Directed Current due to Broken Time-Space Symmetry

S. Flach, O. Yevtushenko, and Y. Zolotaryuk
Max-Planck-Institute for the Physics of Complex Systems, Nöthnitzer Strasse 38, D-01187 Dresden, Germany
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We consider the classical dynamics of a particle in a one-dimensional space-periodic potential $U(X) = U(X + 2\pi)$ under the influence of a time-periodic space-homogeneous external field $E(t) = E(t + T)$. If $E(t)$ is neither a symmetric function of t nor antisymmetric under time shifts $E(t \pm T/2) \neq -E(t)$, an ensemble of trajectories with zero current at $t = 0$ yields a nonzero finite current as $t \to \infty$. We explain this effect using symmetry considerations and perturbation theory. Finally we add dissipation (friction) and demonstrate that the resulting set of attractors keeps the broken symmetry property in the basins of attraction and leads to directed currents as well.

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Transport phenomena are at the heart of many problems in physics. Nonlinear effects (as well as their quantized counterparts) may lead to many novel results in this area even for seemingly simple models (see, e.g., [1]). Well-known applications include the dynamics of Josephson junctions [2] and electronic transport through superlattices [3], to name a few. In the bulk of theoretical work on transport phenomena nonzero dc currents are obtained by applying time-dependent fields with nonzero mean. It is normally expected that the opposite case may not lead to a nonzero dc current. However, it has been also known for a long time that nonlinear dynamical systems may allow for the generation of ac fields from external dc fields (Josephson effect) and even vice versa [2]. Of course, what matters is a proper average over initial conditions, so that one has to ask whether there exist (or do not exist) sets of solutions which cancel their contribution to the total current. This question calls for an analysis of the symmetry properties of the system under consideration.

Let us make things more precise by considering a paradigmatic equation of the following type:

$$\ddot{X} + \gamma\dot{X} + f(X) + E(t) = 0\,. \qquad (1)$$

Functions f and E are bounded and periodic with period 2π and $T = 2\pi/\omega$, respectively, and have zero mean, and $\max(|f(X)|) \sim 1$. This equation describes, e.g., a particle moving in a periodic potential $U(X)$ with $f(X) = U'(X)$ in one space dimension under the influence of a periodic external field with friction [4]. It also may describe the current-voltage properties of a small Josephson junction under the action of a time-periodic external current (here X becomes the phase difference of the complex order parameter across the junction). This equation has been considered by numerous authors, however typically with harmonic functions f and E. We will show below that this choice induces symmetries which lead to zero total dc current. The purpose of this Letter is to demonstrate that a proper lowering of the symmetries of even $E(t)$ alone (still keeping its above defined properties) will lead to a nonzero dc current.

Dissipationless case $\gamma = 0$.—We first consider the case of zero friction $\gamma = 0$ in (1). In the limit of large velocities $|\dot{X}| \gg 1$, $f(X)$ can be neglected and the solution $X(t) = X_0 + P_0 t + \int_0^t dt' \int_0^{t'} E(t'')\,dt''$ has a bounded first derivative. Thus the time average over the velocity on a given trajectory is a well-defined nondiverging quantity.

To characterize the relevant symmetries of (1) we have to consider transformations in X, t which lead to a change of sign in P. These are (i) a reflection in $X \to -X$ and a shift in t or (ii) a shift in X and a reflection in $t \to -t$. We need first to characterize the relevant symmetries of $f(X)$ and $E(t)$. For that we expand f and E into a Fourier series: $f(X) = \sum_k f_k e^{ikX}$, $E(t) = \sum_k E_k e^{i\omega kt}$. Zero mean implies $f_0 = E_0 = 0$, and reality yields $f_k = f_{-k}^*$, $E_k = E_{-k}^*$ (A^* means complex conjugation). If $f(X) = U'(X)$ is antisymmetric after some appropriate argument shift $f(X + \mathcal{X}) = -f(-X + \mathcal{X})$ we call $f(X)$ possessing $\hat{f}_a$ symmetry. If $E(t)$ is symmetric after some appropriate argument shift $E(t + \tau) = E(-t + \tau)$ we call $E(t)$ possessing $\hat{E}_s$ symmetry. If $E(t)$ changes sign after a fixed argument shift (which trivially can be equal only to any odd multiple of $T/2$) $E(t) = -E(t + T/2)$, resulting in $E_{2k} = 0$, we call $E(t)$ possessing $\hat{E}_{sh}$ symmetry.

