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Cavity resonances in Josephson ladders

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Electromagnetic waves which propagate along a Josephson junction ladder are shown to manifest themselves by resonant steps in the current-voltage characteristics. We report on experimental observation of resonances in ladders of different geometries. The step voltages are mapped on the wave dispersion relation which we derive analytically for the general case of a ladder of arbitrary anisotropy. Using the developed model, current amplitudes of the resonances are also calculated and their dependence on magnetic field is found to be in good accord with experiment. [S0163-1829(99)06121-4]

Josephson junction ladders have given rise to a great deal of interest in the past few years.[1–7] These quasi-one-dimensional (1D) structures are more complex than already well-understood 1D parallel Josephson junction arrays or long Josephson junctions. In contrast to the latter systems, a ladder contains small Josephson junctions in the direction transverse to the bias current (Fig. 1). The ladder can be viewed as the elementary row of a two-dimensional Josephson junction array which, in general, shows very complex dynamics. Thus, better understanding of electromagnetic properties of ladders may lead to new insight into the dynamics of underdamped 2D Josephson junction arrays.

For experimentally relevant modeling of Josephson ladders it is important to take into account magnetic field screening effects which are related to the finite inductances of elementary cells. Using the simplest model with only self-inductances taken into account, both static[2,3,6,7] and some of the dynamic[4–6] properties of ladders have been recently investigated. However, one of the basic characteristics of these systems such as the dispersion relation for small-amplitude waves remained unstudied until now. As in the case of long junctions and 1D parallel arrays, cavity resonances in underdamped ladders can be important as experimentally measurable ''fingerprints'' of their electrodynamic properties.

In this paper we report on the observation of cavity resonances in Josephson junction ladders with different anisotropy. We also derive the dispersion relation for linear waves in ladders which allows us to consistently explain the measurements and also interpret previously published data by other authors.

The measured ladders consist of Nb/Al-AlO_x/Nb underdamped Josephson tunnel junctions. We investigated both the annular geometry and the linear geometry ladders, sketched in Fig. 1. Each cell of a ladder contains four small junctions. The bias current I_{ext} is injected uniformly at every node via the external resistors. Here, we call *vertical* (JJ_V) the junctions placed in the direction of the external bias current, and *horizontal* (JJ_H) the junctions transverse to the bias. The ladder voltage is read across the vertical junctions.

We have measured current-voltage (I-V) characteristics of annular ladders with two different values of the ratio of the JJ_H's area S_H to JJ_V's area S_V. This ratio is referred in the following as anisotropy factor $\eta = I_{cH}/I_{cV}$, defined in terms of the junction critical currents. Our annular ladders have either $\eta = 0.5$ ($S_H = 16\ \mu m^2$, $S_V = 32\ \mu m^2$) or $\eta = 1$ ($S_H = S_V = 16\ \mu m^2$). The number of cells is $N = 12$, the cell size $A = 135\ \mu m^2$. The studied linear ladders have $\eta = 1$ ($S_H = S_V = 16\ \mu m^2$), $N = 15$, and $A = 140\ \mu m^2$. The discreteness of the ladder is expressed in terms of the parameter $\beta L = 2\pi I_{cV} L/\Phi_0$, where L is the self-inductance of the elementary cell of the ladder, I_{cV} is the critical current of the single vertical junction, and Φ_0 is the magnetic flux quantum. The cell inductance can be roughly estimated as L

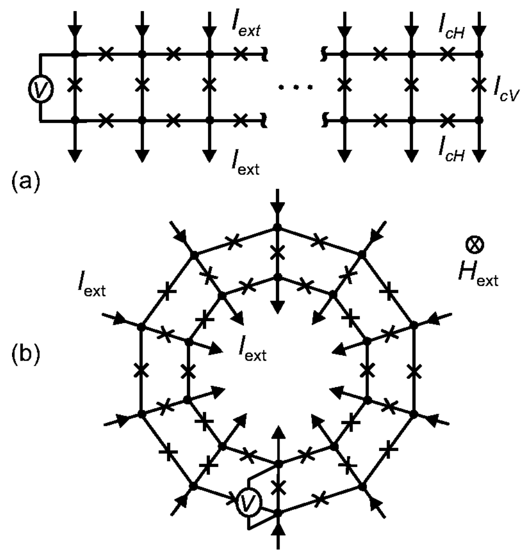

FIG. 1. Sketches of 1D Josephson junction ladders: (a) linear geometry; (b) annular geometry ($N = 10$).

