Flat-band physics in the $S = 1/2$ sawtooth-chain systems

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Outline

1. Three-colorable lattices (observation of H.Changlani et al., 2018)
2. Flat band: one-magnon spectrum and flat-band states
3. $S = 1/2$ $XXZ0$ triangle and localized states
4. $S = 1/2$ $XXZ0$ sawtooth chain: Counting the number of localized states
5. Magnetothermodynamics in a weak magnetic field
   - analytical results
   - numerics (ED and FTL), comparison to other f.-b. $S = 1/2$ sawtooth chains
6. Conclusions

Three-colorable lattices (constructed of triangles, $H_{3c} \equiv H_{XXZ}[J^z = -1/2]$)

PHYSICAL REVIEW LETTERS **120,** 117202 (2018)

Editors' Suggestion

Macroscopically Degenerate Exactly Solvable Point in the Spin-1/2 Kagome Quantum Antiferromagnet

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Frustrated quantum magnets are a central theme in condensed matter physics due to the richness of their phase diagrams. They support a panoply of phases including various ordered states and topological phases. Yet, this problem has defied a solution for a long time due to the lack of controlled approximations which make it difficult to distinguish between competing phases. Here we report the discovery of a special *quantum* macroscopically degenerate point in the *XXZ* model on the spin-1/2 kagome quantum antiferromagnet for the ratio of Ising to antiferromagnetic transverse coupling $J_z/J = -1/2$. This point is proximate to many competing phases explaining the source of the complexity of the phase diagram. We identify five phases near this point including both spin-liquid and broken-symmetry phases and give evidence that the kagome Heisenberg antiferromagnet is close to a transition between two phases.

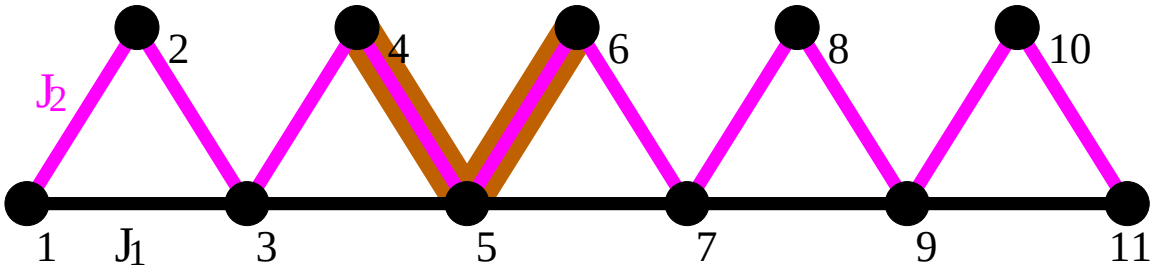
A special point in the $S = 1/2$ XXZ model on a kagome lattice is $\Delta = -1/2$:
It is proximate to <u>all</u> competing g.-s. phases and explains the g.-s. phase diagram including the isotropic case $\underline{\Delta = 1}$

- H.J.Changlani et al., Phys.Rev.B **99**, 104433 (2019)
- S.Pal et al., Phys.Rev.B **103**, 144414 (2021)
- G.Palle and O.Benton, Phys.Rev.B **103**, 214428 (2021)

At the special point the g.-s. degeneracy $\mathcal{W}(N) \propto \exp(\alpha N)$, any relation to flat-band business?

O.Derzhko and J.Richter, Finite low-temperature entropy of some strongly frustrated quantum spin lattices in the vicinity of the saturation field, Phys.Rev.B 70, 104415 (2004)

1D three-colorable lattice: $S = 1/2$ $XXZ0$ sawtooth-chain lattice



$$H = \sum_{j=1}^{\mathcal{N}} H_j = \sum_{j=1}^{\mathcal{N}} H_{\triangle_j}$$

$$H_j = J_1 \left(\mathbf{S}_{j,1} \cdot \mathbf{S}_{j+1,1}\right)_{\Delta_1} + J_2 \left(\mathbf{S}_{j,1} \cdot \mathbf{S}_{j,2}\right)_{\Delta_2} + J_2 \left(\mathbf{S}_{j,2} \cdot \mathbf{S}_{j+1,1}\right)_{\Delta_2} - h\left(S^z_{j,1} + S^z_{j,2}\right)$$

$$H_{\triangle_j} = J_1 \left(\mathbf{S}_{j,1} \cdot \mathbf{S}_{j+1,1}\right)_{\Delta_1} + J_2 \left(\mathbf{S}_{j,1} \cdot \mathbf{S}_{j,2}\right)_{\Delta_2} + J_2 \left(\mathbf{S}_{j,2} \cdot \mathbf{S}_{j+1,1}\right)_{\Delta_2} - \frac{h}{2} S^z_{j,1} - h S^z_{j,2} - \frac{h}{2} S^z_{j+1,1}$$

$(\mathbf{S}_n \cdot \mathbf{S}_m)_\Delta \equiv S^x_n S^x_m + S^y_n S^y_m + \Delta S^z_n S^z_m$ $\qquad$ $J_1 = J_2 = 1$, $\Delta_1 = \Delta_2 = -1/2$, $[S^z, H] = 0$

One-magnon spectrum:

$$E_k^{(1,2)} - E_0 = \left\{0, \frac{3}{2} + \cos k\right\},$$

Flat-band states:

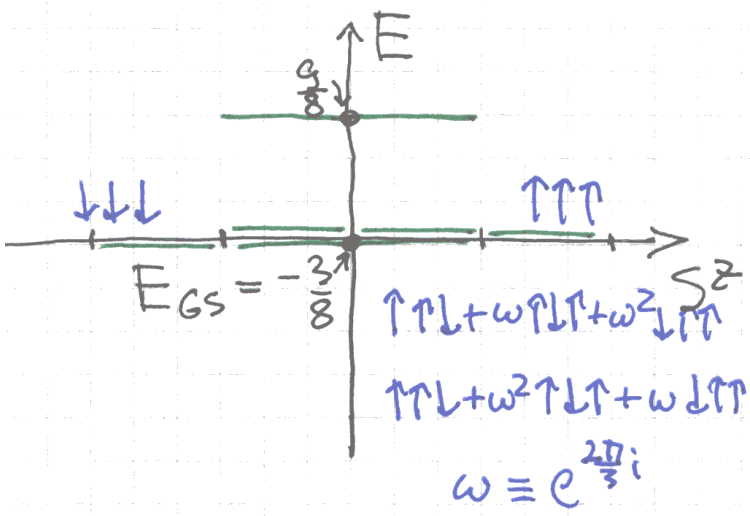
$$I_i|0\rangle, \ |0\rangle \equiv |\ldots\uparrow\ldots\rangle, \ I_i = \underbrace{S^-_{2i-2}}_{j-1,2} - \underbrace{S^-_{2i-1}}_{j,1} + \underbrace{S^-_{2i}}_{j,2}$$

HOW TO COUNT THE LOCALIZED GROUND STATES IN MANY-MAGNON SUBSPACES?

Single triangle eigenstates (a brick for lattices built of triangular motifs)

eight eigenstates of the $S = 1/2$ $XXZ0$ triangle

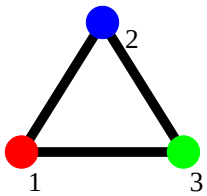
$$H_{123} = \frac{1}{2}\left(s_1^+ s_2^- + s_1^- s_2^+ - s_1^z s_2^z\right) + \frac{1}{2}\left(s_2^+ s_3^- + s_2^- s_3^+ - s_2^z s_3^z\right) + \frac{1}{2}\left(s_1^+ s_3^- + s_1^- s_3^+ - s_1^z s_3^z\right)$$

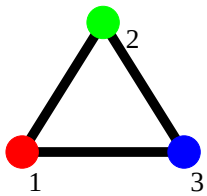


three-coloring states on each site

$$|r\rangle \equiv |\uparrow\rangle + |\downarrow\rangle, \quad |b\rangle \equiv |\uparrow\rangle + \omega|\downarrow\rangle, \quad |g\rangle \equiv |\uparrow\rangle + \omega^2|\downarrow\rangle; \quad \omega \equiv \exp\frac{2\pi i}{3}$$

$$|r_1 b_2 g_3\rangle = (|\uparrow\rangle + |\downarrow\rangle)_1 (|\uparrow\rangle + \omega|\downarrow\rangle)_2 (|\uparrow\rangle + \omega^2|\downarrow\rangle)_3$$
$$= |\uparrow\uparrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle + \omega|\uparrow\downarrow\uparrow\rangle + \omega^2|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle + \omega^2|\downarrow\uparrow\downarrow\rangle + \omega|\downarrow\downarrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle$$





$$H_{123}|r_1 b_2 g_3\rangle = -\frac{3}{8}|r_1 b_2 g_3\rangle, \quad H_{123}|r_1 g_2 b_3\rangle = -\frac{3}{8}|r_1 g_2 b_3\rangle$$

three-colorable lattice:

its vertices can be colored by three colors with no two connected vertices being assigned the same color

$S = 1/2$ $XXZ0$ sawtooth chain: Flat-band states = localized magnons – another point of view

open sawtooth chain of $N = 2\mathcal{N}_\triangle + 1$ sites (of $\mathcal{N}_\triangle$ triangles)

The six ground states of a triangle (with the energy $-3/8$):

$$|1\rangle = |\uparrow\uparrow\uparrow\rangle, \quad |2\rangle = |\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle, \quad |3\rangle = |\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle,$$
$$|5\rangle = |\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle, \quad |6\rangle = |\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle, \quad |8\rangle = |\downarrow\downarrow\downarrow\rangle$$

The localized magnons $l_i|0\rangle \equiv |l_i\rangle$, $i = 2, 3, \ldots, \mathcal{N}_\triangle$:

$$|l_2\rangle = (|\downarrow_2\uparrow_3\uparrow_4\rangle - |\uparrow_2\downarrow_3\uparrow_4\rangle + |\uparrow_2\uparrow_3\downarrow_4\rangle)\,|\ldots\uparrow\ldots\rangle;$$
$$H|l_2\rangle = \left(H_{123} + H_{345} + H_{567} + \ldots + H_{N-2,N-1,N}\right)|l_2\rangle$$
$$= \left(-\frac{3}{8} - \frac{3}{8} - \frac{3}{8} - \ldots - \frac{3}{8}\right)|l_2\rangle = -\frac{3}{8}\mathcal{N}_\triangle|l_2\rangle$$