Now we can define the two relevant symmetry cases of (1) called $\hat{S}_a$ and $\hat{S}_b$ below. If functions $f(X)$ and $E(t)$ possess $\hat{f}_a$ and $\hat{E}_{sh}$ symmetries, respectively, then (1) is invariant under symmetry $\hat{S}_a$: $X \to (-X + 2\mathcal{X})$, $t \to t + T/2$. If function $E(t)$ possesses $\hat{E}_s$ symmetry, (1) is invariant under symmetry $\hat{S}_b$: $t \to (-t + 2\tau)$.

Given a trajectory $X(t; X_0, P_0)$, $P(t; X_0, P_0)$, with $X(t_0; X_0, P_0) = X_0$ and $P(t_0; X_0, P_0) = P_0$, the presence of any of the two symmetries $\hat{S}_a$, $\hat{S}_b$ allows one to generate new trajectories given by

$$\hat{S}_a\colon -X(t + T/2; X_0, P_0 + 2\mathcal{X}, \quad -P(t + T/2; X_0, P_0)\,, \qquad (2)$$

$$\hat{S}_b\colon\ X(-t + 2\tau; X_0, P_0), \qquad -P(-t + 2\tau; X_0, P_0)\,. \qquad (3)$$

Note that these transformations change the sign of the velocity P. Consequently the time average of P on the

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original trajectory will be opposite to the time averages of P on the generated new trajectories. There can be more symmetry operations generating other trajectories, but those will not change the sign of P and are thus not of interest here.

The dynamical evolution of (1) allows both for quasiperiodic solutions (cyclic in X for large P_0 and periodic in X for small P_0) and chaotic trajectories embedded in a stochastic layer [1]. Assuming that ergodicity holds in the stochastic layer we conclude that the average velocity will be one and the same for all trajectories of the layer. Since $\hat{S}_a$ and $\hat{S}_b$ when applied to a trajectory inside the layer generate again trajectories inside the layer, the presence of any of these symmetries implies that the time-averaged velocity of any trajectory in the layer will be zero. Note that we cannot obtain such a conclusion if both symmetries are absent. Indeed in Fig. 1 we show the long-time run $X(t)$ for a trajectory in the layer for several cases with and without symmetries $\hat{S}_a, \hat{S}_b$. While with $\hat{S}_a, \hat{S}_b$ we find zero average velocities, we observe that the loss of $\hat{S}_a, \hat{S}_b$ leads to a nonzero average velocity which is independent of the initial conditions but whose sign depends on the way the symmetry is broken. The dynamics is characterized by anomalous transport, i.e., by Lévy flights of different length interrupted by direction-changing perturbations. Nonzero current appears due to a desymmetrization between Lévy flights to the left and right, respectively. Especially trajectory 2 in Fig. 1 yields a nonzero velocity for a spatially symmetric $U(X)$.

To answer the question of how to invert the direction of a nonzero current in the stochastic layer, we note that considering the equation $\ddot{X} + f(X) + E(-t) = 0$ we arrive back at (1) by substitution $t' = -t$. So the current can be inverted by applying $E(-t)$ instead of $E(t)$ in (1). A second way is to consider equation $\ddot{X} - f(-X) - E(t) = 0$ which after substitution $X' = -X$ again is mapped onto (1). Thus another way of inverting the current is to apply $-f(-X)$ instead of $f(X)$ and $-E(t)$ instead of $E(t)$ in (1). There is no simple way to invert the current by just inverting space, i.e., by considering $f(-X)$.

