Journal of Physics B: Atomic, Molecular and Optical Physics

PAPER

Continuous-wave solutions and modulational instability in spinor condensates of positronium

IOP Publishing Journal of Physics B: Atomic, Molecular and Optical Physics

J. Phys. B: At. Mol. Opt. Phys. **51** (2018) 045006 (13pp) https://doi.org/10.1088/1361-6455/aaa399

Continuous-wave solutions and modulational instability in spinor condensates of positronium

Ishfaq Ahmad Bhat[1], T Mithun[2,3], B A Malomed[4,5] and K Porsezian[1]

[1] Department of Physics, Pondicherry University, Puducherry 605014, India
[2] Department of Physics, SP Pune University, Pune 411007, India
[3] Center for Theoretical Physics of Complex Systems, Institute for Basic Science (IBS), Daejeon 34051, Republic of Korea
[4] Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel
[5] ITMO University, St. Petersburg 197101, Russia

E-mail: ponzsol@yahoo.com

Received 28 August 2017, revised 8 December 2017
Accepted for publication 21 December 2017
Published 26 January 2018

Abstract

We obtain general continuous-wave (CW) solutions in the model of a spinor positronium condensate in the absence of magnetic field. The CW solutions with both in-phase ($n = 0$) and out-of-phase ($n = 1$) spin components exist, with their ranges limited by the total particle density, ρ. In the limit of negligible population exchange between the spin components, the CW solutions are found to be stable or unstable, depending on the particle density of the para-positronium. Ortho-positronium, in the $F = 1$ spinor state, forms a ferromagnetic condensate with stable in-phase CW solutions only. Subsequent examination of the modulational instability is carried out both in the limit case of identical wavenumbers in the spin components, $\Delta k \equiv k_1 - k_{-1} = 0$, and in the more general case of $\Delta k \neq 0$ too. The CW solutions with $n = 0$ and 1 solutions, which are stable in the case of $\Delta k = 0$, are unstable for $\Delta k \neq 0$, for the natural repulsive sign of the nonlinearities. The total particle density, ρ, in the limit of $\Delta k = 0$ is found to have a significant role in the stability of the condensate, which is determined by the sign of the self-interaction nonlinearity.

Supplementary material for this article is available online

Keywords: continuous-wave solutions, modulational instability, spinor condensates, positronium

(Some figures may appear in colour only in the online journal)

1. Introduction

Positronium (Ps) is a commonly known bound state of an electron and a positron, with the total angular momentum $F = 0$ or $F = 1$. The spin configurations with $F = 0$ and $F = 1$, viz., the para- ($^1S_0$) and ortho-Ps ($^3S_1$), are separated by an energy gap, $\Delta E = 8.44 \times 10^{-4}$ eV, and annihilate, respectively, by emitting two or three gamma-quanta [1, 2]. This makes positronium a useful source for 511 keV gamma-ray lasers [3, 4]. The lifetime of the para-Ps is 0.125 ns, while the ortho-Ps lives much longer, 142 ns. Referring to its longer lifetime, Platzmann and Mills [5] proposed, in 1994, the possibility of making ortho-Ps Bose–Einstein condensate (BEC) in a cold silicon cavity. As the condensation temperature is inversely proportional to the mass of the species under consideration, and proportional to its density, for positronium the condensation must occur at a much higher temperature and/or density compared to that for the usual bosonic atoms [6, 7]. Currently, spin-polarized ensembles of Ps atoms are available, but with densities of at least two orders of magnitude less than that required to form the condensate [8]. Nevertheless, a specific method has been recently proposed for the realization of the Ps BEC [6]. The method relies on the original cooling of Ps atoms through interaction with a cold silica cavity and Ps–Ps two-body collisions, followed by laser cooling.

Atomic condensates of rubidium (^{87}Rb) [9], sodium (^{23}Na) [10], and lithium (^{7}Li) [11] were first created in magnetic traps. In such BECs, the spin degree of freedom is frozen due to its coupling to the field, hence the system is defined by a scalar order parameter. Unlike magnetic traps, optical ones hold all spin components of a given hyperfine state, without forcing the atoms to align their spins in a specific direction. Spinor BECs with $2F+1$ spin components have been experimentally created in optical traps [12–14], thus providing an opportunity to explore intrinsic spin dynamics in the condensate. Spinor BECs display various phenomena, such as spin domains [15], skyrmions [16], magnetism and its dependence on scattering lengths [17], interaction-dependent ferromagnetic phase transitions [18], modulational instability (MI) in the case of repulsive nonlinearity [19–21], the existence of multicomponent solitons [22], oscillatory coherent spin mixing [23, 24], etc. Apart from these facts, spinor condensates, including nonlinear spin-exchange interactions, find uses in magnetometry [25] and atom interferometry [26].

The objective of the present work is to examine continuous-wave (CW) solutions and their MI in the spinor-BEC model of positronium. This is a relevant aim, as flat CW backgrounds support various dynamical phenomena, including modulational instability, dark and anti-dark solitons, vortices, etc [27]. MI is the exponential growth of the Bogoliubov modes of the miscible condensates [28]. In scalar BECs, MI solely depends on the sign of the nonlinear interactions, taking place for the attractive sign. However, in the case of $F=1$ [21, 29] and $F=2$ [30] spinor condensates, it was reported that the MI depends not only on the interaction, but is also sensitive to phase shifts between components and population ratios. Mixing the $F=0$ and $F=1$ spin components in positronium may lead to new interesting properties. Detailed experimental studies of the MI in self-attractive BEC were recently reported in [31] and [32].

The presentation in the paper is organized as follows. Section 2 introduces the theoretical model, based on the Gross–Pitaevskii (GP) equations governing the mean-field dynamics of the spinor condensate, ignoring effects produced by the finite lifetime (spontaneous annihilation) of the ortho-Ps. Section 3 addresses CW solutions, focusing on conditions for their existence and stability. Section 4 discusses the MI and dispersion relations in different cases. Section 5 summarizes the results of the analysis of the MI in the spinor positronium, and section 6 concludes the work. Section 6 also includes a brief discussion of possible manifestations of the obtained results in experimental studies of the MI.

2. The model

We consider a uniform spinor BEC in an optical trap, without external magnetic fields. Accordingly, positronium is free to realize any of its four spin states, labeled as $|p\rangle$ and $|1\rangle$, $|0\rangle$, $|-1\rangle$, with $|p\rangle$ representing the para state and $|-1, 0, 1\rangle$ standing for three values of the magnetic quantum number M in the ortho-state. Thus, different states $|F, M\rangle$ of the ortho positronium are designated by $|1, M\rangle$, while the para state corresponds to $|0, 0\rangle$. The Hamiltonian of this system is a combination of the non-interacting single-particle Hamiltonians and the interaction energy ($\hat{H}_{\text{int}}$) [33, 34]:

$$\hat{H} = \int \mathrm{d}^3\mathbf{r} \sum_{j=0,\pm1,p} \psi_j^{\dagger}\left[\frac{\mathbf{p}^2}{2m} + V_{\text{ext}}(\mathbf{r}) + \epsilon_j\right]\psi_j + \hat{H}_{\text{int}}\,, \quad (1)$$

where $\mathbf{p}$ is the momentum operator, $V_{\text{ext}}(\mathbf{r})$ the trapping potential, and ϵ_j the internal energy of the spin state j. In the mean-field approximation, the interaction Hamiltonian is (see its detailed derivation in [33]):

$$\hat{H}_{\text{int}} = \frac{1}{2}\int \mathrm{d}^3\mathbf{r}(\tilde{g}_0\rho^2 + \tilde{g}_1|2\psi_1\psi_{-1} - \psi_0^2 + \psi_p^2|^2), \quad (2)$$

where $\rho = \sum_{j=0,\pm1,p}|\psi_j|^2$ is the total atomic density, $\tilde{g}_0$ and $\tilde{g}_1$ are nonlinearity coefficients, defined by $\tilde{g}_0 = 4\pi\hbar^2 a_1/m$ and $\tilde{g}_1 = \pi\hbar^2(a_1 - a_0)/m$, while a_0 and a_1 are the scattering lengths of the para ($S=0$) and ortho ($S=1$) positronium, respectively (these parameters were theoretically calculated in several theoretical works [35–38]). An independent para-ortho scattering length does not appear in equation (2), as the interaction between the ortho- and para-positronium vanishes for an odd spin channel. This fact follows from the presentation of the atomic spin in the form of $F_{\text{collidingpair}} = F_{\text{ortho}} + F_{\text{para}} \equiv 1$ [17, 34].