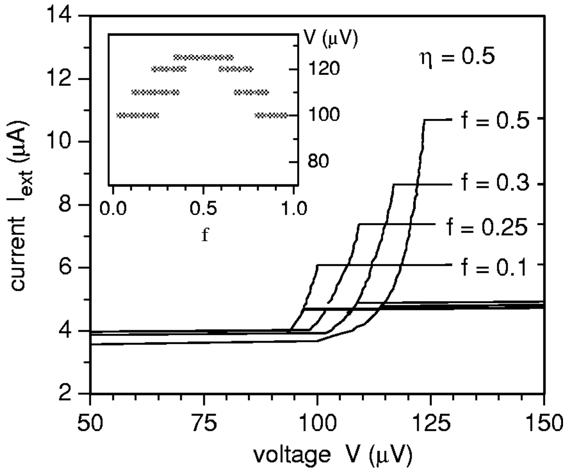

FIG. 2. Current-voltage characteristics of an annular Josephson ladder with anisotropy $\eta=0.5$ for different values of frustration f. Temperature $T=4.2$ K. Inset: Measured V_+ vs f at the same temperature; the horizontal branches indicate the step stability range.

$=1.25\mu_0\sqrt{A}$, where μ_0 is the magnetic permeability. At $T=4.2$ K, the values of β are between 0.1 and 0.2, depending on the vertical junction critical current. The damping of the ladder is given in terms of the junction McCumber parameter, defined as $\beta_c=2\pi I_{cV,H}R^2{}_{V,H}C_{V,H}/\Phi_0$. Here $R_{V,H}$ is the subgap resistance; $C_{V,H}$ is the junction capacitance, calculated from the Fiske modes in a long junction on the same chip ($C/S=3.4$ μF/cm^2). At $T=4.2$ K, typical values of β_c for annular ladders are around 200. The applied field H_{ext} is transverse to the cells plane and is expressed in terms of the frustration f, defined as the magnetic flux threading the cell normalized to Φ_0.

In the presence of frustration, the I-V curves of the ladders show steps with resonant behavior. These steps occur at fixed voltage positions and are split in two voltage domains denoted by V_+ and V_- (upper and lower voltage resonances). Figure 2 shows an enlargement of the I-V curves of the anisotropic annular ladder in the voltage region where the upper resonances V_+ appear. The curves were recorded at different values of frustration for $T=4.2$ K. At this temperature, in both the anisotropic and isotropic annular ladders, only four resonances V_+ are present in the region of frustration $0\leqslant f\leqslant 0.5$. A slight increase of the temperature leads to the appearance of the fifth resonance, at about $f=0.4$. Each step is located at a given voltage position and shows a resonant dependence of its magnitude on f (see the inset of Fig. 2). In contrast to 1D parallel arrays (no horizontal junctions, $\eta=\infty$), in ladders the reduction of the Josephson critical current due to frustration is rather small,[2] even in the case of low β. As a consequence of this, at any f the critical current of a ladder is always larger than the amplitude of the resonances, and the ladder switches from the zero voltage state directly to the gap voltage state. The only way to bias the ladder on one of the resonances is to follow the backward hysteretic branch of the I-V curve to the bottom part of the resonance. The voltage spacing between the higher steps is slightly reduced. Inversely, the current amplitude of the resonances increases with voltage, and the resonance at $f=0.5$ has the maximum height.

