Interaction-induced fractional Bloch and tunneling oscillations

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We study the dynamics of few interacting bosons in a one-dimensional lattice with dc bias. In the absence of interactions the system displays single-particle Bloch oscillations. For strong interaction the Bloch oscillation regime re-emerges with fractional Bloch periods which are inversely proportional to the number of bosons clustered into a bound state. The interaction strength affects the oscillation amplitude. Excellent agreement is found between numerical data and a composite particle dynamics approach. For specific values of the interaction strength, a particle will tunnel from the interacting cloud to a well-defined distant lattice location.

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Bloch oscillations [1] in dc biased lattices are due to wave interference and have been observed in a number of quite different physical systems: atomic oscillations in Bose-Einstein condensates (BECs) [2], light intensity oscillations in waveguide arrays [3], and acoustic waves in layered and elastic structures [4], among others.

Quantum many-body interactions can alter the above outcome. A mean-field treatment makes the wave equations nonlinear and typically nonintegrable. For instance, for many atoms in a Bose-Einstein condensate, a mean-field treatment leads to the Gross-Pitaevsky equation for nonlinear waves. The main effect of nonlinearity is to deteriorate Bloch oscillations, as recently studied experimentally [5] and theoretically [6–8].

In contrast, we explore the fate of Bloch oscillations for quantum interacting few-body systems. This exploration is motivated by a recent experimental advance [9] in monitoring and manipulating few bosons in optical lattices. Few-body quantum systems are expected to have finite eigenvalue spacings, consequent quasiperiodic temporal evolution, and phase coherence. In a recent report on interacting electron dynamics, spectral evidence for a Bloch frequency doubling was reported [10]. On the other hand, it was also recently argued that Bloch oscillations are effectively destroyed for few interacting bosons [11].

In this Brief Report, we show that for strongly interacting bosons a coherent Bloch oscillation regime re-emerges. If the bosons are clustered into an interacting cloud at time $t = 0$, the period of Bloch oscillations will be a fraction of the period of the noninteracting case, scaling as the inverse number of interacting particles (Fig. 1). The amplitude (spatial extent) of these fractional Bloch oscillations decreases with increasing interaction strength. For specific values of the interaction, one of the particles leaves the interacting cloud and tunnel for a possibly distant and well-defined site of the lattice. For few particles, the dynamics is always quasiperiodic, and a decoherence similar to the case of a mean-field nonlinear equation [7] does not take place.

We consider the Bose-Hubbard model with a dc field:

$$\hat{\mathcal{H}} = \sum_j \left[t_1(\hat{b}^+_{j+1}\hat{b}_j + \hat{b}^+_j\hat{b}_{j+1}) + Ej\hat{b}^+_j\hat{b}_j + \frac{U}{2}\hat{b}^+_j\hat{b}^+_j\hat{b}_j\hat{b}_j \right], \quad (1)$$

where $\hat{b}^+_j$ and $\hat{b}_j$ are standard boson creation and annihilation operators at lattice site j, the hopping $t_1 = 1$, and U and E are the interaction and dc field strengths, respectively. To study the dynamics of n particles, we use the orthonormal basis of states $|\mathbf{k}\rangle \equiv |k_1,k_2,\ldots,k_n\rangle = b^+_{k_1}b^+_{k_2}\cdots b^+_{k_n}|0\rangle$, where $|0\rangle$ is the zero-particle vacuum state, and $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$ are lattice site indices (for instance, in the case of two particles the state representation is mapped to the triangle). The eigenvectors $|\nu\rangle$ of Hamiltonian (1) with eigenvalues λ_ν are then given by

$$|\nu\rangle = \sum_{\mathbf{k}} A^\nu_{\mathbf{k}}|\mathbf{k}\rangle, \quad \hat{\mathcal{H}}|\nu\rangle = \lambda_\nu|\nu\rangle, \quad (2)$$

where the eigenvectors $A^\nu_{\mathbf{k}} \equiv \langle\mathbf{k}|\nu\rangle$ and the time evolution of a wave function $|\Psi(t)\rangle$ is given by

$$|\Psi(t)\rangle = \sum_\nu \Phi_\nu e^{-i\lambda_\nu t}|\nu\rangle, \quad \Phi_\nu \equiv \langle\nu|\Psi(0)\rangle. \quad (3)$$