For the BEC confined in a cigar-shaped optical trap, the corresponding Hamiltonian $\hat{H}$ with $\epsilon_j = \epsilon_o$ for $j = 0, \pm1$ produces a set of coupled one-dimensional GP equations, following the usual procedure of the dimensional reduction [33, 39, 40]:

$$\mathrm{i}\hbar\dot{\psi}_1 = (H_0 + \epsilon_o + 2g_1|\psi_{-1}|^2)\psi_1 + g_1\psi_{-1}^{*}(\psi_p^2 - \psi_0^2), \quad (3a)$$

$$\mathrm{i}\hbar\dot{\psi}_0 = (H_0 + \epsilon_o + g_1|\psi_0|^2)\psi_0 - g_1\psi_0^{*}(2\psi_1\psi_{-1} + \psi_p^2), \quad (3b)$$

$$\mathrm{i}\hbar\dot{\psi}_{-1} = (H_0 + \epsilon_o + 2g_1|\psi_1|^2)\psi_{-1} + g_1\psi_1^{*}(\psi_p^2 - \psi_0^2), \quad (3c)$$

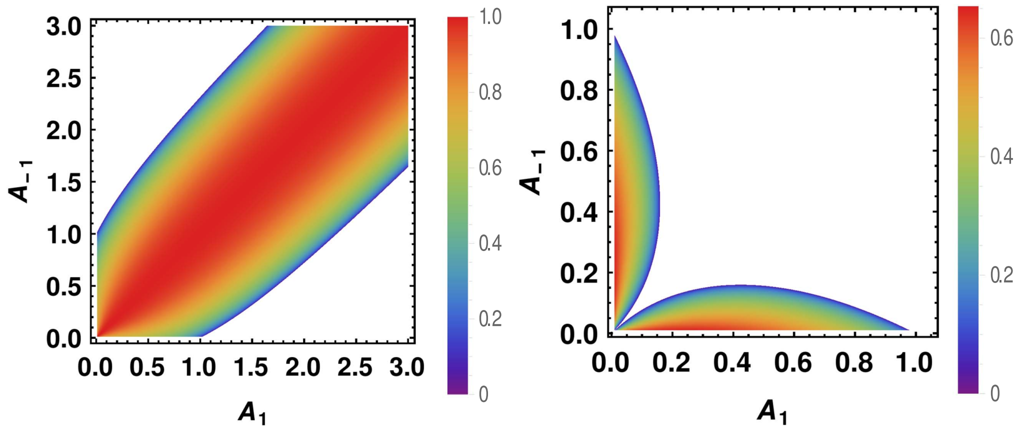
$$\mathrm{i}\hbar\dot{\psi}_p = (H_0 + \epsilon_p + g_1|\psi_p|^2)\psi_p + g_1\psi_p^{*}(2\psi_1\psi_{-1} - \psi_0^2) \quad (3d)$$

where the overdot stands for $\partial/\partial t$, and

$$H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + V_{\text{ext}}(z) + g_0\rho. \quad (4)$$

The annihilation of the positronium is disregarded here, hence the GP equations are valid on a limited time scale. The nonlinear coefficients are now defined by $g_0 = \frac{2\hbar^2}{ma_\perp^2}a_1$ and $g_1 = \frac{\hbar^2}{2ma_\perp^2}(a_0 - a_1)$ with $a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}}$, and $\omega_\perp$ is the transverse trap frequency. The terms $\sim g_1$ in equation (3) govern the population exchange between different spin states, while the total density remains constant. Equations (3) may be normalized to a dimensionless form by means of rescaling:

$$t' = t\omega_\perp \quad (5a)$$



(a) (b)

Figure 1. Existence ranges of CW solutions in positronium for $\rho = 1$ and $\gamma = 1$ ($g_1' > 0$). Different colors represent the corresponding amplitude A_0 in the CW solution. (a) The existence ranges for even n, represented by $n = 0$. (b) The existence ranges for odd n, represented by $n = 1$.

$$z' = \frac{z}{a_\perp} \tag{5b}$$

$$V'_{\mathrm{ext}}(z) = V_{\mathrm{ext}}(z)\hbar\omega_\perp \tag{5c}$$

$$\psi'_j(z) = \frac{\psi_j(z)}{\sqrt{2|a_\perp|}} \tag{5d}$$

The resulting dimensionless GP equations have the same form as above, but with $\hbar = 1$ and $m = 1$ and new dimensionless nonlinear coefficients $g_0' = a_1/a_\perp$ and $g_1' = (a_0 - a_1)/a_\perp$, while the internal energy of the jth state is measured in units of $\hbar\omega_\perp$. In the rest of the paper, the energies are shifted to $\epsilon_p = 0$ and $\epsilon_o \equiv \epsilon$. These scaled variables are used in figures displayed below, while equations are written in dimensional units.

3. CW solutions

We begin by considering the following general CW solutions [41]:

$$\psi_j = A_j \exp[\mathrm{i}(k_j z + \theta_j - \omega_j t)], \tag{6}$$

where $j = 0, \pm 1, p$ represents, as above, the respective spin components. Wavenumbers, phase shifts, and frequencies (chemical potentials) k_j, θ_j and ω_j are real, while amplitudes A_j are positive.

The substitution of the CW ansatz (6) for $j = 0, \pm 1, p$ into governing equations (3) yields

$$\hbar\omega_1 = \frac{\hbar^2 k_1^2}{2m} + g_0\rho + \epsilon + 2g_1 A_{-1}^2 + (-1)^n g_1 \frac{A_{-1}}{A - 1}(A_p^2 - A_0^2), \tag{7a}$$

$$\hbar\omega_0 = \frac{\hbar^2 k_0^2}{2m} + g_0\rho + \epsilon + g_1 A_0^2 - g_1(2(-1)^n A_{-1}A_1 + A_p^2), \tag{7b}$$

$$\hbar\omega_{-1} = \frac{\hbar^2 k_{-1}^2}{2m} + g_0\rho + \epsilon + 2g_1 A_1^2 + (-1)^n g_1 \frac{A_1}{A_{-1}}(A_p^2 - A_0^2), \tag{7c}$$

$$\hbar\omega_p = \frac{\hbar^2 k_0^2}{2m} + g_0\rho + g_1 A_p^2 + g_1(2(-1)^n A_{-1}A_1 - A_0^2), \tag{7d}$$

where n and s are integers, which are defined below in equation (8c), and the following relations between the wavenumbers, frequencies, and phase shifts of the different components must hold:

$$k_p = k_0 = \frac{1}{2}(k_1 + k_{-1}), \tag{8a}$$

$$\omega_p = \omega_0 = \frac{1}{2}(\omega_1 + \omega_{-1}), \tag{8b}$$

$$\theta_p + s\pi = \theta_0 = \frac{1}{2}(\theta_1 + \theta_{-1} + n\pi) \tag{8c}$$

For the compatibility of equation (8) with the other equations, amplitude A_0 of the ψ_0 component must satisfy the condition

$$A_0^2 = \frac{1}{2}\left[\rho - (A_1 - (-1)^n A_{-1})^2 + \frac{2\gamma(-1)^n A_1 A_{-1}}{(A_1^2 + A_{-1}^2)}\right], \tag{9}$$

where

$$\gamma \equiv \frac{\hbar^2(k_1 - k_{-1})^2}{8mg_1} + \frac{\epsilon}{2g_1}. \tag{10}$$

The second term on the right hand side of equation (10) displays the competition between the internal energy difference ϵ and the spin-mixing interaction with the effective 1D strength g_1. The ortho- to para- interconversion is substantial for $g_1 \gg \epsilon$.

From the CW ansatz it follows that the condition of A_0 being positive makes the left hand side of equation (9) real and non-negative, thus giving a criterion for the existence of

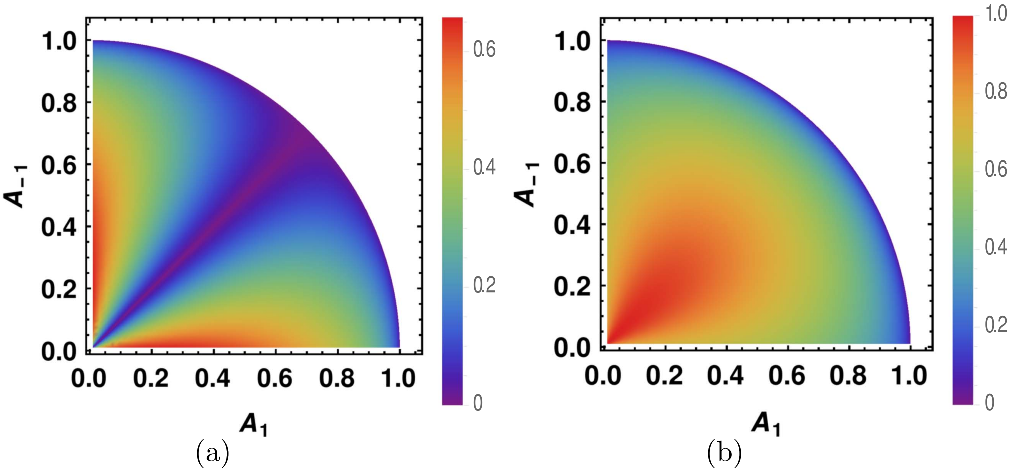


Figure 2. Existence ranges of CW solutions in positronium for $\rho = 1$ and $\gamma = -1$ ($g_1' < 0$). Different colors represent the corresponding amplitude A_0 in the CW solution. (a) The existence ranges for even n, represented by $n = 0$. (b) The existence ranges for odd n, represented by $n = 1$.

