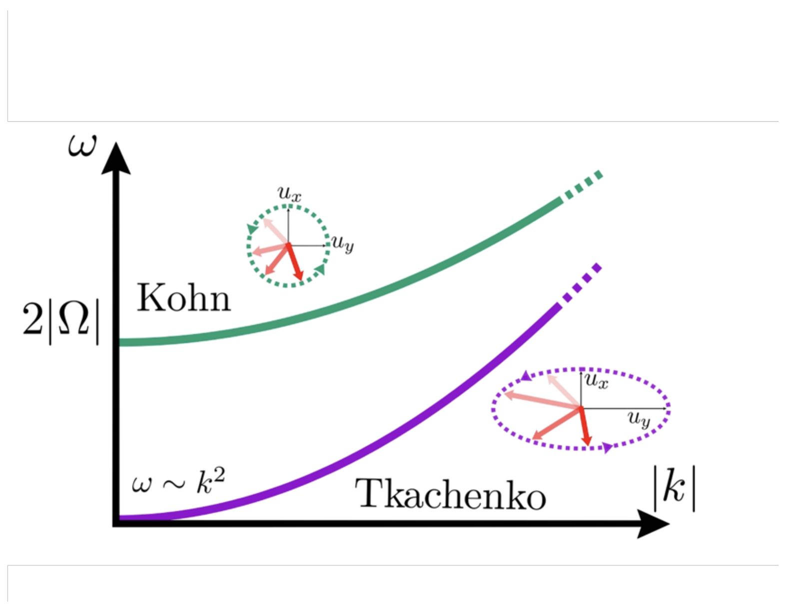
$$\mathcal{L} = \frac{n_0}{2} n_s \epsilon^{ij} u_i \dot{u}_j + \dot{\mathbf{u}}^2 - G(\partial_i u_j + \partial_j u_i)^2 + \mathbf{u}\cdot\mathbf{e} + \frac{1}{2}\left(\mathbf{e}^2 + \mathbf{b}^2\right)$$

- n_0 and n_s are vortex density and superfluid density respectively, G determines the elastic energy.
- Dual electric field and dual magnetic field related to superfluid current and desity respectively $e_i = \partial_t a_i - \partial_i a_0 = \epsilon^{ij} j^j_s, b = \epsilon^{ij}\partial_i a_j = n_s$, vortices are charged under the emergent $u(1)$ gauge.
- At the low energy limit, $\omega \ll 2|\Omega|$, the power counting determined by the dispersion relation $\omega \sim k^2$, $\mathbf{e}^2$ and $\dot{\mathbf{u}}^2$ terms are ignored.
- $\frac{\delta S}{\delta a_0} = \nabla\cdot\mathbf{u} = 0 \to u^i = \epsilon^{ij}\partial_j\phi$, the special phonon only has the transverse polarisation.
- Integrating out the dual gauge field a_μ gives the Lifshitz action

$$\mathcal{L} = \dot{\phi}^2 - c^2\left(\nabla^2\phi\right)^2$$

- The Lifshitz model describes the system with dipole global symmetry $\phi \to \phi + \vec{\alpha}\cdot\vec{x}$.





Dynamical degree of freedom of the vortex lattice and Tkachenko mode.

- u^i denotes the displacement of vortices, a_μ denotes the dynamical degree of the superfluid in the *dual description* with the replacement $j^\mu_s = \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$.
- Ω is the angular velocity of the rotating container.
- With the low energy less than the energy gap of Kohn mode, $\omega \ll 2|\Omega|$, Tkachenko mode is the only dynamical degree of freedom.
- Tkachenko wave is the **combination** of the lattice vibration (phonon) and the dual photon.

Non-commutative field theory of Tkachenko wave

- Deformed lattice described by Lagrangian coordinate

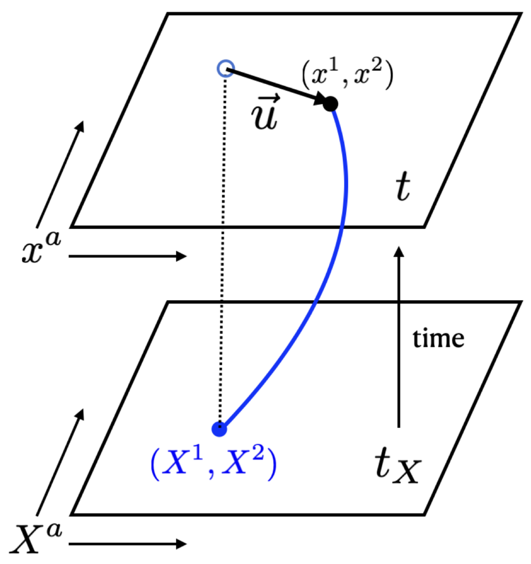
$$x^i \leftrightarrow X^a = \delta^a_i x^i - u^a$$

