Dynamical mechanisms of dc current generation in driven Hamiltonian systems

S. Denisov and S. Flach
Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Strasse 38, D-01187 Dresden, Germany

(Received 25 July 2001; published 30 October 2001)

Recent symmetry considerations [Flach *et al.*, Phys. Rev. Lett. **84**, 2358 (2000)] have shown that dc currents may be generated in the stochastic layer of a system describing the motion of a particle in a one-dimensional potential in the presence of an ac time-periodic drive. In this paper we explain the dynamical origin of this current. We show that the dc current is induced by the presence and desymmetrization of ballistic channels inside the stochastic layer. The existence of these channels is due to resonance islands with nonzero winding numbers. The characterization of the flight dynamics inside ballistic channels is described by distribution functions. We obtain these distribution functions numerically and find very good agreement with simulation data.

DOI: 10.1103/PhysRevE.64.056236 PACS number(s): 05.45.Ac, 05.60.Cd

Transport in driven systems has received widespread interest for several years because of its potential applicability to nonequilibrium processes [1]. One canonical model reduces to the motion of a particle in a one-dimensional space-periodic potential in the presence of friction and a time-dependent stochastic force $\chi(t)$. If $\chi(t)$ contains correlations, a nonzero current may be realized even in the case of zero average $\langle \chi(t)\rangle=0$ [1]. Despite an enormous accumulation of results in this area [2] we are still lacking a full understanding of the *microscopic* mechanisms of current occurrence. If such an understanding is realizable, it should make use of the true dynamical evolution of the system rather than of properties of equations for probabilities. A first step in this direction requires separation of the essential time correlations from the pure Gaussian white noise in $\chi(t)$. The simplest way is to assume that $\chi(t)=E(t)+\xi(t)$ where $E(t)=E(t+T)$ is a time-periodic function with zero mean $\langle E(t)\rangle=0$ and $\xi(t)$ is a Gaussian white noise term.

The next step is to skip the $\xi(t)$ term which leaves us with a regular dynamical problem. In [3] such a case was considered and the relevant space-time symmetries of the dynamical problem were obtained. It was shown that a breaking of those symmetries leads to a nonzero dc current. The mechanism of current occurrence for the dissipative case was identified with a desymmetrization of attractor basins. In contrast, the nondissipative (Hamiltonian) case is much less well understood [4]. A time-dependent Hamiltonian system is usually nonintegrable [5]. A strong dc current component was found in the corresponding stochastic layer of such a system [3]. While its presence or absence was clearly connected to the above mentioned absence or presence of symmetries, the dynamical nature of directed transport in the stochastic layer is not fully understood. The importance of this understanding can be seen from, e.g., the results in [6], where kinetic equations for probability functions were studied. In particular, it was found that the approaching of the Hamiltonian (dissipationless) limit leads to an increase of the dc current value by 2–3 orders of magnitude. Thus the description of the dynamical mechanisms of directed current generation in the stochastic layer of a driven Hamiltonian system will provide very useful information for dissipative systems as well.

Let us consider the canonical example of a particle moving in a spatially periodic nonlinear potential $U(x)=-\cos x$ under the influence of a time-periodic zero-mean force $E(t)$. The Hamiltonian and the equation of motion are given by

$$H=\frac{p^2}{2}-\cos x-xE(t),\quad \ddot{x}=-\sin x+E(t). \tag{1}$$

Here p and x are the canonically conjugated momentum and coordinate and $\ddot{x}\equiv d^2x/dt^2$.

We restrict our consideration to the choice

$$E(t)=E_1\cos(t)+E_2\cos(2t+\phi). \tag{2}$$

According to [3] for $E_2\neq 0$ and $\phi\neq 0,\pi$ all possible symmetries that yield zero dc current are broken. Note that the phase space dimension d of Eq. (1) is $d=3$.