the CW solutions. Thus, for the ground state of positronium with $g_1 > 0$, CW solutions for both even n and odd n exist, but with different existence ranges, as shown in figure 1. These ranges for different CW solutions are found to depend on the total number density, ρ, in addition to the magnitude and sign of g_1, which, in turn, can be tuned by means of the Feshbach-resonance techniques [42]. In particular, the CW solutions with even n (represented by $n = 0$) exist if

$$\gamma \geqslant \frac{(A_1^2 + A_{-1}^2)}{2A_1A_{-1}}[(A_1 - A_{-1})^2 - \rho]. \tag{11}$$

There are CW solutions with odd n (represented by $n = 1$) if

$$\gamma \leqslant \frac{(A_1^2 + A_{-1}^2)}{2A_1A_{-1}}[\rho - (A_1 + A_{-1})^2]. \tag{12}$$

Clearly, values which A_1 and A_{-1} must assume for the fulfillment of the conditions for the existence of the respective CW solutions are limited by the total density, ρ.

The case of $g_1 < 0$, which may be realized, as mentioned above, with the help of the Feshbach resonance, modifies the existence ranges of the CW solutions, as shown in figure 2. There are CW solutions for even n when

$$\gamma \leqslant \frac{(A_1^2 + A_{-1}^2)}{2A_1A_{-1}}[\rho - (A_1 - A_{-1})^2]. \tag{13}$$

However, for odd n, CW solutions exist at

$$\gamma \leqslant \frac{(A_1^2 + A_{-1}^2)}{2A_1A_{-1}}[(A_1 + A_{-1})^2 - \rho]. \tag{14}$$

In accordance with equation (8*b*), and combining equations (7*b*) and (7*d*) in the limit of $g_1 \gg \epsilon$, we arrive at an equation similar to the one obtained in [33], which relates the ortho- and para- populations of the condensate:

$$A_p^2 = A_0^2 - 2(-1)^n A_1^2 A_{-1}^2. \tag{15}$$

The ground state is found to have the maximum para-population for odd ($n = 1$) CW solutions, which, for equal amplitudes of the $M = \pm 1$ components, becomes equal to the density of the ortho- component:

$$\rho_p = \rho_o - 2(A_1 + (-1)^n A_{-1})^2 \tag{16}$$

where $\rho_p = A_p^2$ and $\rho_o = A_1^2 + A_0^2 + A_{-1}^2$ are densities of the para- and ortho- components, respectively. The ortho- sector of the condensate for equal densities of ortho- and para-components is equivalent to a polar state of the $F = 1$ condensate [17, 33].

For the stability analysis of the CW solutions, we neglect the ortho-to-para interconversion, assuming $g_1 \ll \epsilon$. Except for the wavenumbers k_j with $j = 0, \pm 1, p$, CW parameters, such as the amplitudes A_j and the phases θ_j, are made time-dependent. The linearization of equations (3), taking into account the conservation of total density, $\rho = \sum_{j=0,\pm 1,p} A_j^2$, and magnetization, $(A_{-1}^2 - A_1^2)$, and subsequent linearization with respect to amplitude and phase perturbations (the latter reduce to $\delta\theta(t) = (\theta_1(t) + \theta_{-1}(t) - 2\theta_0(t))$) shows that the CW solutions are either stable or unstable, depending on the density of the para-component relative to that of the one with $M = 0$ in the ortho-component. Oscillation eigenfrequencies of infinitesimal perturbations are then given by

$$\omega_{n=0,1}^2 = 2\left(\frac{g_1 A_0}{\hbar}\right)^2 \left[(A_1 + (-1)^n A_{-1})^2 + \frac{(A_{-1}^2 - A_1^2)^2 (A_0^2 - A_p^2)}{4A_1^2 A_{-1}^2}\right]. \tag{17}$$

These frequencies are real, and hence the CW solutions are stable, if $A_0^2 > A_p^2$. If $A_0^2 < A_p^2$, then, for $n = 0$, the CW solutions are stable if

$$A_p^2 < A_0^2 + \left(\frac{2A_1A_{-1}}{A_{-1} - A_1}\right)^2. \tag{18}$$

Likewise, for $n = 1$ and $A_1 \neq A_{-1}$, the CW solutions are stable if

$$A_p^2 < A_0^2 + \left(\frac{2A_1A_{-1}}{A_{-1} + A_1}\right)^2. \tag{19}$$

Table 1. Possible CW solutions with respective conditions for their existence in the spinor condensate of positronium for different values of $\gamma = \frac{1}{8mg_1}[\hbar^2(k_1 - k_{-1})^2 + 4m\epsilon]$. A_1 and A_{-1} are allowed amplitudes of the respective spin components, and ρ is the total density.

γ	ψ_p	CW ($n = 0$)	CW ($n = 1$)
$\geqslant 0$	0	$A_1, A_{-1} \in R \geqslant 0$	No Solutions
	$\neq 0$	$\gamma \geqslant \frac{(A_1^2 + A_{-1}^2)}{2A_1A_{-1}}[(A_1 - A_{-1})^2 - \rho]$	$\gamma \leqslant \frac{(A_1^2 + A_{-1}^2)}{2A_1A_{-1}}[\rho - (A_1 + A_{-1})^2]$
<0	$\neq 0$	$\gamma \leqslant \frac{(A_1^2 + A_{-1}^2)}{2A_1A_{-1}}[\rho - (A_1 - A_{-1})^2]$	$\gamma \leqslant \frac{(A_1^2 + A_{-1}^2)}{2A_1A_{-1}}[(A_1 + A_{-1})^2 - \rho]$

The respective spin oscillations with real frequency ω represent the exchange of populations between the different spin states [43, 44].

In the limit of $\Psi_p = 0$, we are left with the ortho-Ps condensate only, the system being equivalent to the $F = 1$ spinor condensate. The spinor ortho-Ps condensate with $g_1 > 0$ supports solely even ($n = 0$) CW solutions for all values of A_1 and A_{-1} with

$$A_0^2 = 2(-1)^n A_1 A_{-1}\left(1 + \frac{\frac{\hbar^2}{2mg_1}\left(\frac{\Delta k}{2}\right)^2}{(A_1 + (-1)^n A_{-1})^2}\right), \tag{20}$$

where $\Delta k \equiv k_1 - k_{-1}$. This is a peculiarity of the ferromagnetic states of the $F = 1$ spinor condensate [41]. Thus, the CW states of the ortho-Pt only represent a ferromagnetic condensate. The conditions for the existence of the possible CW solutions in spinor Ps are summarized in table 1.

The stability analysis of the CW solutions by means of the linearization with respect to the amplitude and the phase perturbations show that the CW solutions are stable against the infinitesimal perturbations, with real perturbation eigenfrequencies

$$\omega_{n=0}^2 = \left(\frac{g_1 A_0^2}{\hbar}\right)^2\left[\frac{(A_{-1}^2 - A_1^2)^2}{A_1^2 A_{-1}^2} + \frac{8(A_1 + A_{-1})^2}{A_0^2}\right]. \tag{21}$$

These are frequencies of the coherent spin mixing, which account for the exchange of populations between the different spin states of the ortho-Ps.

4. Modulational instability (MI)

To investigate MI in the Ps BEC, described by the dynamical equations (3), we start by addressing small perturbations $\delta\phi(z, t)$ added to the CW solutions [20, 21, 28, 41, 45, 46]:

$$\phi_j = [A_j + \delta\phi_j(z, t)]\exp[\mathrm{i}(k_j z + \theta_j - \omega_j t)] \tag{22}$$

where j defines the spin index 0, ± 1 and p.

Assuming that the system size is greater than the healing length, which determines the characteristic length scale for MI, we assume the perturbations to be in the form of plane waves,

$$\delta\phi_j(z, t) = \lambda_j \cos(kz - \omega t) + \mathrm{i}\,\eta_j \sin(kz - \omega t), \tag{23}$$

where λ_j and η_j are perturbation amplitudes, while k and ω are the wavenumber and the (generally complex) frequency, respectively.