- Vortex current in the Lagrangian coordinate formalism

$$j^\mu_v = \frac{n_0}{2}\epsilon^{\mu\nu\lambda}\epsilon^{ab}\partial_\nu X^a \partial_\lambda X^b$$

- The low energy dynamics is constrained by the area-preserving mapping (incompressibility condition)

$$\frac{\delta S}{\delta a_0} = 0 \to \frac{1}{2}\epsilon^{ij}\epsilon^{ab}\partial_i X^a \partial_j X^b = 1$$



Non-commutative coordinate

- The angular velocity breaks time reversal T and plays the role of the magnetic field in the rotating frame.
- In the lowest Landau level projected Hilbert space, $\omega \ll 2|\Omega|$, the geometry becomes non-commutative due to the *Aharonov-Bohm phase*

$$[\hat{x}, \hat{y}] = -i\ell^2 = i\theta, \qquad \ell^{-2} = B = 2\Omega m_{He}$$

- The *non-commutative* version of the *area-preserving mapping*

$$[\hat{X}, \hat{Y}] = i\theta$$

- The area-preserving can be satisfied by an unitary transformation using a non-commutative scalar field $\hat{\phi}$ as the generator

$$\hat{X}^a = e^{i\hat{\phi}}\hat{x}^a e^{-i\hat{\phi}}$$

- Tkachenko mode = non-commutative scalar field $\hat{\phi}(\hat{x})$.
- Under the global magnetic translation ($\hat{\phi}(\hat{x})$ transformed as a dipole symmetry)

$$e^{i\vec{P}\cdot\vec{c}}\;\hat{\phi} \to \hat{\phi} + \vec{\alpha}\cdot\hat{x} + \frac{1}{2}\vec{c}\cdot\vec{\nabla}\phi \qquad (c^i = \ell^2\epsilon^{ij}\alpha_j)$$

- Under the $U(1)$ transformation, $\hat{\phi}$ transformed as superfluid phase $\hat{\phi} \to \hat{\phi} + \lambda$
- **$\hat{\phi}$ is the shared Goldstone boson of magnetic translations and $U(1)$ global symmetries**

Moyal product

- Non-commutative field theory can be mapped to a familiar commutative field theory by replacing usual product by Moyal product

$$\hat{A}\hat{B} \to A\star B = A\exp\left(\frac{i}{2}\epsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)B = AB + \frac{i}{2}\theta\epsilon^{ij}\partial_i A\partial_j B + \cdots$$

- We trade non-commutativity for non-locality.
- Physics motivation: Since the coordinates do not commute, one can't define the position of operator, operators are blur objects.

Non-linear effective theory and Tkachenko decay rate

- Non-commutative field theory

$$X^a = e^{i\phi}\star x^a \star e^{-i\phi} = x^a + \theta D_i\phi, \qquad D_\mu\phi = -i\left(\partial_\mu e^{i\phi}\right)\star e^{-i\phi}$$

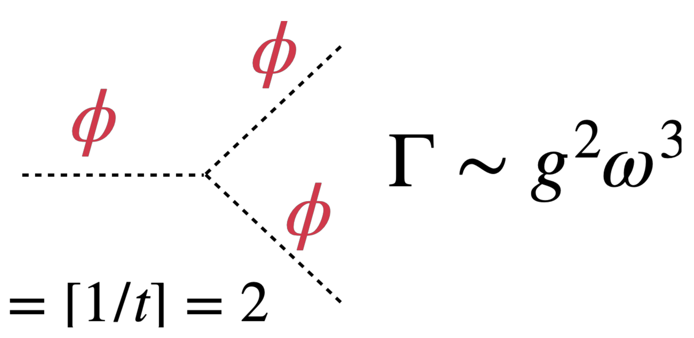
- The symmetry under magnetic translations constraints the action $\mathcal{L} = \mathcal{L}\left(D_t\phi, \partial_i D_j\phi\right)$
- Non-linear action up to cubic order of ϕ with also the rotational symmetry

$$S = \int dt\, d^2\mathbf{x}\left[\dot{\phi}^2 - c^2\left(\nabla^2\phi\right)^2 + g_1\dot{\phi}^3 + g_2\dot{\phi}\left(\nabla^2\phi\right)^2 + g_3\left(\nabla^2\phi\right)^3\right]$$

- Simple dimensional analysis

$$[t] = [x^2] = -2, \qquad [\omega] = [k^2] = 2 \to [g_i] = -2$$

Fixes the Beliaev decay of Tkachenko quanta



- **Tkachenko is a well defined excitation at low energy**

Summary and conclusion

- We developed a non-commutative field theory model for vortex lattice in the lowest Landau level limit.
- Nambu Goldstone bosons counting can be more complicated than the number of broken symmetry generators.
- Tkachenko is a NGB of a non commutative field theory.
- Tkachenko is a stable quasi particle $\tau \sim \omega^{-3}$.

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