In the case of a nonzero field $E(t)$ the phase space of Eq. (1) is characterized by the presence of a stochastic layer which originates from the destroyed separatrix of the undriven system [5]. For $\phi=0,\pi$ this layer is invariant under the transformation $(p\to -p,t\to -t,x\to x)$. At the same time the average velocity for any trajectory in this layer vanishes, so we find zero dc current. The symmetry will be broken when tuning ϕ away from the values $(0,\pi)$. The stochastic layer will deform. Most importantly any trajectory in the layer will then be characterized by a nonzero value of the average velocity. Due to ergodicity inside the layer this value will be unique for all trajectories from the layer. While the fact that it may become nonzero is understandable using symmetry analysis, its appearance and magnitude are due to dynamical mechanisms of motion inside the stochastic layer. In this paper, we show that the dc current is induced by the presence and desymmetrization of ballistic channels inside the stochastic layer. The existence of these channels is due to resonances. The characterization of the realization of flights inside ballistic channels is described by distribution functions. We obtain these distribution functions numerically and find very good agreement with simulation data.

System (1) has a mixed phase space, which contains chaotic areas and regular resonance islands [7]. These islands

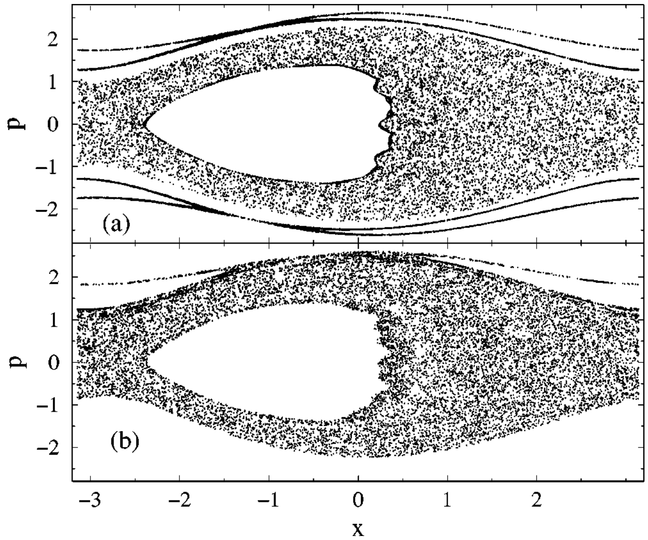

FIG. 1. Poincaré map for (a) $\phi=0$ and (b) $\phi=\pi/2$.

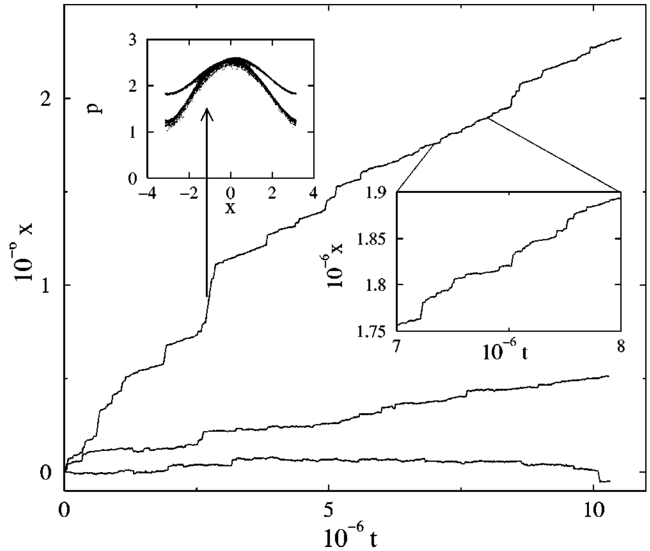

FIG. 2. $x(t)$ for $\phi=0,\pi/5,\pi/2$ (lower, middle, and upper curves, respectively). Left upper inset: Poincaré map for ballistic flight with $\phi=\pi/2$ as indicated by arrow. Right inset: enlargement of $x(t)$ for $\phi=\pi/2$.