In both annular and linear ladders the voltage of the resonances is field dependent and approaches its maximum at $f=0.5$. We show these dependencies for isotropic and anisotropic annular ladders in Fig. 3 and for a linear ladder in Fig. 4. All curves are found to be nearly symmetric with respect to $f=0.5$. Thus, for the isotropic case we show data only up to $f=0.5$.

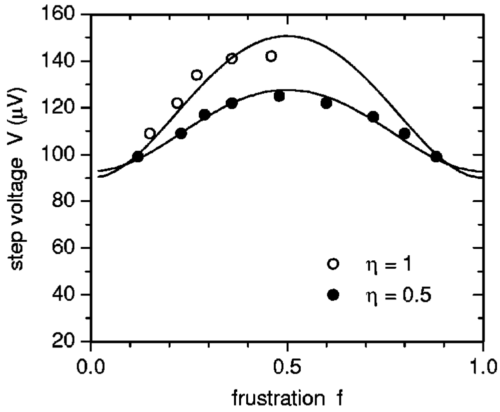

FIG. 3. Experimental (circles) and theoretical (solid lines) dependencies of the resonance voltages V_+ on the frustration f for two annular Josephson ladders. Results are shown for isotropic (open circles) and anisotropic (solid circles) ladders. Temperature $T\approx 5.7$ K. To fit the experimental data by Eq. (3), we have used the parameters $\eta=1$ with $\beta=0.16$ and $\eta=0.5$ with $\beta=0.3$.

In Fig. 4 the two resonances V_+ and V_- for the linear ladder are compared with the resonant step in a linear 1D parallel array. The 1D array has the same number of cells and cell and junction areas as the ladder. A small difference in their critical currents gives $\beta=0.12$ for the ladder, and $\beta=0.17$ for the array. The responses of the ladder and 1D array to the frustration are very different. In the 1D array, there is only one resonance (V_{PA}), and the voltage of this resonance follows the well known sin-like dependence on f. Above a critical value of frustration ($f\approx 0.1$) the resonance appears, by increasing f it moves to higher voltages, and at $f=0.5$ it saturates at the value $V_{PA}=135$ μV. The ladders, instead, have two branches V_- and V_+, which are confined in two different voltage regions. Both V_- and V_+ have the same f periodicity as V_{PA}. In contrast to the annular case, in

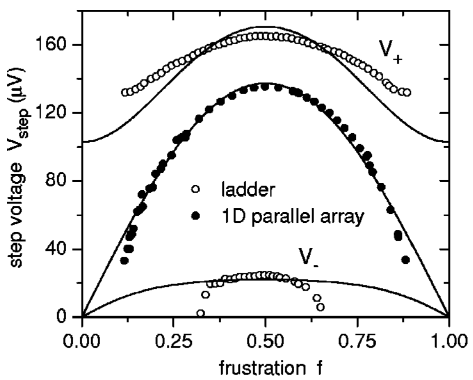

FIG. 4. Experimental (circles) and theoretical (lines) dependencies of the voltage position of two resonances $V_\pm(f)$ on the frustration for the linear isotropic ladder. The dependence of the resonance voltage position on the frustration for the parallel array ($\eta=\infty$) is also shown (solid circles).

the linear ladders V_- and V_+ were found to be continuously tuned by field in the ranges $0 \leqslant V_- \leqslant 25$ μV and $135 \leqslant V_+ \leqslant 165$ μV, respectively.

In the following, we present a theory of I-V resonances for a Josephson junction ladder. The theory allows us to predict the voltages and the current magnitudes of resonances observed in experiment.