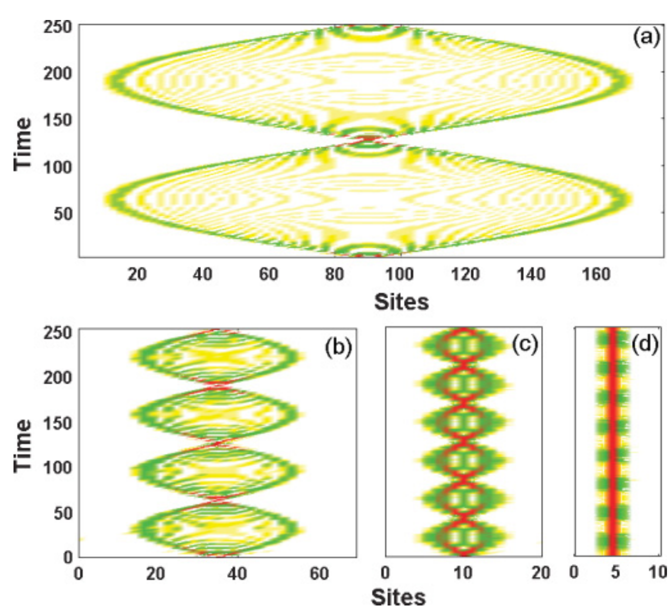

FIG. 1. (Color online) Time evolution of the probability density function $P_j(t)$ for the interaction constant $U = 3$ and dc field $E = 0.05$ and different particle numbers initially occupying a single site at $t = 0$. (a) One-particle Bloch oscillations with the conventional Bloch period $2\pi/E$, and (b) two-, (c) three-, and (d) four-particle oscillations with periods $2\pi/(2E)$, $2\pi/(3E)$, and $2\pi/(4E)$, respectively.

We monitor the probability density function (PDF) $P_j(t) = \langle\Psi(t)|\hat{b}_j^+\hat{b}_j|\Psi(t)\rangle/n$, which can be also computed using the eigenvectors and eigenvalues:

$$P_j(t) = \frac{1}{n}\sum_{\nu,\mu}\Phi_\nu\Phi_\mu^* e^{i(\lambda_\mu-\lambda_\nu)t}\langle\mu|\hat{b}_j^+\hat{b}_j|\nu\rangle. \quad (4)$$

In Fig. 1 we show the evolution of $P_j(t)$ for $U = 3$, $E = 0.05$, and $n = 1,2,3,4$ with initial state $k_1 = k_2 = \cdots = k_n \equiv p$, that is, when all particles are launched on the same lattice site p. For $n = 1$ we observe the usual Bloch oscillations with period $T = 2\pi/E$ [Fig. 1(a) and below]. Due to the small value of E, the amplitudes of oscillations are large. However, with increasing number of particles, we find that the oscillation period is reduced according to $2\pi/(nE)$, and at the same time the amplitude of oscillations is also reduced.

In the one-particle case, for $n = 1$ the interaction term in Eq. (1) does not contribute. The eigenvalues $\lambda_\nu = E\nu$ (where ν is an integer) form an equidistant spectrum which extends over the whole real axis: the Wannier-Stark ladder. The corresponding eigenfunctions obey the generalized translational invariance $A_{k+\mu}^{\nu+\mu} = A_k^\nu$ [1] and are given by the Bessel function $J_k(x)$ of the first kind [12,13]:

$$A_k^\nu = J_k^\nu \equiv J_{k-\nu}(2/E). \quad (5)$$

All eigenvectors are spatially localized with an asymptotic decay $|A_{k\to\infty}^0| \to (1/E)^k/k!$, giving rise to the well-known localized Bloch oscillations with period $T_B = 2\pi/E$. The localization volume $\mathcal{L}$ of a single-particle eigenstate characterizes its spatial extent. It follows that $\mathcal{L} \propto -[E\ln E]^{-1}$ for $E \to 0$ and $\mathcal{L} \to 1$ for $E \to \infty$ [7]. For $E = 0.05$ the single particle oscillates with amplitude of the order of $2\mathcal{L} \approx 160$ [Fig. 1(a)]. According to Eqs. (4) and (5), the probability density function is given by

$$P_j(t) = \sum_{\nu,\mu} J_p^\nu J_p^\mu J_j^\nu J_j^\mu e^{iE(\mu-\nu)t}. \quad (6)$$