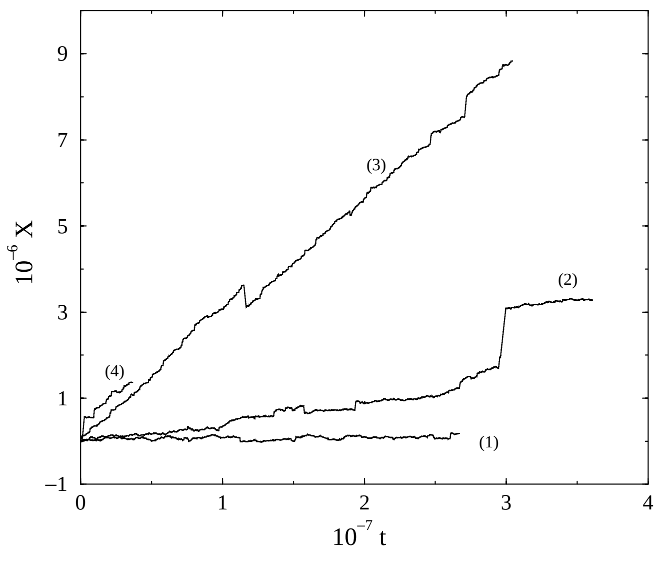

FIG. 1. Dependence $X(t)$ versus t for different realizations of (1) and $\gamma = 0$ with $f(X) = \cos X + v_2 \cos(2X + 0.4)$, $E(t) = E_1 \sin(\omega t) + E_2 \sin(2\omega t + 0.7)$ and $\omega = 2.4$. (1): $v_2 = 0$, $E_1 = -2.3$, $E_2 = 0$; (2): $v_2 = 0$, $E_1 = -2.3$, $E_2 = -1.38$; (3): $v_2 = 0.6$, $E_1 = -2.3$, $E_2 = -1.38$; (4): same as (3) but with $f(-X)$ instead of $f(X)$. Note that in this case the direction of the current is not inverted as explained in the text.

To get a grasp of this result we consider the quasiperiodic cyclic regime for $U(X) = -\cos X$ and $E(t) = E_1 \cos\omega t + E_2 \cos(2\omega t + \alpha)$. Note that $\hat{S}_a$ symmetry is present if $E_2 = 0$ or $E_1 = 0$ and $\hat{S}_b$ symmetry is present if $\alpha = 0, \pi$ or $E_1 = 0$ or $E_2 = 0$. Each individual trajectory for sufficiently large P_0 gives a nonzero average velocity. The question is whether we obtain a nonzero velocity after averaging over initial conditions with some distribution function $\rho(X_0, P_0, t_0)$ reflecting equilibrium properties, at least, of course, $\rho(X_0, P_0, t_0) = \rho(X_0, -P_0, t_0)$. Here t_0 is the time when the trajectories with initial conditions X_0, P_0 are started. In the simplest case we might assume that ρ is independent of t_0. Consider the case $P_0 \gg 1$ and $\omega \gg P_0$. In that case we can separate the solution $X(t)$ into a slow part $X_s(t)$ and a small fast part $\xi(t)$. Expanding to linear order in the fast variable yields

$$\ddot{X}_s + \ddot{\xi} - \sin X_s - \cos(X_s)\xi + E(t) = 0\,. \tag{4}$$

Collecting the fast variables we find $\ddot{\xi} - \cos(X_s)\xi + E(t) = 0$. This equation has to be solved by assuming that X_s is constant and skipping the slow homogeneous solution part. We find $\xi = A_1 \cos\omega t + A_2 \cos(2\omega t + \alpha)$ with $A_1 = -E_1/[\omega^2 - \cos X_s]$ and $A_2 = -E_2/[4\omega^2 - \cos X_s]$. Final averaging over the fast variables in (4) gives $\ddot{X}_s - \sin X_s = 0$. The crucial point is to observe that the initial condition is now $X_0 = X_s(t_0) + \xi(t_0)$, $P_0 = \dot{X}_s(t_0) + \dot{\xi}(t_0)$. Since $\xi(t)$ is a completely defined function, defining the initial conditions for X, P we obtain initial conditions for the slow variables. The symmetry breaking will be hidden there. Indeed, averaging over time we find $\langle P(t)\rangle = \langle \dot{X}_s(t)\rangle$. Assuming, e.g., large values of P_0 the time-average velocity of the slow variable will be simply $\langle \dot{X}_s(t)\rangle = \mathrm{sgn}(P_0)\sqrt{2H_s}\,[1 - 1/(4H_s^2) + 0(P_0^{-8})]$ with $2H_s = P_s^2 - 2\cos X_s$. Expanding $\langle \dot{X}_s(t)\rangle$ in powers of $1/P_0$ we will encounter terms $P_0^{-6}\dot{\xi}^3(t_0)\cos^2[X_0 - \xi(t_0)]$. Averaging over X_0 and t_0 we obtain in leading order for the average velocity

$$-\sqrt{2}\,\frac{25}{32}\,\frac{1}{P_0^6}\,\frac{E_1^2 E_2}{\omega^3}\sin\alpha \tag{5}$$