are impermeable for chaotic trajectories and, at a first glance, may be excluded from the consideration of the phase space flow inside the stochastic layer. In reality the phase space topology inside the stochastic layer is very complex precisely at the boundary between chaotic and regular regions [7]. Close to resonances the stochastic layer shows a hierarchical set of cantori, which form partial barriers for a trajectory from the layer. Due to the presence of these barriers a chaotic trajectory can be trapped for a long time near a regular island resonance. This trapping or sticking effect leads to the appearance of strongly nonergodic episodes during the overall chaotic motion. Regular islands are characterized by a corresponding rational winding number $\omega=\Delta x/T$ which defines the distance Δx traveled during one period $T=2\pi$ of the drive. If the winding number ω is nonzero, the corresponding sticking episode of the chaotic trajectory is a ballisticlike unidirectional flight. For $\omega=0$ the sticking episode corresponds to trapped oscillations.

Thus the complicated evolution of a trajectory in the stochastic layer can be subdivided into several parts [7]. The first one is a fast diffusion in the bulk of the layer, while the other ones are stickings to the above mentioned regular islands and correspond to propagation in ballistic channels. The switching from the diffusion process to a ballistic flight will be described by some probability distribution. The same will be true for the actual residence or sticking time inside a given channel. We will show for the cases studied that the fast diffusion alone is not capable of explaining the observed dc current. The main point is that the leading mechanism of current generation in the *stochastic layer* is related to the desymmetrization of the strongly *nonstochastic* part of the overall stochastic dynamics inside the layer. Note that our kinetic energy choice $p^2/2$ in Eq. (1) implies that the stochastic layer is bounded in p, so we will always expect ballistic channels to appear.

Let us study the case of weak driving $E_1=0.252$ and $E_2=0.052$. A Poincaré map of the phase space flow for $\phi=0$ is shown in Fig. 1(a) [8]. The main stochastic layer (central location) shows up with zero average velocity due to symmetry arguments (Fig. 2). The large hole in the middle of this layer corresponds to regular trapped motion in the well of $U(x)$. Additional resonances are seen above and below the central layer. These thin ballisticlike but yet stochastic channels have no overlap with the central layer.

A weak asymmetry $\phi=\pi/5$ leads to a slight deformation of the main stochastic layer and to a desymmetrization of the overlap of the chaotic layer with higher-order resonances and to the appearance of a positive current in the system. Note that it still does not overlap with the thin ballistic channels seen in Fig. 1(a). Most importantly, we observe a nonzero average velocity $\langle \dot{x} \rangle \approx 0.05$ (Fig. 2).

Further increase of the asymmetry, $\phi=\pi/2$, results in an overlapping of the main stochastic layer with the upper ballistic resonance Fig. 1(b). Note that at the same time the lower ballistic resonance is not overlapping. The average velocity increases to $\langle \dot{x} \rangle \approx 0.2$, which is four times larger than the result for $\phi=\pi/5$. Standard harmonic mixing theories (see [2,9]) would predict a dependence $\langle \dot{x} \rangle \sim \sin\phi$ and thus an increase by a factor of only 1.7.

In the $x(t)$ curves in Fig. 2 we observe many ballistic flights. For one of them an inset shows the corresponding Poincaré map result, which verifies that these flights correspond to stickings of the chaotic trajectory to the upper ballistic resonances. A zooming of the $x(t)$ curves shows *self-similarity*, i.e., the seemingly random dynamics between observable long flights is actually again composed of shorter flights and seemingly random dynamics, etc. (see the insets in Fig. 2).