The substitution of equations (22) and (23) into equation (3), for the four spin indices, gives a set of eight homogeneous equations, with respect to λ_j and η_j, in the matrix form:

$$\mathcal{M}\Psi = \left([-\hbar\omega + \frac{\hbar^2 k}{2m}(k_1 + k_{-1})]\check{\mathbf{I}} + \mathbf{X} + \mathbf{Y} + \mathbf{Z}\right)\Psi = 0 \tag{24}$$

where $\Psi = (\eta_1, \lambda_1, \eta_0, \lambda_0, \eta_{-1}, \lambda_{-1}, \eta_p, \lambda_p)^T$, $\check{\mathbf{I}}$ is the usual unit matrix and

$$X = \frac{\hbar^2}{2m}k\begin{pmatrix} \Delta k & k & 0 & 0 & 0 & 0 & 0 & 0 \\ k & \Delta k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta k & k & 0 & 0 \\ 0 & 0 & 0 & 0 & k & -\Delta k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 & 0 & 0 & k & 0 \end{pmatrix} \tag{25a}$$

$$Y = 2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_0 A_1^2 & 0 & g_0 A_0 A_1 & 0 & (g_0 + 2g_1)A_1 A_{-1} & 0 & g_0 A_1 A_p & 0 \\ 0 & 0 & 0 & g_1 A_p^2 & 0 & 0 & 0 & -g_1 A_0 A_p \\ g_0 A_1 A_0 & 0 & (g_0 + g_1)A_0^2 & 0 & g_0 A_0 A_{-1} & 0 & (g_0 - g_1)A_0 A_p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (g_0 + 2g_1)A_1 A_{-1} & 0 & g_0 A_0 A_{-1} & 0 & g_0 A_{-1}^2 & 0 & g_0 A_{-1} A_p & 0 \\ 0 & 0 & 0 & -g_1 A_0 A_p & 0 & 0 & 0 & g_1 A_0^2 \\ g_0 A_1 A_p & 0 & (g_0 - g_1)A_0 A_p & 0 & g_0 A_1 A_p & 0 & (g_0 + g_1)A_p^2 & 0 \end{pmatrix} \tag{25b}$$

$$Z = (-1)^n g_1 \begin{pmatrix} 0 & \frac{A_{-1}}{A_1}(A_0^2 - A_p^2) & 0 & -2A_0A_{-1} & 0 & (A_0^2 - A_p^2) & 0 & 2A_{-1}A_p \\ \frac{A_{-1}}{A_1}(A_0^2 - A_p^2) & 0 & -2A_0A_{-1} & 0 & -(A_0^2 - A_p^2) & 0 & 2A_{-1}A_p & 0 \\ 0 & -2A_0A_{-1} & 0 & 4A_1A_{-1} & 0 & -2A_0A_1 & 0 & 0 \\ -2A_0A_{-1} & 0 & 0 & 0 & -2A_0A_1 & 0 & 0 & 0 \\ 0 & (A_0^2 - A_p^2) & 0 & -2A_0A_1 & 0 & \frac{A_1}{A_{-1}}(A_0^2 - A_p^2) & 0 & 2A_1A_p \\ -(A_0^2 - A_p^2) & 0 & -2A_1A_0 & 0 & \frac{A_1}{A_{-1}}(A_0^2 - A_p^2) & 0 & 2A_1A_P & 0 \\ 0 & 2A_{-1}A_p & 0 & 0 & 0 & 2A_1A_p & 0 & -4A_1A_{-1} \\ 2A_{-1}A_p & 0 & 0 & 0 & 2A_1A_p & 0 & 0 & 0 \end{pmatrix} \tag{25c}$$

Frequency ω of the perturbations, the average wavenumber, $k_0 = k_p = \frac{(k_1 + k_{-1})}{2}$, and the wavenumber difference, Δk, are diagonal entries of the stability matrix. In the most general case, the wavenumber difference assumes a nonzero value, $\Delta k \neq 0$, while $\Delta k = 0$ corresponds to the limit case, in which all the wavenumbers are equal. The off diagonal elements depend upon various CW parameters, either explicitly or implicitly via A_0 (see equation (9)) and A_p (see equation (15)).

Equation (24) represents an eigenvalue problem for matrix $\mathcal{M}$. The solution to the problem aims to identify all possible eigenvalues $\hbar\omega$ and the corresponding eigenvectors $(\eta_j, \lambda_j; j = 1, 2, \cdots, 8)$. In this study, we produce solutions for eigenvalues $\hbar\omega$ for both cases of $\Delta k = 0$ and $\Delta k \neq 0$. Complex eigenvalues $\hbar\omega$, if obtained by solving the dispersion equation, $\det(\mathcal{M}) = 0$, for some positive values of k^2, render the spinor condensate modulationally unstable. In the present case, this is the eighth-order equation with respect to $\hbar\omega$, with the dependence on the CW parameters implied in the coefficients, $C^{n=0,1}_{k^\alpha,\omega^\beta}$, while α and β are integer numbers:

$$\begin{aligned} &(\hbar\omega)^8 + (\hbar\omega)^7 C^{n=0,1}_{k,\omega^7} k + (\hbar\omega)^6 \sum_{j=1}^{3} C^{n=0,1}_{k^{2j-2},\omega^6} k^{2j-2} \\ &+ (\hbar\omega)^5 \sum_{j=1}^{3} C^{n=0,1}_{k^{2j-1},\omega^5} k^{2j-1} + (\hbar\omega)^4 \sum_{j=1}^{4} C^{n=0,1}_{k^{2j},\omega^4} k^{2j} \\ &+ (\hbar\omega)^3 \sum_{j=1}^{4} C^{n=0,1}_{k^{2j+1},\omega^3} k^{2j+1} + (\hbar\omega)^2 \sum_{j=1}^{5} C^{n=0,1}_{k^{2j+2},\omega^2} k^{2j+2} \\ &+ (\hbar\omega) \sum_{j=1}^{5} C^{n=0,1}_{k^{2j+3},\omega^3} k^{2j+3} + \sum_{j=1}^{6} C^{n=0,1}_{k^{2j+4},\omega^0} k^{2j+4} = 0 \end{aligned} \tag{26}$$

The coefficients $C^{n=0,1}_{k^\alpha,\omega^\beta} = \frac{1}{\alpha!\,\beta!}\frac{\partial^\alpha}{\partial k^\alpha}\frac{\partial^\beta}{\partial\,\omega^\beta}|\mathcal{M}|$ represent a blend of nonlinearity coefficients and various CW parameters, being too cumbersome to be included in the main text. They are explicitly displayed in Supplement 1, which is available online at stacks.iop.org/JPB/51/045006/mmedia.

Equation (26) can be solved analytically only in a few limit cases, as shown below.

For the ortho-Ps condensate ($\psi_p = 0$), for which only the in-phase CW solutions (with $n = 0$) exist, the sixth-order dispersion equation is

$$\begin{aligned} &(\hbar\omega)^6 + (\hbar\omega)^5 C^{n=0}_{k,\omega^5} k + (\hbar\omega)^4 \sum_{j=1}^{3} C^{n=0}_{k^{2j-2},\omega^4} k^{2j-2} \\ &+ (\hbar\omega)^3 \sum_{j=1}^{3} C^{n=0}_{k^{2j-1},\omega^3} k^{2j-1} + (\hbar\omega)^2 \sum_{j=1}^{4} C^{n=0}_{k^{2j},\omega^2} k^{2j} \\ &+ (\hbar\omega) \sum_{j=1}^{4} C^{n=0}_{k^{2j+1},\omega} k^{2j+1} + \sum_{j=1}^{5} C^{n=0}_{k^{2j+2},\omega^0} k^{2j+2} = 0. \end{aligned} \tag{27}$$

Expressions for coefficients, $C^{n=0}_{k^\alpha,\omega^\beta}$, are again too cumbersome for the main text and can be found in the online supporting information, Supplement 2.

In the case of zero fields with $M = \pm 1$, the dynamics of the system is governed by a pair of coupled GP equations, and the resulting characteristic polynomial of the fourth order is analytically soluble:

$$\begin{aligned} &(\hbar\omega)^4 + (\hbar\omega)^3 C_{k,\omega^3} k + (\hbar\omega)^2 \sum_{j=1}^{3} C_{k^{2j-2},\omega^2} k^{2j-2} \\ &+ (\hbar\omega) \sum_{j=1}^{3} C_{k^{2j-1},\omega} k^{2j-1} + \sum_{j=1}^{4} C_{k^{2j},\omega^0} k^{2j} = 0. \end{aligned} \tag{28}$$

Here coefficients $C_{k^\alpha,\omega^\beta}$, being functions of the nonlinearity coefficients g_0 and g_1, and can be found in the online supporting information, Supplement 3. These coupled GP equations have been extensively studied in terms of optics [47, 48] and BEC [49, 50].