which remains nonzero and will contribute to an average nonzero current after further averaging over P_0. Note that the directed current disappears if $E_1 = 0$ or $E_2 = 0$ or $\alpha = 0, \pi$ when the mentioned symmetries are restored. The current direction is defined in this perturbation limit by the sign of the product $E_2 \sin\alpha$. Finally in the limit $P_0 \rightarrow \infty$ the current amplitude tends to zero, although the symmetries are not restored. The reason is that in this limit we recover the problem of a free particle moving under the influence of an external field $E(t)$ which can be easily solved [5]. Averaging over t_0 in this case yields zero total current. It follows that nonzero total currents occur if symmetries $\hat{S}_a$ and $\hat{S}_b$ are violated and if we provide a mechanism of mixing of different harmonics as it happens in nonlinear equations of motion (see also [6]).

We checked the above statements of the perturbation theory for the quasiperiodic regime by computing numerically the average velocity $\langle \dot{X}_s \rangle$ for two initial conditions with opposite initial velocities $\pm P_0$, taking their half sum, and finally averaging over all possible initial positions X_0 and over the initial time t_0. We observe a nonzero current except for the symmetric values of α. Finally we did the same direct computation in the initial equation (1). The results are similar.

In order to keep the dc current nonzero the value of α should be kept fixed with time, or at least to be allowed to fluctuate only with small amplitude. Additional averaging over α will lead to a disappearance of the dc current. To our understanding this should not pose a technical difficulty, since one can take a monochromatic field source, and then experimentally generate a second harmonic from it such that the phase α is fixed.

The case with dissipation.—Consider now a small but nonzero value of γ in (1) (see [7]). Generically the phase space of the system will separate into basins of attraction of low-dimensional attractors. There exist strong hints that when being close to the Hamiltonian case these attractors will be periodic orbits or limit cycles (cyclic in X) [8]. The stochastic layer is transformed into a complex transient part in phase space, where the basins of attraction of different limit cycles are entangled in a complicated way. For stronger deviations from the conservative limit the periodic attractors undergo (period doubling) bifurcations, and finally possibly chaotic attractors are generated, which are however not directly related to the stochastic layer of the conservative limit (see also [1]).

Of the two symmetries $\hat{S}_a$, $\hat{S}_b$ in the conservative case only $\hat{S}_a$ may survive for nonzero dissipation. Consider such a case when (2) holds. Suppose we find a limit cycle which is characterized by $X(t + T) = X(t) + 2\pi m$ and $P(t + T) = P(t)$, $m \in Z$. Because of the external time-periodic field $E(t)$ we have $T = n2\pi/\omega$, $n \in Z$. The average velocity $\langle P \rangle = \frac{1}{T} \int_0^T \dot{X}\, dt$ on such a cycle will be given by $\langle P \rangle = \omega m/n$. Because of the required symmetry there will also be a limit cycle with $\langle P \rangle = -\omega m/n$. Moreover the symmetry presence also implies that the basins of attraction of the two symmetry related limit cycles are equivalent.

Assume now that we violate $\hat{S}_a$. The two cycles previously related by symmetry to each other will generically continue to exist, but there is no obvious symmetry which relates them to each other. However after computing the average velocities, we will still find that they equal each other up to a sign. The symmetry breaking is, in fact, hidden in a *desymmetrization* of the two basins of attraction. It is this asymmetry which after averaging over initial condition distributions (symmetric in P) will lead to a different number of particles attracted to both cycles and thus to a nonzero current. To observe the desymmetrization of the basins locally we may tune some parameter of the equation to such a value that one of the cycles becomes unstable. In that case its basin of attraction shrinks to zero and disappears. If the other (previously symmetry related) cycle will still be stable, i.e., if its basin of attraction still exists, the asymmetry in the basins becomes obvious—one of them completely disappeared, and the other one still exists. We tested these predictions and found complete agreement. We used

$$f(X) = \sin X + v_2 \sin(2X + 0.4)\,, \tag{6}$$

$$E(t) = E_1 \sin\omega t + E_2 \sin(2\omega t + 0.7) \tag{7}$$

with $\gamma = 0.005$ and $\omega = 1.1$. The two symmetry related limit cycles ($n = 1$ and $m = \pm 1$) have been computed with a Newton method (see, e.g., [9]) for $v_2 = E_2 = 0$, $E_1 = -2.0$. Then the parameters were changed to $v_2 = 0.02$, $E_1 = -2.017$, and $E_2 = -0.06051$, and the two limit cycles were traced again with a Newton method. Finally the eigenvalue problem (3×3 matrix) of the linearized phase space flow around each of the cycles has been evaluated in order to check the stability (see [9] for details). For the given parameter values the $m = -1$ cycle is stable (all Floquet eigenvalues inside the unit circle) while the $m = 1$ cycle is unstable (one Floquet eigenvalue is outside the unit circle).

