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Shape profile of compactlike discrete breathers in nonlinear dispersive lattice systems

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We study the spatial decay profile of compactlike discrete breathers in nonlinear dispersive lattices. We show that the core region of such nonlinear localized excitations can be described by a cosinelike spatial shape while the tail region decays with a faster than exponential law, such as a superexponential one. We discuss the relation of the tail decay to properties of space-time separability.

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Since the publication of the paper by Rosenau and Hyman [1] showing that solitary wave solutions supported by nonlinear wave equations may compactify under nonlinear dispersion, the idea of compact localized solutions of nonlinear systems has gained considerable interest. They showed that a Korteweg–de Vries–type equation with nonlinear dispersion supports exact compacton solutions of the form

$$u(\xi)=A\cos^{\nu}(B\xi) \tag{1}$$

for $|B\xi|\leqslant\pi/2$, $u(\xi)=0$ otherwise, where $\xi=x-vt$ and the parameter ν depends on the order of the nonlinear dispersion in the equations.

It is well known that nonlinear equations with linear dispersion admit solitary waves, called solitons, that are infinite in extent. On the other hand, Rosenau and Hyman's studies showed that nonlinear dispersion can cause qualitative changes to the nature of genuinely nonlinear phenomena. The interaction of nonlinear dispersion with nonlinear convection generates exact compact structures (compactons), free of exponential tails. The stability analysis has shown that compactons are stable structures [2]. Numerical simulations of the nonlinear dispersive equations have also revealed the existence of compact traveling breathers [3].

It is known that many nonlinear lattice models with linear dispersion give rise to energy localization effects and support stable soliton/kink structures. Kivshar first conjectured that intrinsic localized modes in a nonlinear dispersive lattice may exhibit compactlike properties [4]. As a model of nonlinear dispersion, he considered a one-dimensional lattice with purely anharmonic nearest-neighbor interaction for which the equation of motion of the nth atom is given by

$$\frac{d^2u_n}{dt^2}=[(u_{n+1}-u_n)^3+(u_{n-1}-u_n)^3] \tag{2}$$

and obtained a solution of this equation similar to Eq. (1). Later Flach argued that even for such a nonlinear dispersive model (2), the nonlinear localized excitations cannot have an exact compact structure [5]. Analysing the result in the tail of the nonlinear localized excitations , it was shown that the amplitudes of the lattice displacements are not exactly zero outside a finite volume of the solution, and in leading order the amplitudes decay according to the *superexponential* law $e^{-a\exp bn}$, where a and b are positive numbers that depend on the parameters of the model Hamiltonian. On the other hand, more recently, Dinda and Remoissenet [6] demonstrated that a class of exact continuous compacton solutions [of the form as in Eq. (1)] of a low-order continuum approximation of the discrete equations survives on finite times when substituted in the corresponding lattice equations of motion. They also studied the ability of the compactlike kink to propagate and obtained the parameter region in which a compactlike kink can show up with stable ballistic propagation [7]. Since all these methods for obtaining compactons in discrete nonlinear lattices are approximate ones, Eleftheriou *et al.* [8] generated numerically exact discrete compactlike breathers of nonlinear dispersive lattices starting from the anticontinuous limit and by using the Newton scheme [9,10]. They observed that compactlike breathers exist in the whole range from strong to intermediate to small coupling constants provided the interaction between the nonlinear oscillators is purely nonlinear, i.e., that there exists nonlinear dispersion. However, while in the continuous limit compact breathers are exact compacton solutions with strict compact support, in the other extreme, i.e., when close to the anticontinuous limit, breathers become compactlike by acquiring a very small tail. The latter was shown to decay in a faster than exponential fashion and was fitted in Ref. [8] to a stretched exponential law $e^{-\gamma n^s}$ with exponent $s>4$.

All previously mentioned investigations clearly point to the existence of discrete compact (or compactlike) breathers in nonlinear dispersive lattices whose main feature is the faster than exponential tail decay. Nevertheless, there is some lingering ambiguity regarding the overall shape profile and in particular the spatial configuration of the core and tail regions of the breather. On one side, Kivshar's solution [4] is an approximation in the tails, since breather amplitudes do not exactly vanish there. On the other side, cosinelike discrete breathers were shown to have finite-time persistence in lattices [6], and although they describe proper bounded solutions in the regime close to the continuum limit, they do not have the feature of long-term stability [8], even though for a relatively short time they seem to be quite stable. The superexponential decay law obtained in [5] for localized excitations in similar nonlinear dispersive lattices is valid only in the tail region of the solutions provided space-time separability holds. Finally, the issue of the spatial variation in the core region of compactlike discrete breathers has not been fully addressed.