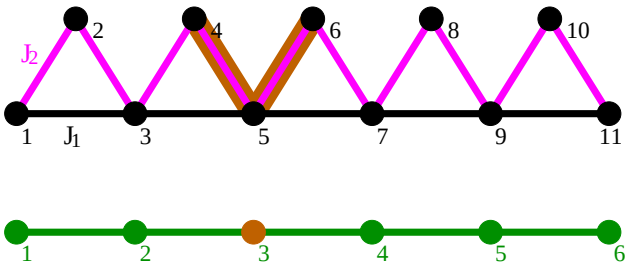
Two more states are localized on the left and right boundaries:

$$|l_1\rangle = (-|\downarrow_1\uparrow_2\uparrow_3\rangle + |\uparrow_1\downarrow_2\uparrow_3\rangle)\,|\ldots\uparrow\ldots\rangle;$$
$$H|l_1\rangle = \left(H_{123} + H_{345} + H_{567} + \ldots + H_{N-2,N-1,N}\right)|l_1\rangle$$
$$= \left(-\frac{3}{8} - \frac{3}{8} - \frac{3}{8} - \ldots - \frac{3}{8}\right)|l_1\rangle = -\frac{3}{8}\mathcal{N}_\triangle|l_1\rangle$$

$S^z = N/2 - 1$: $\mathcal{N}_\triangle + 1$ localized magnons are the ground states

$$g_{\mathcal{N}_\triangle}(1) = \mathcal{C}^0_{\mathcal{N}_\triangle} + \mathcal{C}^1_{\mathcal{N}_\triangle}$$

$S = 1/2$ $XXZ0$ sawtooth chain: $S^z = N/2 - 1$. Calculations



$$|I_3\rangle = |\uparrow_1\uparrow_2\uparrow_3\,(\downarrow_4\uparrow_5\uparrow_6 - \uparrow_4\downarrow_5\uparrow_6 + \uparrow_4\uparrow_5\downarrow_6)\,\uparrow_7\uparrow_8\uparrow_9\uparrow_{10}\uparrow_{11}\rangle$$

$$H = H_{123} + H_{345} + H_{567} + H_{789} + H_{9,10,11}$$

$$H_{123}|I_3\rangle = |\uparrow_1\uparrow_2\uparrow_3\ldots\rangle = -\frac{3}{8}|I_3\rangle$$

$$|\uparrow\uparrow\uparrow\rangle = |1\rangle$$

$$H_{345}|I_3\rangle = |\uparrow_1\uparrow_2\uparrow_3\,(\downarrow_4\uparrow_5\uparrow_6 - \uparrow_4\downarrow_5\uparrow_6 + \uparrow_4\uparrow_5\downarrow_6)\ldots\rangle = -\frac{3}{8}|I_3\rangle$$

$$|\uparrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle = -|2\rangle + |3\rangle, \quad |\uparrow\uparrow\uparrow\rangle = |1\rangle$$

$$H_{567}|I_3\rangle = |\ldots(\downarrow_4\uparrow_5\uparrow_6 - \uparrow_4\downarrow_5\uparrow_6 + \uparrow_4\uparrow_5\downarrow_6)\uparrow_7\ldots\rangle = -\frac{3}{8}|I_3\rangle$$

$$|\uparrow\uparrow\uparrow\rangle = |1\rangle, \quad -|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle = -|2\rangle$$

$S = 1/2$ $XXZ0$ sawtooth chain: $S^z = N/2 - 2$

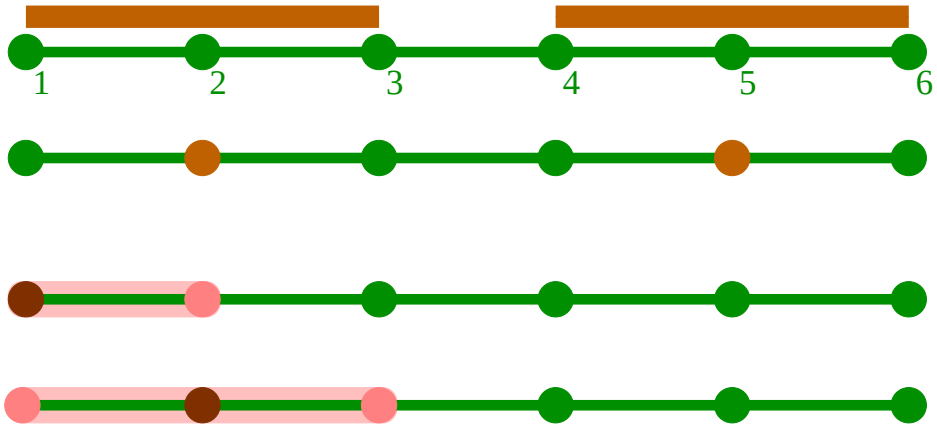
Ground states in the subspace with $S^z = N/2 - 2$?
Two independent localized magnons
AND localized two-magnon complexes!

$$I_1 \left(-\frac{1}{2} I_1 + I_2 \right) |0\rangle = \left(-S_1^- + S_2^- \right) \left(\frac{1}{2} S_1^- + \frac{1}{2} S_2^- - S_3^- + S_4^- \right) |0\rangle;$$

$$\left(H_{123} + H_{345} + H_{567} + \ldots + H_{N-2,N-1,N} \right) I_1 \left(-\frac{1}{2} I_1 + I_2 \right) |0\rangle = -\frac{3}{8} \mathcal{N}_\triangle I_1 \left(-\frac{1}{2} I_1 + I_2 \right) |0\rangle$$

$$I_2 \left(I_1 - \frac{1}{2} I_2 + I_3 \right) |0\rangle = \left(S_2^- - S_3^- + S_4^- \right) \left(-S_1^- + \frac{1}{2} S_2^- + \frac{1}{2} S_3^- + \frac{1}{2} S_4^- - S_5^- + S_6^- \right) |0\rangle;$$

$$\left(H_{123} + H_{345} + H_{567} + \ldots + H_{N-2,N-1,N} \right) I_2 \left(I_1 - \frac{1}{2} I_2 + I_3 \right) |0\rangle = -\frac{3}{8} \mathcal{N}_\triangle I_2 \left(I_1 - \frac{1}{2} I_2 + I_3 \right) |0\rangle$$



$$g_{\mathcal{N}_\triangle}(2) = \mathcal{C}^0_{\mathcal{N}_\triangle} + \mathcal{C}^1_{\mathcal{N}_\triangle} + \mathcal{C}^2_{\mathcal{N}_\triangle}$$