5. Results and discussions

Below we discuss cases for which the dispersion relations of different orders obtained above are solved for the perturbation

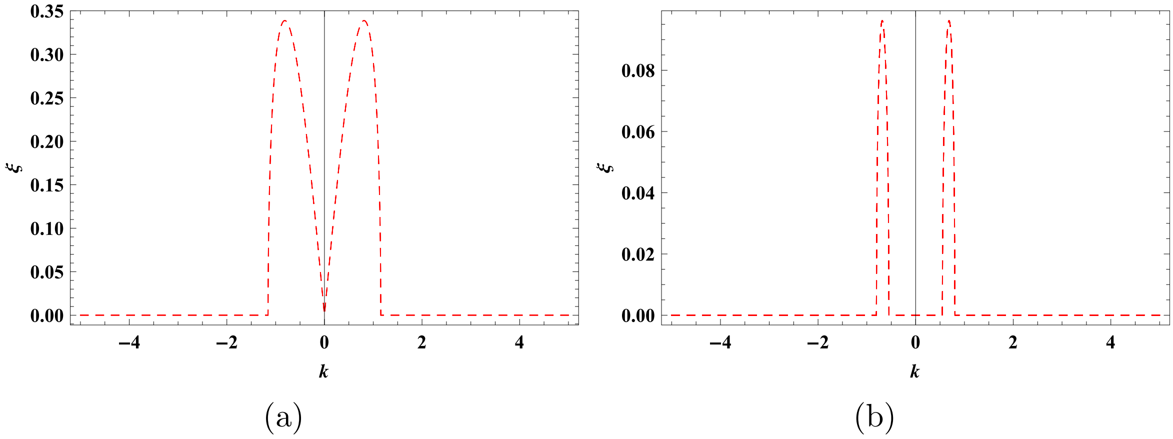


Figure 3. The MI gain for the CW states in the two-component positronium for $A_0 = \frac{1}{3}$ and $A_p = \frac{1}{2}$. The CW background is modulationally unstable for a proper choice of the amplitudes of the spin components if either of the nonlinearities are repulsive. (a) The MI gain for $g_0' = -1$ and $g_1' = 1$ (b) The MI gain for $g_0' = 1$ and $g_1' = -1$.

frequency in the limiting case of equal wavenumbers, $\Delta k = 0$, when simple analytical solutions are obtained, as well as in the general case when wavenumbers of different components are not equal, and related solutions are not available. In the latter case, results are plotted for difference $\Delta k = 1$ between scaled wavenumbers of the spin components with $M = \pm 1$.

5.1. Limit cases with simple analytic solutions

The simplest and the most generic case corresponds to the single-component model with three of the four spin components equal to zero. In this case, the CW frequency (chemical potential) is

$$\hbar\omega_j = \frac{\hbar^2 k_j^2}{2m} + \epsilon + gA_j^2, \tag{29}$$

and the eigenfrequency of the perturbations is given by

$$\hbar\omega = \frac{\hbar^2 k}{m} k_j \pm \sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2gA_j^2\right)}, \tag{30}$$

with $g = g_0$ for spins $j = \pm 1$, $g = g_0 + g_1$ for spins $j = 0$, p, and $\epsilon = 0$ for the para component. Equation (30) is the Bogoliubov dispersion relation [51] for the propagation of small perturbations (sound waves) on top of CW solutions (6). The results obtained here are found to be in complete agreement with the previously reported ones [52–54]. Namely, for the attractive nonlinearity ($g < 0$), there is MI against the perturbations in the wavenumber range $0 < k < 2\sqrt{|g|m}A_j/\hbar$, characterized by the MI gain, $\xi = \mathrm{Im}(\omega)$. The maximum MI gain,

$$\xi_{\max} = |g|A_j/\hbar, \tag{31}$$

is attained at

$$k_{\max} = \sqrt{2|g|m}A_j/\hbar. \tag{32}$$

In the case of zero components with $M = \pm 1$, the BEC consists of a mixture of $|1, 0\rangle$ and $|0, 0\rangle$ spin components with vanishing interspecific interaction. The frequency of the perturbations is in fact solutions of equation (28) for the same wavenumbers ($k_0 = k_p = k$) and the Bogoliubov dispersion relation with $\rho_2 = A_0^2 + A_p^2$ and $\alpha = 8A_0^2 A_p^2 g_1^2$ modifies as

$$\hbar\omega = \frac{\hbar^2 k^2}{m} \pm \left[\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + \rho_2(g_0 + 2g_1)\right) + \alpha \pm \left\{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m}\left(\rho_2 g_0^2 + 2\alpha\left(1 - \frac{g_0}{g_1}\right)\right) + 2\rho_2\alpha\left(2 - \frac{g_0}{g_1}\right)\right) + \alpha^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{33}$$

From the above equation (33), it is found that MI in the system for a proper choice of A_0 and A_p, as shown in figure 3, is possible if either of the nonlinearities, g_0 or g_1, is attractive ($g_0 < 0$ or $g_1 < 0$), which in fact does not hold for the Ps BEC, hence the system is modulationally stable. The computation of relations for $k_{\max}$, and maximum growth rate ξ in terms of the nonlinearities and the respective wavenumbers, is too complex to be performed analytically.

In accordance with the above result, and aiming to properly choose A_0 and A_p, we studied the variation of the largest MI gain for a range of values of A_0 and A_p. In the limit case of equal wavenumbers, figure 4 shows the peak MI gain versus amplitudes of the two spin components for $g_0 < 0$ and $g_0 > 0$, respectively. In the latter case, the system is modulationally unstable if and only if condition $g_1 < 0$ holds, and amplitudes of the spin components are different, as seen in figure 4(b).

The analytic solutions for the perturbation frequency around the $n = 0$ CW background in the case of the ortho Ps condensate, i.e., the three-component system with identical

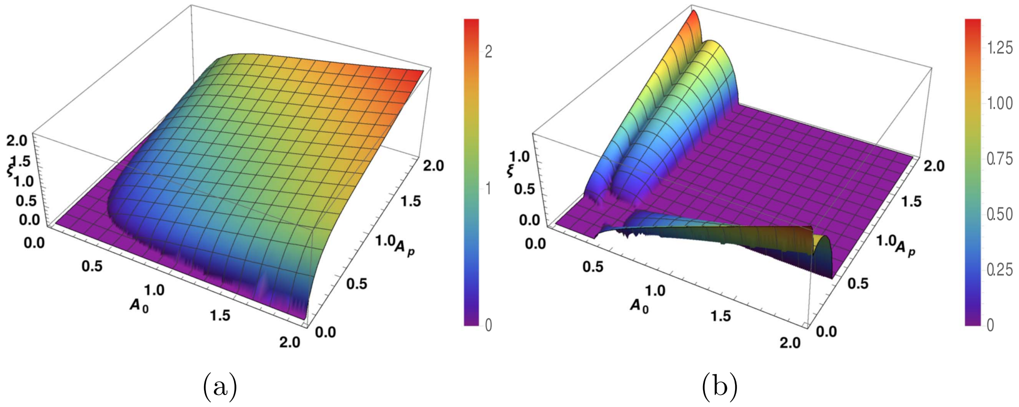


Figure 4. The dependenece of the largest MI gain on the amplitudes of the two CW components with identical wavenumbers, $k_p = k_0$. (a) The dependence of the maximum gain on A_0 and A_p for $g_0' = -1$ and $g_1' = 1$. (b) The dependence of the maximum gain on A_0 and A_p for $g_0' = 1$ and $g_1' = -1$.

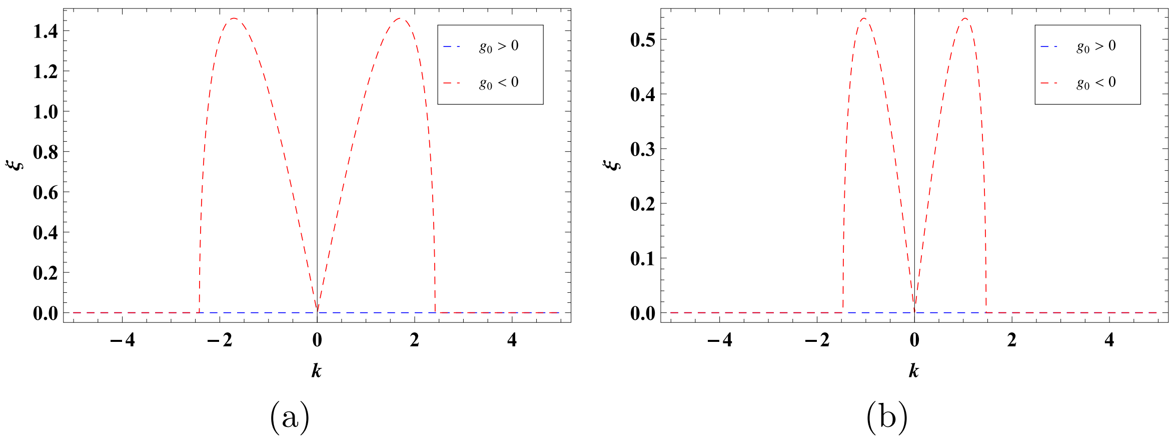


Figure 5. The MI gain for the CW solutions with $n = 0,1$, for $\rho = 1$, $A_1 = \frac{1}{2}$ and $A_{-1} = \frac{1}{3}$. The CW solutions with both $n = 0$ and $n = 1$ are modulationally unstable for $g_0' < 0$. (a) The MI gain for $n = 0$, $g_1' = 1$ or $n = 1$, $g_1' = -1$. (b) The MI gain for $n = 0$, $g_1' = -1$ or $n = 1$, $g_1' = 1$.