We use the time-dependent Josephson phases of vertical φ_n and horizontal $\psi_{n1,2}$ junctions. The indices 1 and 2 refer to the lower and upper branches of horizontal junctions, and n is the index of the cell in the ladder. For the upper and lower horizontal junctions we use the symmetry condition that $\psi_{n1} = -\psi_{n2} = \psi_n$.[2,5,8] According to the derivation done in Refs. 2,8 for the isotropic case, we use the resistively shunted model for Josephson junctions and the usual analysis for superconductive loops with only self-inductances taken into account to derive a set of normalized dynamical equations for the phases φ_n and ψ_n for the anisotropic case

$$\ddot{\varphi}_n + \alpha\dot{\varphi}_n + \sin\varphi_n = \frac{1}{\beta}[2\psi_n - 2\psi_{n-1} + \varphi_{n-1} - 2\varphi_n + \varphi_{n+1}] + \gamma\ ,$$

$$\ddot{\psi}_n + \alpha\dot{\psi}_n + \sin\psi_n = \frac{1}{\eta\beta}[\varphi_n - \varphi_{n+1} - 2\psi_n] + \frac{2\pi f}{\eta\beta},$$

$$n = 1,\ldots,N. \quad (1)$$

Here, the unit of time is $\omega_p^{-1} = \sqrt{\hbar C_V/(2eI_{cV})}$, the inverse of plasma frequency of the ladder. The parameter $\alpha = 1/\sqrt{\beta_c}$ determines the damping in the ladder. We have used the fact that the anisotropy in typical Josephson circuits is realized by choosing different areas of horizontal and vertical junctions. Thus, the condition $I_{cH}/I_{cV} = C_H/C_V = R_V/R_H$ is assumed in Eq. (1). Note that, in this case, ω_p and α do not depend on anisotropy of the ladder. Finally, $\gamma = I_{\rm ext}/I_{cV}$ is the normalized bias current. In the finite voltage state we impose a whirling solution along the vertical junctions and oscillations with a small amplitude for the horizontal junctions.[9] Moreover, the phase of vertical junctions increases from cell to cell due to the presence of frustration. In this case, we quite naturally decompose the Josephson phases as follows:

$$\varphi_n = \omega t + 2\pi f n + \varphi e^{i(\omega t + 2\pi q n)},$$

$$\psi_n = \psi e^{i(\omega t + 2\pi q n)}, \quad (2)$$

where ω and q are the angular frequency and the wave number of the electromagnetic wave in the ladder. The time average Josephson current of vertical junctions is zero for this kind of a solution. In the limit of small amplitudes φ and ψ, we obtain the spectrum of electromagnetic wave propagating along the ladder. This spectrum consists of two branches $\omega_\pm(q)$ and is given by (in the usual units)

$$\omega_\pm = \omega_p\sqrt{F \pm \sqrt{F^2 - G}}, \quad (3)$$

where $F = \frac{1}{2} + 1/(\eta\beta) + (2/\beta)\sin^2(\pi q)$, and $G = (4/\beta)\sin^2(\pi q)$. This equation generalizes, for the case of the arbitrary wave number, the dynamical checkerboard state

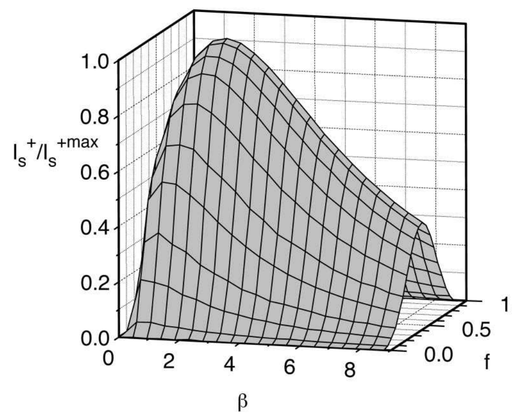

FIG. 5. Dependence of the maximum magnitude I_s^+ of the resonance V_+ on the self-inductance parameter β and frustration f for the isotropic ladder.

($q = 0.5$) considered in Ref. 5. The presence of the horizontal junctions leads to two branches in the spectrum of electromagnetic waves. Moreover, the maximum value of ω_+ is higher than that of 1D parallel array and the dependence $\omega_+(q)$ is more flat.[10]

Note, that in the case of another particular ground state ($\langle\varphi_n\rangle = \langle\psi_n\rangle = 0$) instead of Eq. (2), the spectrum (3) is substituted by $\omega_+ = \omega_p\sqrt{2F}$ and $\omega_- = \omega_p$. This dispersion relation is important, e.g., for understanding the radiation by fluxon moving in the ladder.