$S = 1/2$ $XXZ0$ sawtooth chain: $S^z = N/2 - 3$. 1

Ground states in the subspace with $S^z = N/2 - 3$?
Three independent localized magnons
Localized magnon + localized two-magnon complex
AND two types of localized three-magnon complexes!

one-bracket type three-magnon complex:

$$
\begin{aligned}
& I_2 I_4 \left(I_1 - \frac{1}{2} I_2 + I_3 - \frac{1}{2} I_4 + I_5 \right) |0\rangle \\
& = \left(S_2^- - S_3^- + S_4^- \right) \left(S_6^- - S_7^- + S_8^- \right) \\
& \times \left(-S_1^- + \frac{1}{2} S_2^- + \frac{1}{2} S_3^- + \frac{1}{2} S_4^- - S_5^- + \frac{1}{2} S_6^- + \frac{1}{2} S_7^- + \frac{1}{2} S_8^- - S_9^- + S_{10}^- \right) |0\rangle; \\
& \left(H_{123} + H_{345} + H_{567} + \ldots + H_{N-2,N-1,N} \right) I_2 I_4 \left(I_1 - \frac{1}{2} I_2 + I_3 - \frac{1}{2} I_4 + I_5 \right) |0\rangle \\
& = -\frac{3}{8} \mathcal{N}_{\triangle} I_2 I_4 \left(I_1 - \frac{1}{2} I_2 + I_3 - \frac{1}{2} I_4 + I_5 \right) |0\rangle
\end{aligned}
$$

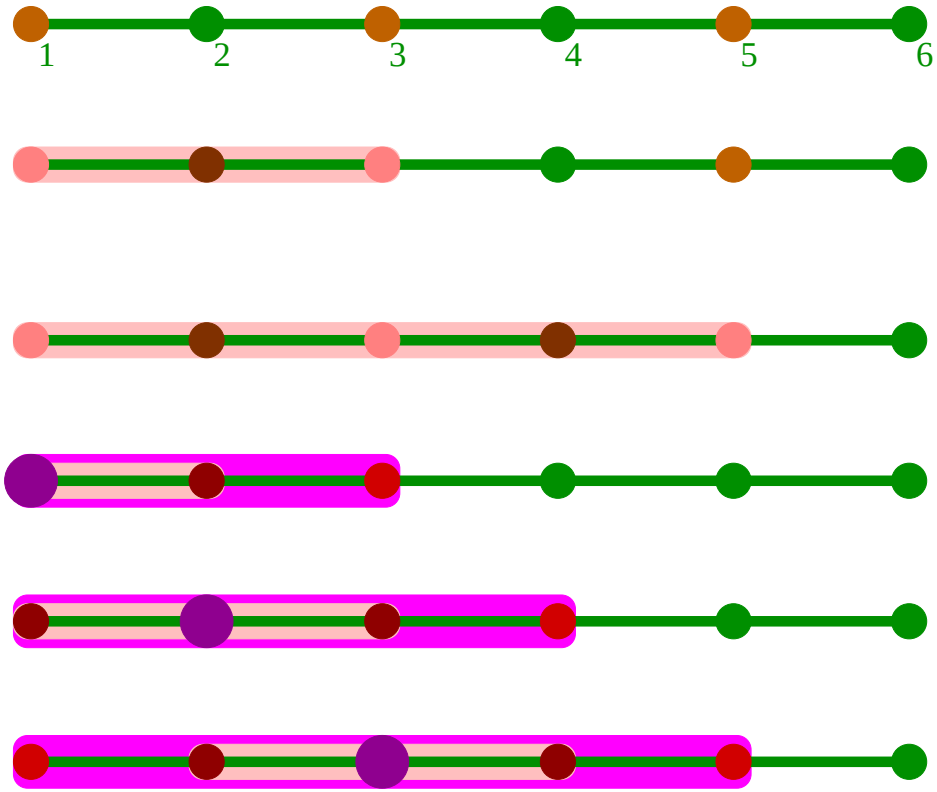
$S = 1/2$ $XXZ0$ sawtooth chain: $S^z = N/2 - 3.\ 2$

two-bracket type three-magnon complex:

$$
\begin{aligned}
I_1\left(-\frac{1}{2}I_1 + I_2\right)\left(\frac{1}{4}I_1 - \frac{1}{2}I_2 + I_3\right)|0\rangle + \frac{1}{8}I_1^3|0\rangle,\\
I_2\left(I_1 - \frac{1}{2}I_2 + I_3\right)\left(-\frac{1}{2}I_1 + \frac{1}{4}I_2 - \frac{1}{2}I_3 + I_4\right)|0\rangle + \frac{1}{8}I_2^3|0\rangle,\\
I_3\left(I_2 - \frac{1}{2}I_3 + I_4\right)\left(I_1 - \frac{1}{2}I_2 + \frac{1}{4}I_3 - \frac{1}{2}I_4 + I_5\right)|0\rangle + \frac{1}{8}I_3^3|0\rangle,\\
\vdots
\end{aligned}
$$