wavenumbers are

$$\hbar\omega = (2 \pm 1)\frac{\hbar^2k^2}{2m}, \tag{34a}$$

$$\hbar\omega = \left[(2 \pm 1)\frac{\hbar^2k^2}{2m} \pm 2g_1(A_1 + A_{-1})^2\right]. \tag{34b}$$

$$\hbar\omega = \frac{\hbar^2k^2}{m} \pm \sqrt{\frac{\hbar^2k^2}{2m}\left[\frac{\hbar^2k^2}{2m} + 2g_0(A_1 + A_{-1})^2\right]}. \tag{34c}$$

The results given by equation (34) are in exact agreement with the previously known results for the ferromagnetic $F = 1$ spinor condensates [41]. Thus, the $n = 0$ CW background of the ferromagnetic ortho Ps with $g_0 > 0$ is stable against the exponential growth of perturbations when the wavenumbers are equal, and it is modulationally unstable if $g_0 < 0$, so that the largest MI gain, $\xi_{\max} = -g_0\frac{(A_1 + A_{-1})^2}{\hbar}$, occurs at $k_{\max} = \sqrt{-2mg_0\frac{(A_1 + A_{-1})^2}{\hbar}}$.

Analytical solutions of equation (26) in the form of Bogoliubov dispersion relations are obtained for the perturbations propagating on top of the CWs with $n = 0$ and 1 CW, in the case of identical wavenumbers:

$$\hbar\omega = (2 \pm 1)\frac{\hbar^2k^2}{2m}, \tag{35a}$$

$$\hbar\omega = (2 \pm 1)\frac{\hbar^2k^2}{2m}, \tag{35b}$$

$$\hbar\omega = (2 \pm 1)\frac{\hbar^2k^2}{2m} \pm 2\rho g_1 \pm (-1)^n\frac{2\epsilon A_1A_{-1}}{(A_1^2 + A_{-1}^2)}, \tag{35c}$$

$$\hbar\omega = \frac{\hbar^2k^2}{m} \pm \sqrt{\frac{\hbar^2k^2}{2m}\left[\frac{\hbar^2k^2}{2m} + 2g_0\left(\rho + \frac{(-1)^n\epsilon A_1A_{-1}}{g_1(A_1^2 + A_{-1}^2)}\right)\right]}. \tag{35d}$$

From the above equations it is clear that, in the limit case of identical wavenumbers, the CW backgrounds with $n = 0$ and 1 give rise to MI and, thereby, formation of solitons if and only if condition $g_0 < 0$ holds, with the largest growth

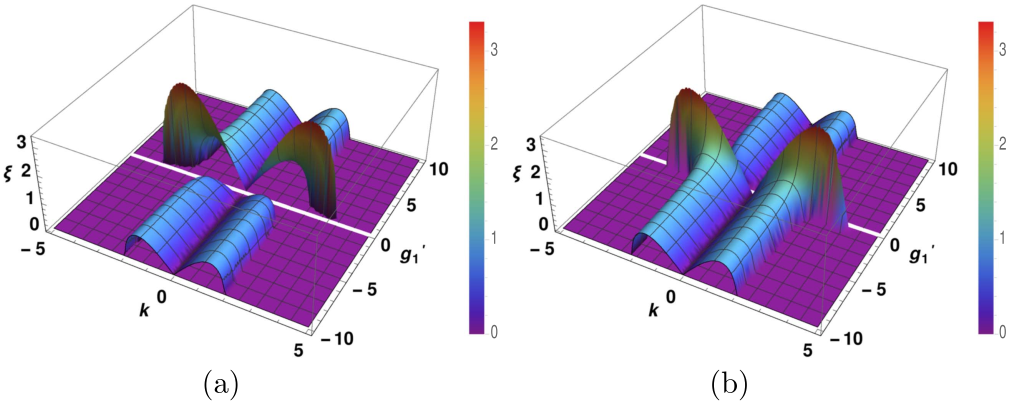


Figure 6. The dependence of the MI gain on g_1 for the CW solutions with $n = 0$ and 1 for $\rho = 1$, $A_1 = \frac{1}{2}$, $A_{-1} = \frac{1}{3}$ and $k_1 - k_{-1} = 0$. Both CW solutions with $n = 0$ and 1 are modulationally unstable for $g_0' < 0$. (a) The dependence of the MI gain on g_1' for $n = 0$ and $g_0' = -1$. (b) The dependence of the MI gain on g_1' for $n = 1$ and $g_0' = -1$.

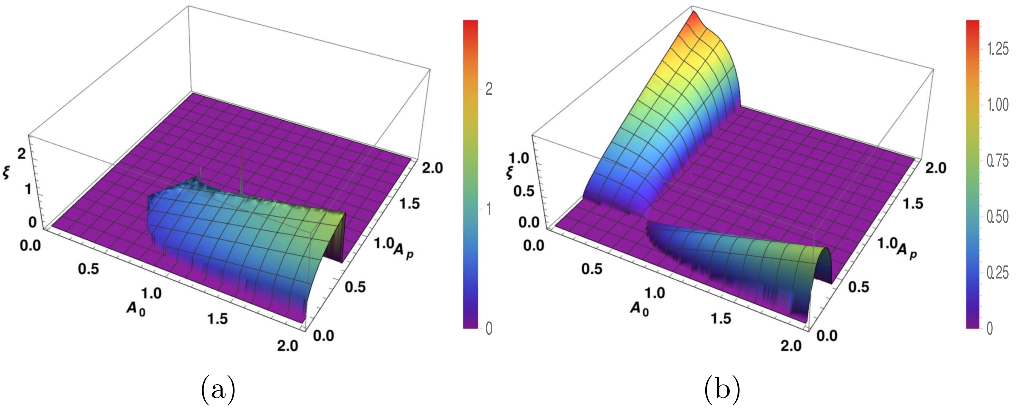


Figure 7. The dependence of the maximum MI gain on amplitudes of two-component condensate with the wavenumbers difference $k_0 - k_p = 1$. (a) The dependence of maximum MI gain on A_0 and A_p for $g_0' = -1$ and $g_1' = 1$. (b) The dependence of maximum MI gain on A_0 and A_p for $g_0' = 1$ and $g_1' = -1$.

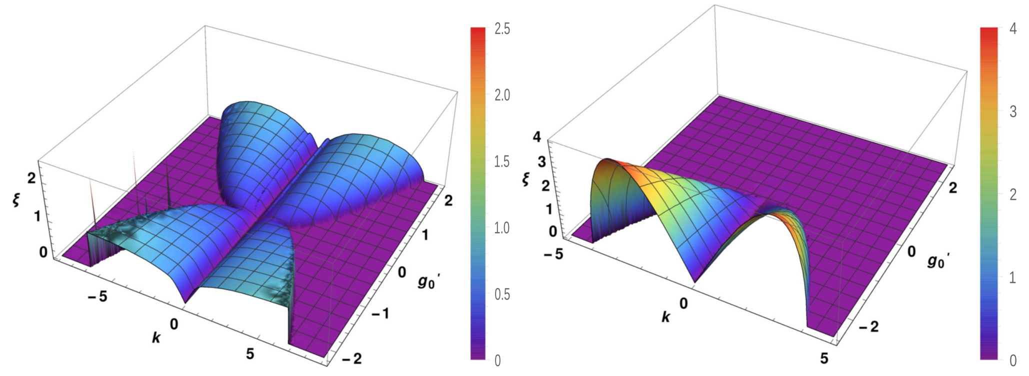
rate, $\xi_{\max} = -\frac{g_0}{\hbar}\left(\rho + (-1)^n \frac{\epsilon A_1 A_{-1}}{g_1(A_1^2 + A_{-1}^2)}\right)$, corresponding to $k_{\max} = \frac{1}{\hbar}\sqrt{-2mg_0\left(\rho + (-1)^n \frac{\epsilon A_1 A_{-1}}{g_1(A_1^2 + A_{-1}^2)}\right)}$. Thus, the Ps condensate with repulsive nonlinearities is modulationally stable in such a case. The total particle density, ρ, together with the densities of the components with $M = \pm 1$, have a significant impact on the stability of the CW solutions, while the order (n) of the CW solution does not affect the stability. However, the structure of the CW solution and the nonlinearity coefficient g_1 affect the instability strength for $g_0 < 0$, with a larger MI gain for $n = 0$, $g_1 > 0$ and $n = 1$, $g_1 < 0$, as shown in figure 5.