As has been mentioned above, the question of the overall shape of the discrete compactlike breather is very important. From the application point of view, it is desirable that the discrete compact breather does not have an infinite tail as do solitons. Accordingly, in this Brief Report we study the shape profile of the lattice displacement patterns of breather solutions in nonlinear dispersive lattices. We compare the shape of the core region with that of the tail region and check if there is an overall shape that fits displacements of all the lattice points in the chain.

We start with a model of oscillators in a one-dimensional chain with nearest-neighbor anharmonic interaction. In contrast to [4], each of the oscillators moves in a nonlinear on-site potential. This is because, as observed in earlier studies [6,8], the presence of the on-site potential is a major requirement for the stability of the compactlike discrete breathers. The equation of motion for the displacement at the site n is

$$\frac{d^2u_n}{dt^2}=k[(u_{n+1}-u_n)^3+(u_{n-1}-u_n)^3]-V'(u_n). \tag{3}$$

We use three different nonlinear substrate potentials, the (soft) double-well potential $V_s(x)=-\frac{1}{2}(x-1)^2+\frac{1}{4}(x-1)^4$, the hard ϕ^4 potential $V_h(x)=\frac{1}{2}x^2+\frac{1}{4}x^4$, and the (soft) Morse potential $V_m(x)=\frac{1}{2}(1-e^{-x})^2$.

First we present the numerical results. Compactlike discrete breathers are generated in the anticontinuous limit (small k) with the help of a Newton scheme and are generally stable. We start with $k=0$ and excite one oscillator, with all others being at rest. The solution is then continued to finite values of k. Here we will consider $k=0.1$ and $k=0.3$. The numerical data clearly show that the decay in the nonlinear dispersive lattice is much faster than the usual exponential decay of the linear dispersive lattice. This is shown in the inset of Fig. 1. At the same time, we again confirm that the breather tail amplitudes are nonzero. This also holds for $V(x)=0$ (cf. [10]). Most importantly, we observe in the main body of Fig. 1 that the shifted double logarithms of the breather amplitudes fall onto a single master curve in the tails. At the same time, the core region (central site and two nearest neighbors) clearly follows a different shape law.

In order to understand the tail behavior, we first consider the V_h case. Similar to the case of homogeneous potential functions [5], we may use the ansatz $u_n(t)=(-1)^n\phi_nG(t)$ which separates space and time. Although $V_h(x)$ is not a homogeneous function, it contains in addition to the quartic term just one quadratic term, which keeps the property of space-time separation. We arrive at the set of equations

$$\ddot{G}(t)+G(t)+CG^3(t)=0, \tag{4}$$

$$k[(\phi_{n+1}+\phi_n)^3+(\phi_{n-1}+\phi_n)^3]+\phi_n^3=C\phi_n. \tag{5}$$

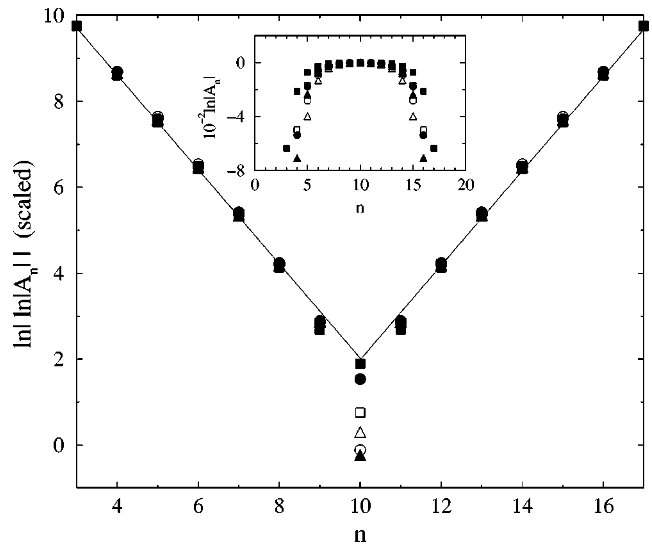

FIG. 1. Dependence of the double logarithm of the breather amplitude for different cases versus n. Note that curves are vertically shifted to observe a master curve in the tails. V_h, circles; V_m, squares; V_s, triangles. Open symbols, $k=0.1$; filled symbols, $k=0.3$. The straight lines have slopes $\pm\ln 3$. Inset: Dependence of the logarithm of the breather amplitude versus n for the same cases. Note the scale of the y axis.