In the two-particle case ($n = 2$), for $U = 0$ the eigenfunctions of the Hamiltonian (1) are given by tensor products of the single-particle eigenstates:

$$|\mu,\nu\rangle = \sqrt{\frac{2-\delta_{\mu,\nu}}{2}}\sum_{k,j} J_k^\mu J_j^\nu \hat{b}_k^+\hat{b}_j^+|0\rangle, \quad \mu \leqslant \nu. \quad (7)$$

The corresponding eigenvalues form an equidistant spectrum which is highly degenerate:

$$\hat{\mathcal{H}}|\mu,\nu\rangle = (\mu+\nu)E|\mu,\nu\rangle. \quad (8)$$

For the above initial condition $k_1 = k_2 \equiv p$, the expression for the PDF (6) is still valid (as it actually is for any number of noninteracting particles) with the same period $2\pi/E$ of Bloch oscillations as in the single-particle case.

For nonvanishing interaction, the degeneracy of the spectrum is lifted, and the eigenvalues of overlapping states are no longer equidistant (Fig. 2). Therefore, we observe quasiperiodic oscillations which, however, are still localizing the particles. For even larger values of U, the basis states with two particles on the same site shift their energies by U exceeding the hopping $2t_1$. Therefore, for $U > 2t_1$, the

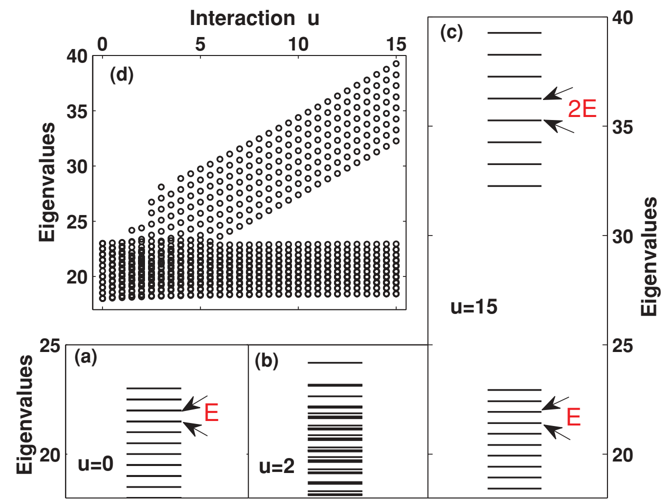

FIG. 2. (Color online) Eigenvalue spectrum for $n = 2$, $E = 0.5$, and different interaction constants U. The eigenvalues are displayed only for eigenvectors localized in the center of the lattice (we select the 32 eigenstates which overlap most strongly with the center of the lattice). (a) $U = 0$: the spectrum is equidistant with spacing E and degenerate. (b) $U = 2$: the degeneracy is lifted. (c) $U = 15$: the spectrum decomposes into two subspectra, with two different equidistant spacings, E and $2E$. (d) Eigenvalue spectrum of the 32 central eigenfunctions as a function of U.

spectrum is decomposed into two nonoverlapping parts: a noninteracting one which excludes double occupancy and has equidistant spacing E, and an interacting part which is characterized by almost complete double occupancy and has corresponding equidistant spacing $2E$, which is the cost of moving two particles from a given site to a neighboring site. Some initial state can overlap strongly with eigenstates from one or the other part of the spectrum and, therefore, result in different Bloch periods. In particular, when launching both particles on the same site, one strongly overlaps with the interacting part of the spectrum and a fractional Bloch period $2\pi/(2E)$ is observed.