For $g_0 < 0$, figure 6 shows the dependence of the gain on the g_1 nonlinearity coefficient in the limit case of $k_1 = k_{-1}$. It is evident that the CW backgrounds with both $n = 0$ and 1 exhibit similar MI regions for opposite signs of nonlinearity g_1. The MI gain is larger at $g_1 > 0$ for the CW solutions with $n = 0$, and at $g_1 < 0$ for the CW with $n = 1$, respectively, for a fixed attractive g_0 nonlinearity. Note that, on either side of $g_1 = 0$ at fixed g_0, the gain attains constant value after an initial change.

5.2. General case with no simple analytic solutions

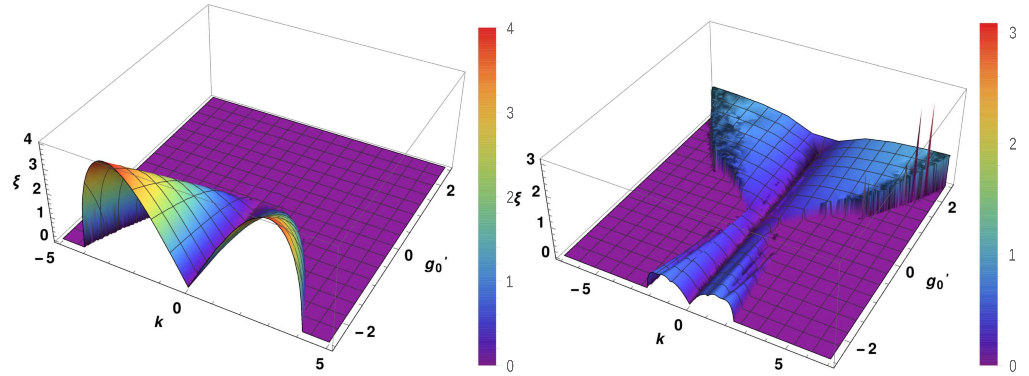
We have derived the dispersion relations for different cases in the limit case of identical wavenumbers in different spin components. However, in the general case, when the wavenumbers of the components with $M = \pm 1$ are different, such simple analytic results cannot be obtained. Note that even if k_1 and k_{-1} assume different values, wavenumbers k_p and k_0 are constrained to be equal, according to equation (8*a*), unless the field components with $M = \pm 1$ vanish. The effect of the CW order n, and of the nonlinearity coefficients g_0 and g_1, on the condensate's MI, for both zero and nonzero differences in the wavenumbers of the $M = \pm 1$ components, were analyzed by plotting the gain, $\xi = \mathrm{Im}(\omega)$, versus the perturbation wavenumber k for suitable values of the parameters, including A_1 and A_{-1}. The possibility of having attractive nonlinearities, achievable via the Feshbach resonance, and their effect on the stability of the condensate, are also discussed in the respective cases.

We chose $\rho = 1$, $A_1 = \frac{1}{2}$ and $A_{-1} = \frac{1}{3}$ for the case when the wavenumbers are identical, and $\rho = 1$ and $A_1 = A_{-1} = \frac{1}{2}$,



(a) (b)

Figure 8. The dependence of the MI gain on g_0 in the out-of-phase CW solution for $\rho = 1$, $A_1 = A_{-1} = \frac{1}{2}$ and $\Delta k = 1$. The CW solutions with $n = 1$ are modulationally stable if the nonlinearities g_0' and g_1' have opposite signs and $g_1' < 0$. (a) The variation of the MI gain for $g_1' = 1$. (b) The variation of the MI gain for $g_1' = -1$.



(a) (b)

Figure 9. The dependence of the MI gain on g_0 for CW solutions with $n = 0$ for $\rho = 1$, $A_1 = A_{-1} = \frac{1}{2}$ and $\Delta k = 1$. These solutions are modulationally stable if the nonlinearities g_0' and g_1' have identical signs and $g_1' > 0$. (a) The variation of the MI gain for $g_1' = 1$. (b) The variation of MI gain for $g_1' = -1$.

as the total particle density and the dimensionless amplitudes of the field components with $M = \pm 1$, respectively, in the case of the wavenumber difference $\Delta k = 1$. The case of $\Delta k \neq 0$ with different amplitudes $A_1 \neq A_{-1}$ is not analytically solvable.

In accordance with the above set parameters, for a CW solution with zero $M = \pm 1$ components, we chose $A_p = \frac{1}{2}$ and $A_0 = \frac{1}{3}$ in the case when the wavenumbers are identical, the MI sets in for $g_1 < 0$ if and only if the amplitudes of the spin components $|1, 0\rangle$ and $|0, 0\rangle$ are different. However, when the wavenumbers of the two spin components are different, MI occurs even if A_0 and A_p share the same values for repulsive g_0 and attractive g_1 nonlinearities, as shown in figure 7.

Comparison of figures 4(a) and 7(a) makes it obvious that, for $g_0 < 0$, the CW states with the zero $M = \pm 1$ component and nonzero difference between the wavenumbers of the other components are less vulnerable to the MI in a larger domain of values of the A_0 and A_p amplitudes, than in the limit case of identical wavenumbers.

As demonstrated in the previous subsection, the CW states of spinor Ps with both $n = 0$ and 1, $g_0 > 0$, and identical wavenumbers are modulationally stable. Nevertheless, when the wavenumbers of the $M = \pm 1$ components are different, the $n = 1$ CW solution is modulationally unstable for the natural repulsive signs of the g_0 and g_1 nonlinearities, as shown in figure 8 . The gain is found to increase with the increase in magnitude of g_0 for fixed $g_1 > 0$.

The effect of the CW parameter n and the nature of the nonlinearity on the stability of the Ps condensate can be understood by comparing the subplots (a) and (b) in figures 8 and 9, respectively. The CW background with $n = 0$ and the attractive sign of g_1 is modulationally unstable for all values of g_0, and for the CW solutions with $n = 1$; the same is true for the repulsive sign of g_1. For small Δk, even CW solutions ($n = 0$) are found to be stable for identical signs of g_0 and g_1 nonlinearities and $g_1 > 0$, while odd CW backgrounds ($n = 1$) are stable for opposite signs of the nonlinearities and $g_1 < 0$. Large differences in the wavenumbers of the spin components with $M = \pm 1$ give rise to the MI even in such backgrounds, which might be expected to be stable. The final

Table 2. Summary of the results of the MI analysis for different combinations of the parameters.

$\Delta k = (k_1 - k_{-1})$	n	g_1	g_0	Inference
0	0	+	+	The $n = 0$, 1 CW backgrounds are modulationally unstable for the attractive g_0 nonlinearity ($g_0 < 0$). It is independent of the nature of g_1.
			−	
		−	+	
			−	
	1	+	+	
			−	
		−	+	
			−	
≠0	0	+	+	stable for small values of Δk
			−	unstable
		−	+	unstable
			−	unstable
	1	+	+	unstable for natural values of nonlinearities
			−	unstable
		−	+	stable for small values of Δk
			−	unstable

conclusion in the case of the nonzero wavenumber difference between the spin components is that the nonlinearities and the CW parameter, n, in equation (35*d*) tend to increase the susceptibility of the CW backgrounds to MI.

Throughout the paper, we have studied CW solutions and their MI in the condensate of positronium, which is a mixture of $F = 1$ and $F = 0$ spinor components with vanishing interaction between them. The $n = 0$, 1 CW solutions in the case of $F = 1$ spinor condensates (CW solutions with $n = 1$ do not exist for the ferromagnetic case) were found to be completely stable, with the existence ranges independent of the density of the condensate. However, for the spinor positronium condensate, the existence ranges of the $n = 0$, 1 CW solutions is found to be limited by the density of the condensate. In addition, such CW solutions can be stable or unstable depending on the relative densities of the different constituents and the order (n) of the CW solution.

The absence of the inter-species interaction between the $F = 1$ and $F = 0$ spinor components invalidates the familiar MI condition ($g_{12}^2 > g_1 g_2$, where g_1, g_2 and g_{12} are, respectively, strengths of the intra-species and inter-species interactions) for the case of vanishing fields with $M = \pm 1$. Note that spin–orbit coupling is also found to alter the MI condition [50]. MI is possible if either of the nonlinearities, g_0 or g_1, are attractive. In the case of the attractive spin-mixing interaction, $g_1 < 0$, binary condensates of $|1, 0\rangle$ and $|0, 0\rangle$ spin components are stable for equal amplitudes of the components and $g_0 > 0$.

The $n = 0$, 1 CW states in the spinor positronium are stable against MI similar to the stability of $n = 0$ CW states in the condensate with $F = 1$ for repulsive nonlinearities. Nevertheless, the instability gain in the $n = 0$ CW states in the case of $F = 1$ spinor condensates depends solely on the interaction strength, while for the spinor condensate of positronium, the order n of the CW state, in the combination with the nonlinear spin-exchange interaction, g_1, modifies the instability gain for $g_0 < 0$. Wavenumber differences between the different spin components are found to have significant destabilizing effects.