Here C is an arbitrary non-negative separation constant. While the function $G(t)$ can be easily found by implicitly integrating Eq. (4), the existence of a spatially localized profile for the amplitudes ϕ_n was proven in [11]. As the spatial decay is obviously faster than exponential, we immediately find the asymptotic law (to the right of the breather center) using $A_n=\phi_n\max(G(t))$:

$$\kappa A_n=A_{n-1}^3. \tag{6}$$

Here κ is a constant which depends on k. This spatial decay is a superexponential one, since

$$\ln|\ln|A_n||\approx n\ln 3. \tag{7}$$

This asymptotic law holds provided κ is bounded. In Fig. 1, we show that Eq. (7) indeed holds in the tails of our breather solutions. In addition, we plot in Fig. 2 the values for κ computed with the help of Eq. (6) at each lattice site independently. We observe that for the V_h case indeed the breather tails are characterized by a well-defined value of κ.

For the cases of V_m and V_s, we cannot separate time and space. Thus we cannot obtain a superexponential decay law in the tails as done above. Yet the numerical results in Fig. 1 show that Eq. (7) holds. At the same time, the data in Fig. 2 suggest that κ did not converge to an asymptotic value as compared to the V_h case. Let us give some reasons for these observations. As we consider time-periodic solutions, we may expand the temporal evolution at each site into a Fourier series with respect to time:

$$u_n(t)=\sum_{k=-\infty}^{+\infty}u_{kn}e^{ik\omega t}. \tag{8}$$

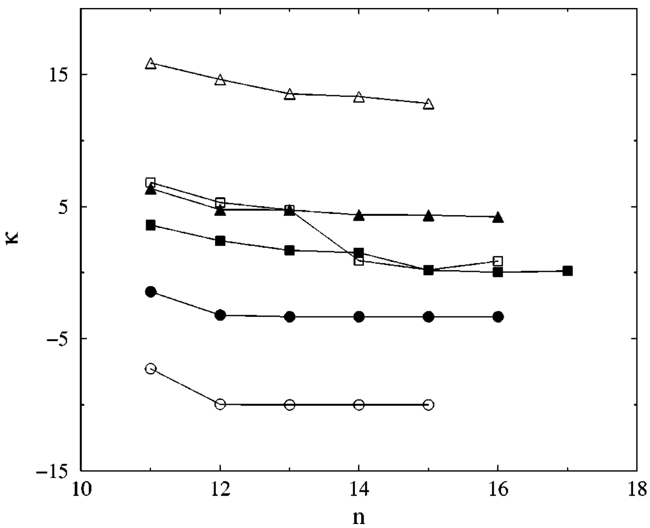

FIG. 2. Dependence of the constant κ on n when computed independently at each lattice site using Eq. (6). Symbols as in Fig. 1. Lines are connecting data and serve as guides to the eye.

In contrast to the case of space-time separation, where each Fourier component u_{kn} shows up with one and the same superexponential decay law, here we have to insert the ansatz (8) into the equations of motion, sort terms with equal exponents, and set the prefactors to zero. The corresponding coupled nonlinear algebraic equations for the coefficients u_{kn} have then to be solved. Considering the tail of a breather and neglecting the interaction in k space, we would obtain k-dependent superexponential decay laws for each component. The one with the weakest decay will be the leading-order asymptotic superexponential decay. Yet the results of Fig. 2 and numerical computations of the u_{kn} from our solutions show that (i) the asymptotic law is not fully reached with our data, and (ii) interactions in k space cannot be neglected for these lattice sites. Thus, we can state that while a single superexponential decay will emerge for large distances from the breather center, for the lattice sites closer to the core a mixture of different superexponential laws should hold [due to Eq. (8)]. The observation of the κ-independent part of this law in Fig. 1 is due to the fact that $\ln \kappa$ is bounded and small compared to $\ln|A_n|$ in the tails.