In order to calculate the amplitude of these fractional Bloch oscillations, we note that for $E = 0$ there exists a two-particle bound-state band of extended states with bandwidth $\sqrt{U^2+16} - U$ [14]. For large U, the bound states are again almost completely described by double occupancy. Therefore, we can construct an effective Hamiltonian for a composite particle of two bound bosons:

$$\hat{\mathcal{H}} \approx \sum_j [t_2(\hat{R}_{j+1}^+\hat{R}_j + \hat{R}_j^+\hat{R}_{j+1}) + 2Ej\hat{R}_j^+\hat{R}_j], \quad (9)$$

where $\hat{R}_j^+$ and $\hat{R}_j$ are creation and annihilation operators at lattice site j of the composite particle (two bosons on the same site) with the effective hopping

$$t_2 = \frac{\sqrt{U^2+16} - U}{4}. \quad (10)$$

The corresponding PDF is given by

$$P_j(t) = \sum_{\nu,\mu} A_p^\nu A_p^\mu A_j^\nu A_j^\mu e^{i2E(\mu-\nu)t}. \quad (11)$$

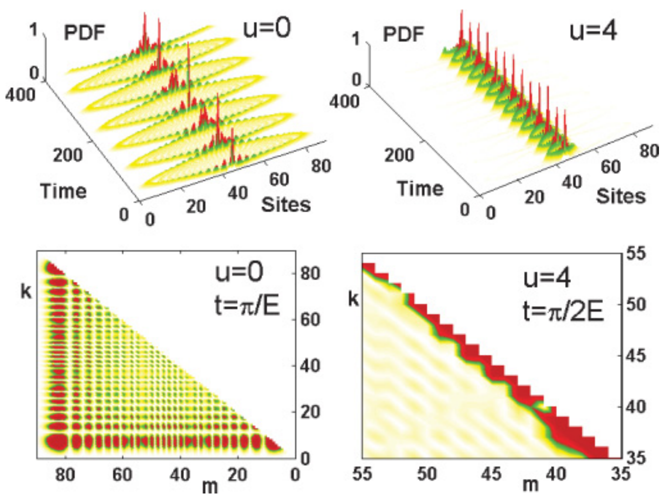

FIG. 3. (Color online) Top: PDF for $E = 0.1$, $n = 2$, single-site initial occupancy, and different interaction constants. For $U = 0$, we find single-particle Bloch oscillations. For $U = 4$, fractional Bloch oscillations take place, in agreement with Eq. (11). Bottom: probability density of the evolved wave function (darker regions correspond to larger probabilities) after one half of the respective Bloch period. For $U = 0$, the two particles are with equal probability close to each other and at maximal separation. For $U = 4$, the two particles avoid separation and form a composite particle which coherently oscillates in the lattice. In the bottom graphs we use triangle $k < m$ mapping for indistinguishable two-particle state representation (index m increases from the right to the left).

The composite particle eigenvectors $A_p^\nu = J_{\nu-p}[2t_2/(2E)]$ are again expressed through Bessel functions, but with a modified argument as compared to the single-particle case. Bloch oscillations evolve with fractional period $2\pi/(2E)$ as observed in Fig. 1(b). The amplitude of the oscillations is reduced with increasing U since the hopping constant t_2 is reduced (Fig. 3). For $U = 3$ it follows that $t_2 = 0.5$, and together with the doubled Bloch frequency the localization volume should be reduced by a factor of 4 as compared to the single-particle case. This is precisely what we find when comparing Figs. 1(a) and 1(b): for $n = 1$, the amplitude is 160 sites, while for $n = 2$ it is 40 sites. In the bottom plots in Fig. 3, we show the probability density of the wave functions $|\langle\Psi(t)|\mathbf{k}\rangle|^2$ after one half of the respective Bloch period in the space of the two-particle coordinates with $k_1 = k$ and $k_2 = m$. For $U = 0$, both particles are with high probability at a large distance from each other. Therefore, the density is large not only for $k = m$ (the two particles are at the same site) but also for $k = 5$, $m = 85$ (the two particles are at maximum distance). However, for $U = 4$, we find that the two particles, which initially occupy the site $p = 45$, do not separate, and the density is large only along the diagonal $k = m$ with $35 \leqslant k \leqslant 55$. (For $U = 4$, the localization volume is $\sim$20.) Therefore, the two particles indeed form a composite state and travel together.