$$
\begin{aligned}
&\left(H_{123} + H_{345} + H_{567} + \ldots + H_{N-2,N-1,N}\right)\left(I_1\left(-\frac{1}{2}I_1 + I_2\right)\left(\frac{1}{4}I_1 - \frac{1}{2}I_2 + I_3\right)|0\rangle + \frac{1}{8}I_1^3|0\rangle\right)\\
&\qquad = -\frac{3}{8}\mathcal{N}_{\triangle}\left(I_1\left(-\frac{1}{2}I_1 + I_2\right)\left(\frac{1}{4}I_1 - \frac{1}{2}I_2 + I_3\right)|0\rangle + \frac{1}{8}I_1^3|0\rangle\right)\\
&\qquad \vdots
\end{aligned}
$$

$S = 1/2$ $XXZ0$ sawtooth chain: $S^z = N/2 - 3$. 3



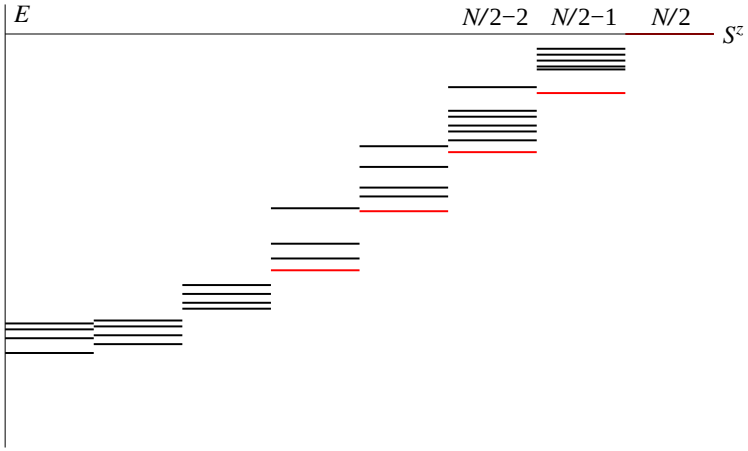
$$g_{\mathcal{N}_\triangle}(3) = \mathcal{C}^0_{\mathcal{N}_\triangle} + \mathcal{C}^1_{\mathcal{N}_\triangle} + \mathcal{C}^2_{\mathcal{N}_\triangle} + \mathcal{C}^3_{\mathcal{N}_\triangle}$$

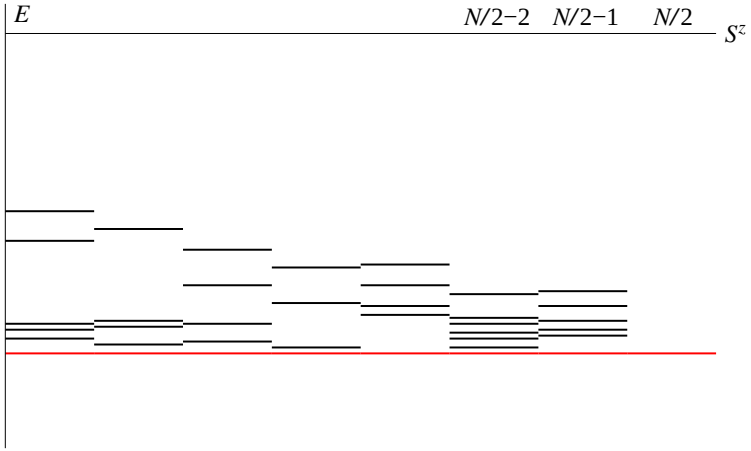
$S = 1/2$ $XXZ0$ sawtooth chain: Magnetothermodynamics. 1

For the subspace with $S^z = N/2 - k$, $k = 0, 1, 2, 3, \ldots, (N-1)/2$ we find

$$g_{\mathcal{N}_\triangle}(k) = \mathcal{C}^0_{\mathcal{N}_\triangle} + \mathcal{C}^1_{\mathcal{N}_\triangle} + \mathcal{C}^2_{\mathcal{N}_\triangle} + \ldots + \mathcal{C}^k_{\mathcal{N}_\triangle}$$

complete (numerics: up to $\mathcal{N}_\triangle = 19$, $k = 0, \ldots, 19$), linearly independent ground states





$J_1 = 1/2$, $J_2 = 1$, $\Delta_1 = \Delta_2 = 1$ (m.3, left) versus $J_1 = J_2 = 1$, $\Delta_1 = \Delta_2 = -1/2$ (m.1, right)

$S = 1/2$ $XXZ0$ sawtooth chain: Magnetothermodynamics. 2

The total degeneracy:

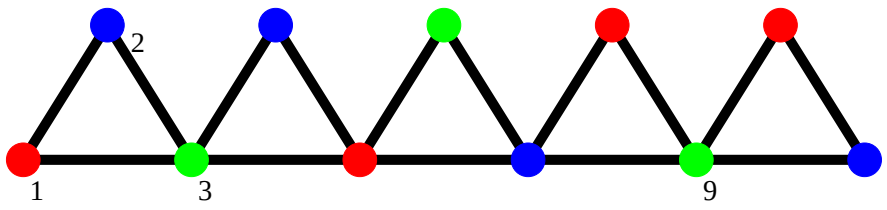
$$\mathcal{W}(N) = \sum g_{\mathcal{N}_\triangle}(k) = 2\sum_{k=0}^{\mathcal{N}_\triangle}\left(\mathcal{C}^0_{\mathcal{N}_\triangle} + \mathcal{C}^1_{\mathcal{N}_\triangle} + \mathcal{C}^2_{\mathcal{N}_\triangle} + \ldots + \mathcal{C}^k_{\mathcal{N}_\triangle}\right) = (\mathcal{N}_\triangle + 2)\, 2^{\mathcal{N}_\triangle}$$