In order to quantify our analysis of the symmetry broken dynamics we compute the distribution of traveling times of ''uniform'' flights to the left $P_-(t_f)$ and to the right $P_+(t_f)$ separately. Here, ''uniform'' means no change of direction of motion [10]. For each separate flight we note both the time t_f spent in this motion and the distance x_f traveled. Note

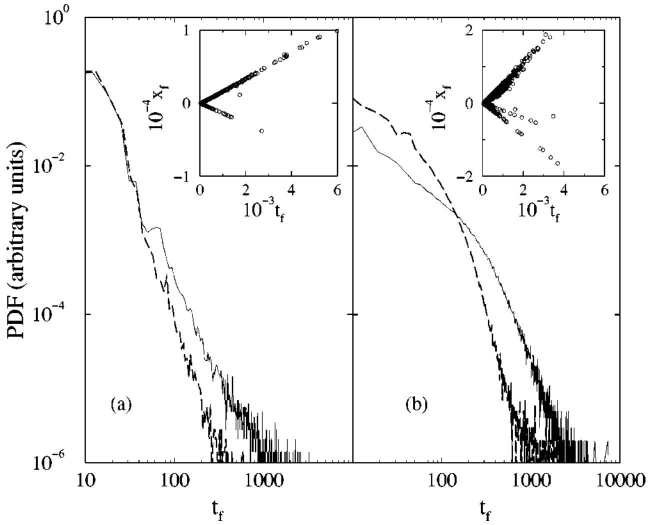

FIG. 3. $P_+(t_f)$ (solid line) and $P_-(t_f)$ (thick dashed line). Insets: x_f versus t_f. For parameters see text.

that such a definition of flights has to be imrpoved if many resonances are involved and especially if ballistic resonances contribute, which are characterized by nonuniform motion.

The dependence of x_f on t_f is shown in the inset of Fig. 3(a) for $\phi=\pi/2$. As in the other cases $\phi=0,\pi/5$ we observe a simple forklike structure. This is due to the fact that any considerable distance covered in the stochastic layer is realized through flights while sticking to the boundary of the stochastic layer. The slopes in the inset of Fig. 3(a) are given by the corresponding winding numbers of the layer boundaries. Note that the two fork parts merge at values of $t_f \approx 10T$. In principle other ballistic channels with different winding numbers might be present. Here they were too weak to be detected.

In Fig. 3(a) we show the corresponding distribution functions $P_\pm(t_f)$ (again for $\phi=\pi/2$). They are obtained by counting the number of flights with t_f falling into a time window of size T. For $t_f<10T$ we observe exponential dependence of $P_\pm$ on t_f. These short flight distributions are in fact independent of the direction of flight. Skipping all longer flights would lead to the prediction of nearly zero average velocity (restricting consideration to flights of length $t_f<10T$ yields about 1% of the numerically observed current). Thus the desymmetrization will manifest itself for longer flights. For $t_f>10T$ a crossover to a power law $P_\pm \sim t_f^{\alpha_\pm}$ takes place. Here we find a significant desymmetrization for $\phi=\pi/5,\pi/2$. Estimating the exponents [11] we find, for $\phi=\pi/5$, $\alpha_-\approx 2.5$, $\alpha_+\approx 2.4$, and for $\phi=\pi/2$, $\alpha_-\approx 3.7$, $\alpha_+\approx 2.3$. It is worthwhile noting that $\alpha<3$ implies unidirectional anomalous diffusion with diverging second moments of $P(t)$. The flights are termed *Levy flights* in such a case [12].

Following the continuous-time random walk (CTRW) formalism [13] we propose a generalized asymmetrical flight model capable of reproducing the above results. The applicability of the CTRW model follows from the assumption that the presence of a random phase with fast decaying correlations leads to the absence of correlations between consecutive flights, since they are almost always separated by dispersive chaotic motion.

Assume that there exist N different resonances with winding numbers w_i, $i=1,\dots,N$. Every resonance is characterized by a probability distribution function (PDF) of sticking times $S_i(t)$. After finishing a random phase event, the probability of sticking to the ith resonance is ρ_i, $\Sigma_{i=1}^N\rho_i=1$. The random phase residing time is characterized by a PDF $S_r(t)$. All functions $S_i(t)$ and $S_r(t)$ must have finite first moments, due to the Kac theorem about finiteness of recurrence times in Hamiltonian systems [7]. With these definitions we obtain the following expression for the current:

$$J=\frac{\sum_{i=1}^{N}\omega_i\rho_i\langle t_i\rangle}{\sum_{i=1}^{N}\rho_i\langle t_i\rangle+\langle t_r\rangle}, \tag{3}$$

where $\langle t_i\rangle=\int tP_i(t)\mathrm{d}t$.