We now derive the lattice displacement patterns of the compactlike discrete breather in the core region. Let us start with the V_h case. Using Eq. (4) we find a solution $G(t) = A\,\mathrm{cn}(wt,s)$, where $\mathrm{cn}(wt,s)$ is the Jacobi elliptic function with the modulus s. For the spatial variation, we assume a solution to Eq. (5) in the form

$$\phi_n = \cos[q(n-n_0)]. \tag{9}$$

Substituting Eq. (9) in Eq. (5) we get an equation for q:

$$2\cos^3 q + 3\cos^2 q - 1 + \frac{1}{4k} = 0. \tag{10}$$

This equation has real q solutions for $1/4k < 1$. For $k=0.3$, the allowed solutions are $q=0.429\pi$ and $q=0.583\pi$. To compare it with the numerical results, we fit the lattice displacement pattern as given in Eq. (9) with the core region (the central and the two nearby sites) and obtain the value of $q=0.403\pi$, which agrees quite well with the analytic value $q=0.429\pi$ as obtained above. For the V_s and the V_m case, however, the space and time variables cannot be separated and thus it is not possible to make any analytic prediction about the spatial variation of the lattice displacement pattern in the core region of the corresponding compactlike discrete breathers. However, even for these cases also if we assume a spatial variation of the form of Eq. (9) in the core region, then the corresponding value of q as obtained by fitting the numerical data points is $q=0.459\pi$ for the V_s case ($k=0.3$). For the Morse potential the similar fitting gives $q=0.431\pi$ for $k=0.1$

In conclusion, we have studied the shape profile of discrete compactlike breathers in nonlinear dispersive lattices. While in a continuous system compact breathers are exact cosine solutions with strict bounded support [Eq. (1)], in lattices compact breathers are characterized by lattice displacements that have two distinct spatial patterns. In the core region, a bounded cosinelike shape prevails while the tail region is clearly characterized by a much faster than exponential spatial decay that is well described by the superexponential decay law. As a result, even though in nonlinear dispersive lattices compactlike breathers have a tail, the latter is very short and decays in an ultrafast fashion unlike the corresponding exponential decay in lattices with linear dispersion. This particular property makes compactons very interesting in the study of pattern formation, as the observed stationary and dynamical patterns in nature are usually finite in extent. Also, compactons can be of interest for energy storage, since due to the lack of an exponential tail they would not interact with each other until the point of contact (short-range interaction), which leads to a much weaker mutual interaction between different compactlike discrete breathers in the same lattice. The lifetime of coexisting discrete breathers is thus substantially increased. Similarly, due to the absence of long-range interaction, the compactons have an advantage over the solitons for data transmission and signal-processing purposes. We would also like to mention that compacton solutions arise in the study of nonlinear dynamics of shear waves in elastic plates. In this case, the nonlinear evolution equation for the shear displacements reduces to the modified Boussinesq equation (MBE) with nonlinear dispersion. The presence of the nonlinear dispersion terms modifies the structure of the soliton solutions of the MBE into compacton or peakons for different materials. Recently, exact dark lattice compactons solutions have been found in a model of Frenkel excitations [12].

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[1] P. Rosenau and J.M. Hyman, Phys. Rev. Lett. **70**, 564 (1993); P. Rosenau, *ibid.* **73**, 1737 (1994).
[2] B. Dey and A. Khare, Phys. Rev. E **58**, R2741 (1998).
[3] P. Rosenau, Phys. Lett. A **252**, 297 (1999), and references therein.
[4] Y.S. Kivshar, Phys. Rev. E **48**, 43 (1993).
[5] S. Flach, Phys. Rev. E **50**, 3134 (1994).
[6] P. Tchofo Dinda and M. Remoissenet, Phys. Rev. E **60**, 6218 (1999).
[7] P. Tchofo Dinda, T.C. Kofane, and M. Remoissenet, Phys. Rev. E **60**, 7525 (1999).
[8] M. Eleftheriou, B. Dey, and G.P. Tsironis, Phys. Rev. E **62**, 7540 (2000).
[9] S. Aubry, Physica D **71**, 196 (1994).
[10] J.L. Marin and S. Aubry, Nonlinearity **9**, 1501 (1996).
[11] S. Flach, Phys. Rev. E **51**, 1503 (1995).
[12] V.V. Konotop and S. Takeno, Phys. Rev. E **60**, 1001 (1999).