For periodic chains of $N = 2\mathcal{N}$ sites, the total degeneracy:

$$\mathcal{W}(N) = \sum G_{\mathcal{N}}(k) = \left(\frac{\mathcal{N}}{3} + 1\right) 2^{\mathcal{N}} + \frac{2\mathcal{N}}{3} + 1$$

In both cases

$$\frac{\ln \mathcal{W}(N)}{N} \to \frac{1}{2}\ln 2 = 0.346\,573\,590\ldots$$



Counting the number of 3-colorings of the vertices of the open sawtooth-chain lattice yields

$$\mathcal{W} = 3 \cdot 2^{\mathcal{N}_\triangle} \to \exp\left(0.346\,573\,590\ldots N\right)$$

Boundary conditions become irrelevant in the thermodynamic limit $N \to \infty$

$S = 1/2$ $XXZ0$ sawtooth chain: Magnetothermodynamics. 3

Localized-states contribution dominates the partition function at low T and small h:

$$
\begin{aligned}
Z_{\rm fbs}(T,h,N) &= \mathrm{e}^{-\frac{E_0}{T}} \sum_{k=0}^{\mathcal{N}} g_{\mathcal{N}}(k) \left[\exp\left(\frac{(\mathcal{N}+\frac{1}{2}-k)h}{T} \right) + \exp\left(-\frac{(\mathcal{N}+\frac{1}{2}-k)h}{T} \right) \right] \\
&= \mathrm{e}^{-\frac{E_0}{T}} \sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} F_k(x,\mathcal{N}), \quad F_k(x,\mathcal{N}) = \frac{\sinh\left[(\mathcal{N}+1-k)x\right]}{\sinh\frac{x}{2}}, \quad x = \frac{h}{T}
\end{aligned}
$$

here $E_0 = -3\mathcal{N}/8$ and $\mathcal{N} = \mathcal{N}_{\triangle}$ for brevity

$$
F(T,h,N) = -T \ln Z(T,h,N), \quad M = -\frac{\partial F(T,h,N)}{\partial h}, \quad X = \frac{\partial M}{\partial h},
$$

$$
S = -\frac{\partial F(T,h,N)}{\partial T}, \quad C = T\frac{\partial S}{\partial T}
$$

$S = 1/2$ $XXZ0$ sawtooth chain: Universal behavior

the magnetization M

$$M_{\mathrm{fbs}}(x, N) = \frac{\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} \frac{\partial F_k(x,\mathcal{N})}{\partial x}}{\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} F_k(x,\mathcal{N})}$$

the susceptibility X

$$TX_{\mathrm{fbs}}(x, N) = \frac{\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} \frac{\partial^2 F_k(x,\mathcal{N})}{\partial x^2} - \left[\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} \frac{\partial F_k(x,\mathcal{N})}{\partial x}\right]^2}{\left[\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} F_k(x,\mathcal{N})\right]^2}$$

the entropy S

$$\begin{aligned} S_{\mathrm{fbs}}(x, N) &= \ln \sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} F_k(x,\mathcal{N}) - x \frac{\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} \frac{\partial F_k(x,\mathcal{N})}{\partial x}}{\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} F_k(x,\mathcal{N})} \\ &= \ln Z_{\mathrm{fbs}}(x, N) - x M_{\mathrm{fbs}}(x, N) \end{aligned}$$

the specific heat C

$$\begin{aligned} C_{\mathrm{fbs}}(x, N) &= x^2 \frac{\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} \frac{\partial^2 F_k(x,\mathcal{N})}{\partial x^2} - \left[\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} \frac{\partial F_k(x,\mathcal{N})}{\partial x}\right]^2}{\left[\sum_{k=0}^{\mathcal{N}} \mathcal{C}_{\mathcal{N}}^{k} F_k(x,\mathcal{N})\right]^2} \\ &= x^2 TX_{\mathrm{fbs}}(x, N) \end{aligned}$$

$S = 1/2$ $XXZ0$ sawtooth chain: Comparison to ED and FTL

obc, $N = 2\mathcal{N}_\triangle + 1$, $N = 17, 19, \ldots, 39$
pbc, $N = 2\mathcal{N}$, $N = 16, 20, \ldots, 32$

magnetization m, entropy s, susceptibility χ, specific heat c (per site) versus $x = h/T$