To observe the effect of asymmetry of basins of attraction globally, we computed the ensemble averaged velocity for a distribution of initial conditions in the phase space of (1) with forces (6) and (7). The distribution was uniform in X and t_0 (40 points on the interval from 0 to 2π for each of them) and 2×20 points symmetrically chosen on the P axis according to a Maxwell distribution with inverse dimensionless temperature $\beta = 0.01$. In total 64 000 trajectories have been computed. The velocity per trajectory averaged over the whole set of trajectories is shown in Fig. 2 as a function of time for the case with $\hat{S}_a$ symmetry (curve 1) and the one without $\hat{S}_a$ symmetry (curve 2). While the first case gives zero current density as $t \rightarrow \infty$, the second case yields nonzero negative current density in this limit.

In order to invert the direction of a nonzero total current we have to apply $-f(-X)$ instead of $f(X)$ and $-E(t)$ instead of $E(t)$ in (1). In contrast to the dissipationless case we cannot just invert time in $E(t)$ but have to perform a combined transformation both in space and time. Taking just $f(-X)$ or $E(-t)$ may or may not lead to a change of the current direction. Recall that directed currents can be

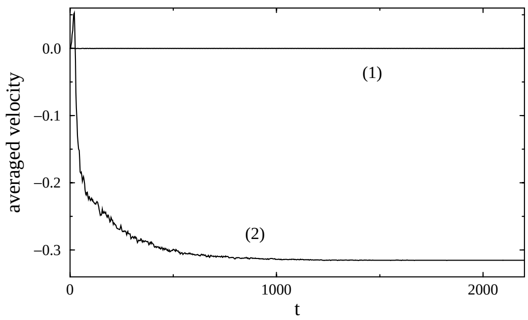

FIG. 2. The averaged velocity (see text) as a function of time for (1) with $\gamma = 0.1$, $\omega = 2.4$, and $E_1 = -5.23$. (1): symmetric case, $v_2 = E_2 = 0$; (2): asymmetric case, $v_2 = 0.6$, $E_2 = -5.23$.

generated by keeping $U(X) = U(-X)$ and lowering the symmetry in $E(t)$ only. In that case the current direction is inverted by applying $-E(t)$.

There exist a lot of publications on the properties of (1) with $\gamma = 0$ (and similar equations reduced to discrete maps), however we did not find studies of such a system when both symmetries $\hat{S}_a$ and $\hat{S}_b$ are broken. Evidently, when taking f and E with only one harmonic, no symmetry broken transport is possible. The closest study in this respect we found in [10], where however, as explicitly stated, the symmetry was kept, leading to zero current when averaging over all possible trajectories. The overdamped case was studied in [11].

Finally we want to discuss the relation of our results to the well-known case of directed currents for particles moving in so-called ratchet potentials under the influence of friction and a stochastic force (see [12], and references therein). These potentials lack inversion symmetry in space and thus lack $\hat{f}_a$ symmetry (see above). However the noise process characterizing the stochastic force has to be nonwhite (see [13] for details). It was then found that proper correlations in the noise allow for directed currents even in the presence of $\hat{f}_a$ symmetry, i.e., for "nonratchet" potentials. In [14] these equations have been modified by adding time-periodic fields. Note that our model allows for an easy treatment of the symmetry analysis, since the symmetry breaking is not hidden in higher order moments of distribution functions.

If we consider corresponding quantum systems, the symmetry breaking will be reflected in the properties of the eigenstates, and nonzero currents can be expected as well. The addition of, e.g., particle-particle interaction or noise can affect only the amplitude of the current, since the broken symmetries cannot be restored by additional interactions. Applications of similar ideas to coherent photocurrents in semiconductors have been reported in [15,16]. Further applications may include driven Josephson junctions or superlattices, electrons in time-dependent magnetic fields, to name a few. Note that it should be much easier to realize experimentally our proposed symmetry breaking rather than to prepare correlated noise as proposed for ratchet transport.

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