As is well known, the electromagnetic waves interact with the oscillating Josephson current and this effect leads to the resonances on the I-V curve.[11,12] More precisely, the resonance conditions are

$$q_0 = f, \quad V_\pm = \frac{\hbar\omega_\pm(q_0)}{2e}. \quad (4)$$

The possibility to observe these resonances on the I-V curve depends on their current amplitudes. In the same limit of small amplitudes of the Josephson phases φ and ψ and by making use of the method elaborated in Refs. 11,12, the magnitudes of the resonances are given by

$$I_s^\pm = NI_{cV}\frac{1}{\alpha}\frac{\pm\sqrt{F^2-G} - F + G}{\sqrt{F \pm \sqrt{F^2-G}}\sqrt{F^2-G}}. \quad (5)$$

The important result of this theory is that the amplitude of the resonance I_s^+ is small in the limits of both small and large β (see Fig. 5). Moreover, it has a maximum at $f = 0.5$. In the limit of very anisotropic ladder, when $\eta = \infty$, the magnitude of I_s^- of the resonance V_- decreases as $1/\eta$ and only the resonance V_+ can be observed. This is consistent with the case of 1D parallel arrays.

Using the developed theory, we can explain all important features of experimentally observed resonances on the current-voltage characteristic of Josephson ladders. First, due to the annular geometry only the electromagnetic waves with quantized wave numbers $q_n = n/N$ can propagate in the ladder. Here $n = 1, 2, \ldots, N$ is an integer. The resonances of sufficiently large magnitude are observed on the I-V curve when the value of frustration matches these wave numbers [see Eq. (4)].

The experimentally observed dependence of the voltage V_+ on the frustration f, shown in Fig. 3, is well described by Eq. (3) for both isotropic and anisotropic annular ladders. We obtain a good quantitative agreement between theory and experiment using the plasma frequency $f=\omega_p/2\pi=14$ GHz (independent measurements give the value of $\omega_p\approx16$ GHz) and the self-inductance parameters $\beta=0.16$ and 0.3, correspondingly, for isotropic and anisotropic cases. We also observe that the current magnitude of the resonances monotonically decreases when the frustration f deviates from the value 0.5 (see Fig. 2). It qualitatively agrees with the Eq. (5) for the maximum magnitude of the resonance.

In the case of annular ladder with $N=12$ cells, we may expect to observe six resonances in the region of $0\leqslant f\leqslant0.5$. In fact, we have observed only four stable resonances corresponding to the values of $n=2,3,4,6$ (see Fig. 2). The resonance corresponding to the value of frustration $f=1/12$ is, apparently, not stable due to its small current magnitude as described by Eq. (5). Another resonance corresponding to the frustration $f=5/12$ is not stable at $T=4.2$ K, but we observed it at slightly higher temperature. The poor stability of this resonance can be due to its small distance from the neighboring resonance at $f\approx1/2$.

Similar results have been obtained for Josephson junction ladder of linear geometry. We have observed two resonances V_+ and V_- in the upper and lower voltage regions. However, in this case, most probably due to a relatively low value of the subgap resistance (larger α) we have found that the dependence of their voltages on frustration is continuous (Fig. 4). Again, the resonance V_+ disappears in the limit of small frustration due to its small amplitude according to Eq. (5).

Finally, we have observed that the dependence $V_-(f)$ deviates from the theory [Eqs. (3) and (4)] in the region of frustration $f\leqslant0.3$. This can be connected with the particular assumption on the state of the horizontal junctions. Our analysis has been carried out for the most simple state ($\langle\psi_n\rangle=0$), but in general one might consider the case when the phases of the horizontal junctions undergo small amplitude oscillations around a finite angle. In this case, the resonance condition on the wave number q is not simply $q_0=f$, but a more complicated relationship can appear (see Ref. 9). The theoretical and experimental study of the properties of such general state, i.e., of the current-voltage characteristics of Josephson ladder in the region of small voltage, is in progress.

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