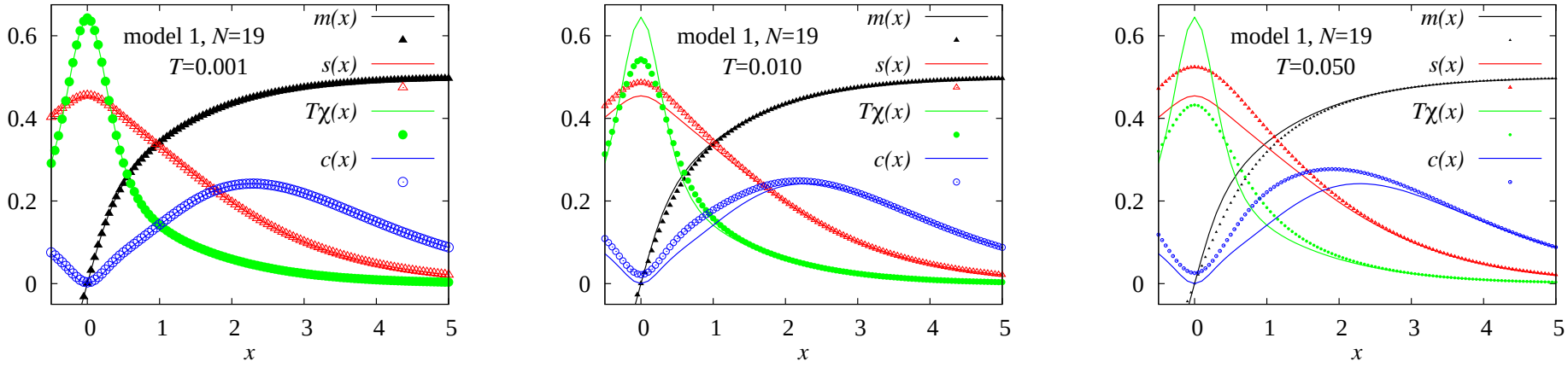


illustration of the region of T and h within which the universal behavior determined by $x = h/T$ emerges

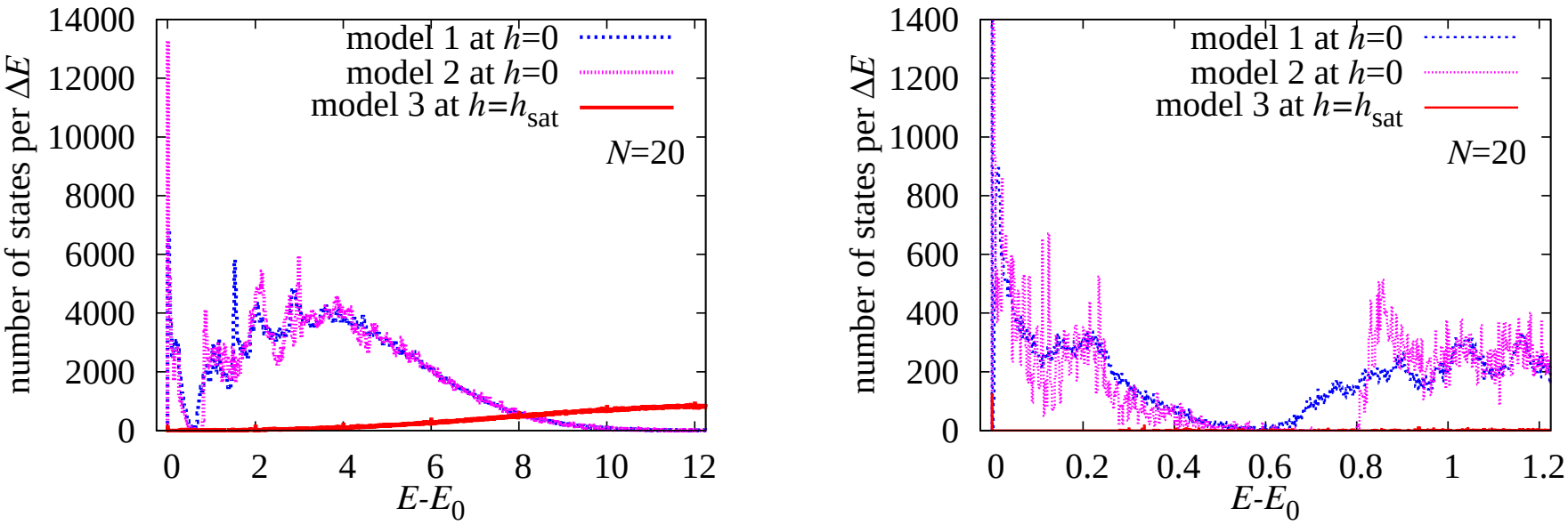
Flat-band $S = 1/2$ sawtooth chains: ED and FTL. 1

- $J_1 = J_2 = 1$, $\Delta_1 = \Delta_2 = -1/2$ [model 1, Eur.Phys.J.B **93**, 161 (2020) = this talk]
- $J_1 = 1/2$, $J_2 = -1$, $\Delta_1 = \Delta_2 = 1$ [model 2, Phys.Rev.B **90**, 014441 (2014)]
- $J_1 = 1/2$, $J_2 = 1$, $\Delta_1 = \Delta_2 = 1$ [model 3, Eur.Phys.J.B **52**, 23 (2006)]

S^z	k	$G^{(1)}(S^z)$	$\Delta^{(1)}(S^z)$	$G^{(2)}(S^z)$	$\Delta^{(2)}(S^z)$
9	1	10	0.5	10	1.0
8	2	45	0.026 996 110 0	45	0.021 776 745 4
7	3	121	0.011 213 200 0	120	0.000 484 876 3
6	4	220	0.005 858 780 0	210	0.000 013 213 8
5	5	297	0.002 110 250 0	252	0.000 000 197 4
4	6	332	0.002 256 090 0	252	0.000 000 064 1
3	7	341	0.003 116 320 0	252	0.000 000 064 1
2	8	342	0.003 828 620 0	252	0.000 000 035 8
1	9	342	0.003 247 900 0	252	0.000 000 007 5
0	10	343	0.003 792 860 0	253	0.000 000 007 5