For the n-particle case, we proceed in a manner similar to the case for $n = 2$ and estimate perturbatively the effective hopping constant for a composite particle of n bosons. For that we use the calculated width of the n-particle bound-state band for $E = 0$ [14]. In leading order of $1/U$, it reads [14]

$$t_n \simeq \frac{n}{U^{n-1}(n-1)!}. \tag{12}$$

For $n = 2$, the above expression gives $t_2 \simeq 2/U$, the first expansion term of the exact relation for two bosons [Eq. (10)]. The corresponding composite particle Hamiltonian is

$$\hat{\mathcal{H}} \approx \sum_j [t_n(\hat{R}^+_{j+1}\hat{R}_j + \hat{R}^+_j\hat{R}_{j+1}) + nEj\hat{R}^+_j\hat{R}_j]. \tag{13}$$

The PDF is given by

$$P_j(t) = \sum_{\nu,\mu} A_p^\nu A_p^\mu A_j^\nu A_j^\mu e^{inE(\mu-\nu)t}, \tag{14}$$

and the composite particle eigenvectors $A_p^\nu = J_{\nu-p}[2t_n/(nE)]$. Bloch oscillations evolve with fractional period $2\pi/(nE)$ as observed in Figs. 1(c) and 1(d). The amplitude of the oscillations is reduced with increasing U since the hopping constant t_n is reduced. For $U = 3$ and $n = 3$, it follows that $t_3 = 0.17$, and for $n = 4$ we have $t_4 = 0.01$. This leads to reduction factors of 18 and 400, respectively, as compared to the single-particle amplitude, and yields amplitudes of the order of 9 and 0.5, respectively, which is in good agreement with the numerically observed amplitudes (10 and 2 sites, respectively) in Figs. 1(c) and 1(d).

With respect to tunneling oscillations, for $n = 1$, the amplitude of Bloch oscillations is less than one site if $E \geqslant 10$ [7]. Thus, for $n \geqslant 2$ and increasing values of U, the amplitude of fractional Bloch oscillations is less than one site if $EU^{n-1}(n-1)! \geqslant 10$. Then, n particles launched on the same lattice site p are localized on that site for all times. The energy of that state is $n((n-1)U/2 + pE)$. However, if one particle is moved to a different location with site q, then the energy would change to $(n-1)((n-2)U/2 + pE) + qE$. For specific values of U, these two energies are equal:

$$(n-1)U = dE, \quad d = q - p. \tag{15}$$

In such a case, one particle leaves the interacting cloud at site p and tunnels to site q at distance d from the cloud, then tunnels back, and so on, following an effective Rabi oscillation scenario between the states $|p,p\rangle$ and $|p,q\rangle$. This process appears as an asymmetric oscillation of a fraction of the cloud either up or down the field gradient (depending on the sign of U). We calculate the tunneling splitting of these two states using higher-order perturbation theory; for an example, see Ref. [15]. The tunneling time is then obtained as

$$\tau_{\text{tun}} \simeq \frac{\pi}{\sqrt{n}} E^{d-1}(d-1)!. \tag{16}$$

In order to observe these tunneling oscillations, we compute the time-averaged second moment $\overline{m_2} = \overline{\sum_j j^2 P_j(t) - [\sum_j jP_j(t)]^2}$ of the PDF P. Then an effective time-averaged volume of the interacting cloud is taken to be $L = \sqrt{12\overline{m_2}} + 1$. We launch $n = 2$ particles at site $p = 40$ and plot the ratio $L(U)/L(U=0)$ in Fig. 4 (solid blue line). We find pronounced peaks at $U = E$, $2E$, $3E$, and $4E$, which become sharper and higher with increasing value of U. As a comparison we also compute the same ratio for the initial condition when both particles occupy neighboring sites

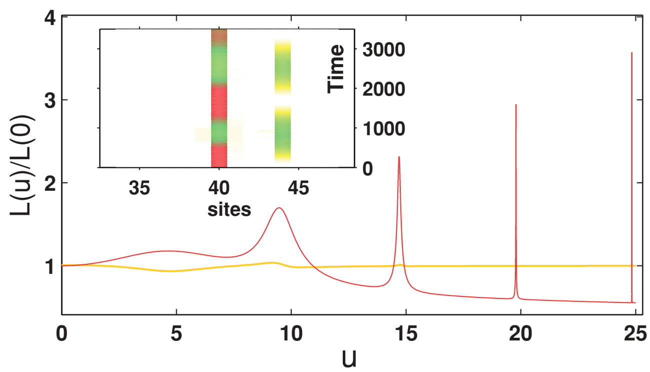

FIG. 4. (Color online) Time-averaged and normalized localization volume L of the wave packet which emerges from two initial distributions as a function of U for $E = 5$. Dark (red) curve: two particles are launched on the same site. Light (orange) curve: two particles are launched on adjacent sites. Inset: PDF for $U = 19.79$, with clearly observed tunneling oscillations.