For the above discussed cases of $\phi=\pi/5,\pi/2$ we find only two relevant ballistic channels—one with positive winding number and a second one with negative winding number. In order to properly obtain $S(t)$, we note that our numerically obtained function $P(t)$ consists of a lot of short ''flights'' as defined through the numerics [10]. These may be either stickings to islands with zero winding number or chaotic motion. We observe that for flight times $t_f>10T$ only ballistic flights with nonzero winding number are obtained. So the functions $S_\pm(t)$ may be easily obtained from $P_\pm(t_f)$ by cutting out the central part $t<10T$ and properly normalizing. In this case the expression for the average current simplifies to

$$J=\frac{1}{\kappa(1+f)}(\omega_+\langle t_+\rangle+f\omega_-\langle t_-\rangle), \tag{4}$$

where the two constants κ and f can be obtained from the total time of a simulation T_{tot} and the numbers $N_\pm$ of ballistic flights, $\kappa=T_{tot}/(N_++N_-)$ and $f=N_-/N_+$.

For $\phi=\pi/5$ we obtain from the numerical runs $f\approx 0.57$, $\kappa\approx 1900$, $\langle t_+\rangle\approx\langle t_-\rangle\approx 220$, and $\omega_+=10/6\approx 1.67$, $\omega_-=-1.5$. In this case of *weak desymmetrization* the main source of a nonzero current is the different probabilities of entering a right or left going flight because $f\neq 1$. At the same time the average flight times in both ballistic channels are nearly identical. With the help of Eq. (4) we find $J\approx 0.056$ which is close to the numerically observed value of 0.05.

For the case $\phi=\pi/2$ we find $f\approx 0.16$, $\kappa\approx 2600$, $\langle t_+\rangle\approx 400$, $\langle t_-\rangle\approx 150$, and $\omega_+=2$, $\omega_-\approx -1.4$. Note that the above discussed overlap with the upper resonance yields a further *strong desymmetrization* in the probabilities of realizing a left or right going flight, and in addition the average flight times in both channels significantly differ. Expression (4) yields $J\approx 0.22$ which is in good agreement with the numerically observed value of 0.2.

For stronger driving amplitude $E_1=3.26$, $E_2=1.2$, and $\phi=\pi/2$ we obtain an average velocity $\langle\dot{x}\rangle\approx 0.85$. The corresponding $x_f(t_f)$ dependence and the PDFs $P_\pm(t_f)$ are shown in Fig. 3(b). The $x_f(t_f)$ dependence shows that more than two ballistic channels are involved. The asymmetry of the PDFs at short flight times indicates that a considerable number of left going flights become dominant at short times, in agreement with the tendency of the previous results. This makes the application of the simplified sum rule (4) impossible; instead the original definition (3) should be used. Careful analysis of the structure of the stochastic layer shows that relevant resonances become embedded in the bulk of the stochastic layer. While these structures are of rather small size, they are frequently visited. A restriction to short flights $t_f < 10T$ now yields a considerable nonzero current, which is, however, *negative*, i.e., opposite to the total current value. Again the long ballistic flights are necessary in order to properly obtain the observed current value.

The last example suggests that the applicability of Eq. (3) is rather limited. While this is to some extent true considering the practical side, it can of course be improved by using refined definitions of ballistic flights. However, the most important property of Eq. (3) is its validity in principle. It represents a dynamical approach to directed transport in driven Hamiltonian systems. This approach states that transport in a chaotic layer is realized through ballistic channels. This has been successfully tested for some cases [Fig. 3(a)] and we conjecture that it has general validity.