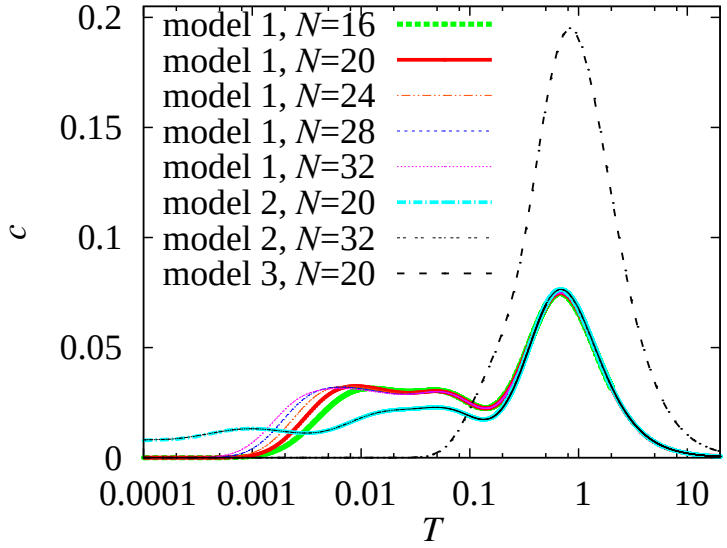
Ground-state degeneracies $G^{(i)}(S^z)$ and excitation gaps $\Delta^{(i)}(S^z) \equiv E_1^{(i)}(S^z) - E_0$ for the periodic sawtooth-chain model 1 ($J_1 = 1$, $i = 1$) and model 2 ($J_1 = 1/2$, $i = 2$) both of $N = 20$ sites ($\mathcal{N} = 10$) in different subspaces S^z. $E_1^{(i)}(S^z)$ is the energy of the lowest excitation in the subspace S^z and $E_0 = -3.75$ is the ground-state energy (which is identical for models 1 and 2). The results in the third and fifth columns coincide with the analytical predictions. For (finite) periodic sawtooth chains of $N = 20$ sites, the total number of ground states for model 1 is 4445 and for model 2 it is 3545.

Flat-band $S = 1/2$ sawtooth chains: ED and FTL. 2

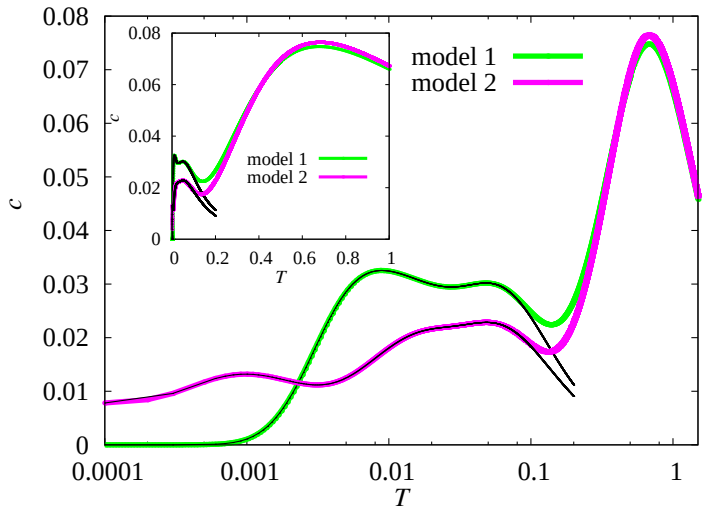


Density of states (histogram) for the models 1 and 2 at $h = 0$ (blue and magenta) and for model 3 at $h = h_{\mathrm{sat}}$ (red) for periodic chains of $N = 20$ sites (ED).
Left: Histogram bar width $\Delta E = 0.02$. Right: Histogram bar width $\Delta E = 0.002$, only low-energy part, where the y-axis is cut at 1400 to improve the visibility.

Flat-band $S = 1/2$ sawtooth chains: ED and FTL. 3



Specific heat (logarithmic temperature scale) for periodic model 1 ($J_1 = 1$) and model 2 ($J_1 = 1/2$) at $h = 0$ as well as for periodic model 3 ($J_1 = 1/2$) at $h = h_{\rm sat}$ ($h_{\rm sat} = 2$). Exact diagonalization data ($N = 16, 20$) and finite-temperature Lanczos data ($N = 24, 28, 32$; $R = 20$ for model 1 and $R = 10$ for model 2). Finite-size effects are small.



Specific heat $c(T)$ (exact diagonalization data, periodic chain $N = 20$) for model 1 ($J_1 = 1$) and model 2 ($J_1 = 1/2$) at $h = 0$: Green and magenta lines – exact data using the full spectrum, black lines – approximate data using only the low-energy spectrum below the quasi-gap. Main panel: Logarithmic temperature scale. Inset: Linear temperature scale.

Conclusions

$S = 1/2$ $XXZ0$ sawtooth chain (antiferro + ferro $\Rightarrow h = 0$)

- 3-colorable lattice but such an approach still needs further studies
- flat-band-spin-system approach can be used for constructing low-temperature magnetothermodynamics
- more localized states: localized magnons and localized n-magnon complexes
- ground-state manifold is well understood (lattice-gas description loses its powerfulness)
- universal behavior (thermodynamic quantities depend on h/T only) emerges at low temperatures and weak fields
- quasi-gap: ground-state manifold dominates the low-temperature thermodynamics in zero/weak magnetic field (ideas for low-lying excitations are in demand)

THANK YOU FOR YOUR ATTENTION