(dashed red line), for which the resonant structures are absent. According to the above discussion, the resonant structures correspond to a tunneling of one of the particles to a site at distance $d = 1,2,3,4$. The width of the peaks is inversely proportional to the tunneling time τ_{tun}, and the height increases linearly with the tunneling distance d. In the inset in Fig. 4, we plot the time evolution of the PDF P_j for $U = 19.79$. We observe a clear tunneling process from site $p = 40$ to site $q = 44$. The numerically observed tunneling time is approximately 1730 time units, while our above prediction (16) yields $\tau_{\text{tun}} \approx 1666$, in very good agreement with the observations.

The above findings can be useful for control of the dynamics of interacting particles. They can be also used as a testbed of whether experimental studies deal with quantum many-body states. One such testbed is the observation of fractional Bloch oscillations; another one is the resonant tunneling of a particle from an interacting cloud. An intriguing question is the way these quantum coherent phenomena disappear in the limit of many particles, where classical nonlinear and nonintegrable wave mechanics are expected to take over.

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[1] F. Bloch, Z. Phys. **52**, 555 (1928); C. Zener, Proc. R. Soc. London A **145**, 523 (1934); G. H. Wannier, Rev. Mod. Phys. **34**, 645 (1962).
[2] B. P. Anderson and M. A. Kasevich, Science **282**, 1686 (1998); M. Ben Dahan, E. Peik, J. Reichel, Y. Castin, and C. Salomon, Phys. Rev. Lett. **76**, 4508 (1996); O. Morsch, J. H. Muller, M. Cristiani, D. Ciampini, and E. Arimondo, *ibid.* **87**, 140402 (2001); G. Ferrari, N. Poli, F. Sorrentino, and G. M. Tino, *ibid.* **97**, 060402 (2006).
[3] T. Pertsch, P. Dannberg, W. Elflein, A. Brauer, and F. Lederer, Phys. Rev. Lett. **83**, 4752 (1999); H. Trompeter *et al.*, *ibid.* **96**, 023901 (2006); F. Dreisow *et al.*, *ibid.* **102**, 076802 (2009).
[4] H. Sanchis-Alepuz, Y. A. Kosevich, and J. Sanchez-Dehesa, Phys. Rev. Lett. **98**, 134301 (2007); L. Gutierrez *et al.*, *ibid.* **97**, 114301 (2006).
[5] M. Gustavsson *et al.*, Phys. Rev. Lett. **100**, 080404 (2008).
[6] P. K. Datta and A. M. Jayannavar, Phys. Rev. B **58**, 8170 (1998).
[7] D. O. Krimer, R. Khomeriki, and S. Flach, Phys. Rev. E **80**, 036201 (2009).
[8] A. R. Kolovsky, E. A. Gomez, and H. J. Korsch, Phys. Rev. A **81**, 025603 (2010).
[9] K. Winkler *et al.*, Nature **441**, 853 (2006); W. C. Bakr *et al.*, *ibid.* **462**, 74 (2006).
[10] W. S. Dias, E. M. Nascimento, M. L. Lyra, and F. A. B. F. deMoura, Phys. Rev. B **76**, 155124 (2007).
[11] A. Buchleitner and A. R. Kolovsky, Phys. Rev. Lett. **91**, 253002 (2003).
[12] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, 1972).
[13] H. Fukuyama, R. A. Bari, and H. C. Fogedby, Phys. Rev. B **8**, 5579 (1973); P. Feuer, Phys. Rev. **88**, 92 (1952).
[14] A. C. Scott, J. C. Eilbeck, and H. Gilhoj, Physica D **78**, 194 (1994).
[15] S. Aubry, S. Flach, K. Kladko, and E. Olbrich, Phys. Rev. Lett. **76**, 1607 (1996).