In summary, we have explained the dynamical mechanisms of current appearance in driven Hamiltonian systems inside a stochastic layer with broken time reversal symmetry. The key source of such directed transport is the desymmetrization of flight probabilities in ballistic channels inside the layer. It follows from our sum rule (3) that the resulting current depends among other parameters on the average time spent in a ballistic channel. These times will sensitively depend on control parameters of the system if the exponent α becomes less than 3. In such cases small changes may significantly alter the current value as shown above.

An important question is whether the analysis presented is robust with respect to weak damping (dissipation). While a complete analysis is beyond the scope of this work, we checked that the mechanisms of dc current generation stay in place. Ballistic resonances are replaced by attractors (typically limit cycles with nonzero winding numbers). The chaotic layer of the Hamiltonian case is replaced by a complicated entanglement of basins of attraction of different attractors (see also [14]).

A recently proposed geometric approach of counting areas and winding numbers is in principle also capable of obtaining the observed mean value for the current [15]. This approach may also require sophisticated studies of the fractal structure of the chaotic layer. It represents a nontrivial complementary result, since, although not explaining the dynamical mechanisms of current generation, it is capable of obtaining the average current value [provided the sums in Eq. (3) of [15] converge fast enough].

We thank M. Fistul, A. A. Ovchinnikov, H. Schanz, O. Yevtushenko, and Y. Zolotaryuk for useful discussions.

[1] P. Hänggi and R. Bartussek, in *Nonlinear Physics of Complex Systems—Current Status and Future Trends*, edited by J. Parisi, S. C. Müller, and W. Zimmermann, Lecture Notes in Physics Vol. 476 (Springer, Berlin, 1997), p. 294; F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997).

[2] P. Reimann, e-print cond-mat/0010237.

[3] S. Flach, O. Yevtushenko, and Y. Zolotaryuk, Phys. Rev. Lett. **84**, 2358 (2000).

[4] I. Goychuk and P. Hänggi, in *Stochastic Processes in Physics, Chemistry and Biology*, edited by J. A. Freund and T. Pöschel, Lecture Notes in Physics Vol. 557 (Springer, Berlin, 2000), p. 7; T. Dittrich, R. Ketzmerick, M.-F. Otto, and H. Schanz, Ann. Phys. (Leipzig) **9**, 755 (2000).

[5] A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, 2d ed. (Springer, Berlin, 1992).

[6] O. Yevtushenko, S. Flach, A. A. Ovchinnikov, and Y. Zolotaryuk, Europhys. Lett. **54**, 141 (2001).

[7] G. M. Zaslavsky, *Physics of Chaos in Hamiltonian Systems* (Imperial College Press, London, 1998).

[8] The starting time of the output was set to zero; subsequent points are drawn after multiples of the drive period. The value of x is mod 2π, so the true trajectory may actually cover large distances in x.

[9] I. Goychuk and P. Hänggi, Europhys. Lett. **43**, 503 (1998).

[10] Flights are monitored on a time grid with spacing T. A flight end is defined by either a change of the velocity sign or a shift in x in the opposite direction.

[11] Here and in the following our estimated numbers from fitting numerical data are given with a relative error of 10% or less.

[12] *Levy Flights and Related Topics in Physics*, edited by M. F. Shlesinger and G. M. Zaslavsky, Lecture Notes in Physics Vol. 450 (Springer, Berlin, 1994).

[13] G. Zumofen, J. Klafter, and A. Blumen, Phys. Rev. E **47**, 2183 (1993); J. Klafter and G. Zumofen, *ibid.* **49**, 4873 (1994).

[14] J. L. Mateos, Phys. Rev. Lett. **84**, 258 (2000); Acta Phys. Pol. B **32**, 307 (2001).

[15] H. Schanz, M.-F. Otto, R. Ketzmerick, and T. Dittrich, Phys. Rev. Lett. **87**, 070601 (2001).