The summary of the results of the MI analysis for the CW solutions with $n = 0$ and 1 for zero and nonzero values of $k_1 \neq k_{-1}$ is presented in table 2.

6. Conclusion

We have obtained the CW solutions in spinor BEC of positronium, composed of para- ($F = 0$) and ortho- ($F = 1$) spin fields in the absence of external magnetic fields. For the condensate without the para- component, $\psi_p = 0$, the BEC is tantamount to a spinor $F = 1$ ferromagnetic condensate with CW solutions existing only for even n, which determines the phase shift, πm, between the different components (i.e., for the in-phase components). The CW solutions in such a case are stable.

In the presence of the para- component ($\psi_p \neq 0$), there exist CW solutions with both $n = 0$ and $n = 1$, whose existence ranges are limited by the total density, ρ. The ground state of positronium for $g_1 \gg \epsilon$ is found to have equal densities of ortho- and para- components for $n = 1$ CW solutions, provided that the amplitudes of the $M = \pm 1$ components are equal, and the condensate is a polar one. Since the ortho-to-para interconversion is minimal, if measured in units of the scaled energy difference, for $g_1 \ll \epsilon$ and can be neglected. In such a case, the CW solutions are stable if the density of the para-Ps obeys either condition $A_p^2 < A_0^2$, or the conditions given by equations (18) and (19).

The obtained CW solutions with $n = 0$ and 1 in spinor Ps were subsequently examined for the MI, using the linear stability analysis for the small perturbations. In the case of zero fields with $M = \pm 1$, the stability of the CW background is found to depend on the amplitudes of the respective components, and on the nonlinearity coefficients g_0 and g_1. For $g_0 > 0$ and $g_1 < 0$, the CW solutions with identical wavenumbers are modulationally unstable only for different amplitudes of the components. In the limit case of identical wavenumbers of all the spinor components, MI for both CW backgrounds with $n = 0$ and $n = 1$ depends on the sign of g_0 alone. The result is that, both for $n = 0$ and 1, the CW backgrounds are stable for $g_0 > 0$. The total particle density, ρ, with the help of individual densities of the $M = \pm 1$ components, suppresses the influence of the CW parity, n, on the MI. For $g_0 < 0$, the gain attains a constant value after a rapid initial change with the variation of the g_1 nonlinearity coefficient.

In the general case of the nonzero differences between the wavenumbers of the $M = \pm 1$ spin components, the $n = 1$ (out-of-phase) CW background is unstable for the natural repulsive signs of g_0 and g_1, with the gain maximum large for larger values of g_0 at a fixed value of g_1. The wavenumber difference $\Delta k \equiv k_1 - k_{-1} \neq 0$, thereby making the CW solutions with $n = 0$ and 1 much more vulnerable to MI.

Finally, it is relevant to briefly discuss peculiarities of possible experimental realization of the MI in the positronium BEC. The MI directly applies if its characteristic growth time, $\sim 1/\xi_{\max}$ (see equation (31)) does not exceed the (ortho-) positronium lifetime, $\approx$140 ns [55]. However, it is possible to realize the MI in a less extreme form if the positronium condensate is permanently replenished from an external source. Then, an essential difference of the expected experimental situation from that typical for usual atomic condensates [31, 32] is the fact that the scattering length for the positronium is smaller by a factor of approximately ten [35–38], and, most essentially, its mass is smaller than the atomic mass of ^{7}Li by a large factor $\approx$6300. For this reason, equation (32) demonstrates that the same range of spatial scales of the MI as in the experiments with atomic gases ($\sim$0.1 mm) may be achieved for the positronium density exceeding the atomic one by approximately five orders of magnitude, which may be achieved in the experiment. Then, equation (31) suggests that, in this region of the densities, the characteristic growth time of the MI may be approximately four orders of magnitude smaller than in the atomic BEC, i.e., roughly on the microsecond scale, which is closer to the above-mentioned positronium lifetime.

Acknowledgments

KP thanks the Department of Science and Technology (DST), Council of Scientific and Industrial Research (CSIR), National Board for Higher Mathematics (NBHM), Indo-French Centre for the Promotion of Advanced Research (IFCPAR), and the Government of India, for the financial support through major projects. We are grateful to Dr R S Tasgal for fruitful discussions. I A Bhat thanks the University Grants Commission (UGC), the Government of India, for financial support via BSR fellowships in Sciences. TM acknowledges the grant IBS-R024-D1 for the support. The work of BAM is supported, in part, by a joint program of the National Science Foundation and Binational (US-Israel) Science Foundation through grant no. 2015616, and by the Israel Science Foundation (grant no. 1287/17).

ORCID iDs

K Porsezian https://orcid.org/0000-0003-0494-9216

References

[1] Rich A 1981 *Rev. Mod. Phys.* **53** 127
[2] López-Castañares R, Olea-Cardoso O, Vázquez-Moreno F, Lizama-Soberans B, Camps-Carvajal E, Angeles-Anguiano E and Castaño V 2002 *Bulg. J. Phys.* **29** 155–78
[3] Mills A P, Cassidy D B and Greaves R G 2004 *Mater. Sci. Forum* **445–446** 424–9
[4] Avetissian H K, Avetissian A K and Mkrtchian G F 2014 *Phys. Rev. Lett.* **113** 023904
[5] Platzman P M and Mills A P 1994 *Phys. Rev.* B **49** 454
[6] Shu K, Fan X, Yamazaki T, Namba T, Asai S, Yoshioka K and Gonokami M K 2016 *J. Phys. B: At. Mol. Opt. Phys.* **49** 10
[7] Pethick C and Smith H 2001 *Bose–Einstein Condensation in Dilute Gases* vol 2 2nd edn (Cambridge: Cambridge University Press)
[8] Cassidy D B, Meligne V E and Mills A P 2010 *Phys. Rev. Lett.* **104** 173401
[9] Anderson M H, Ensher J R, Wieman C E and Cornell E A 1995 *Science* **269** 198–201
[10] Davis K B, Mewes M O, Andrews M R, Van druten N J, Durfee D S, Kurn D M and Ketterle W 1995 *Phys. Rev. Lett.* **75** 3969–73
[11] Bradley C C, Sackett C A and Hulet R G 1997 *Phys. Rev. Lett.* **78** 985–9
[12] Stenger J, Inouye S, Stamper-Kurn D M, Miesner H-J, Chikkatur A P and Ketterle W 1998 *Nature (London)* **396** 345
[13] Stamper-Kurn D M, Andrews M R, Chikkatur A P, Inouye S, Miesner H-J, Stenger J and Ketterle W 1998 *Phys. Rev. Lett.* **80** 10
[14] Barett M D, Sauer J A and Chapman M S 2001 *Phys. Rev. Lett.* **87** 010404
[15] Isoshima T, Machida K and Ohmi T 1999 *Phys. Rev.* A **60** 4857
[16] Marzlin K P, Zhang W and Sanders B C 2000 *Phys. Rev.* A **62** 013602
[17] Ho T L 1998 *Phys. Rev. Letters* **81** 742
[18] Gu Q and Klemm R 2003 *Phys. Rev.* A **68** 031604(R)
Ashhab S 2005 *J. Low Temp. Phys.* **140** 51
[19] Ueda M 2012 *Annu. Rev. Condens. Matter Phys.* **3** 263–83
[20] Li L, Li Z, Malomed B A, Mihalache D and Liu W M 2005 *Phys. Rev.* A **72** 033611
[21] Robins N P, Zhang W, Ostrovskaya E A and Kivshar Y S 2001 *Phys. Rev.* A **64** 021601(R)
[22] Uchiyama M, Ieda J and Wadati M 2006 *J. Phys. Soc. Jpn.* **75** 064002
Uchiyama M, Ieda J and Wadati M 2007 *J. Phys. Soc. Jpn.* **76** 074005
[23] Chang M-S, Qin Q, Zhang W, You L and Chapman M S 2005 *Nature (London)* **1** 111
[24] Mur-Petit J 2009 *Phys. Rev.* A **79** 063603
[25] Vengalattore M, Higbie J M, Leslie S R, Guzman J, Sadler L E and Stamper-Kurn D M 2007 *Phys. Rev. Lett.* **98** 200801
Mussel W, Strobel H, Linnemann D, Hume D B and Oberthaler M K 2014 *Phys. Rev. Lett.* **113** 103004
[26] Linnemann D, Strobel H, Mussel W, Schulz J, Lewis-Swan R J, Kheruntsyan K V and Oberthaler M K 2016 *Phys. Rev. Lett.* **117** 013001
Linnemann D, Schulz J, Mussel W, Kunkel P, Prüfer M, Frölian A, Strobel H and Oberthaler M K 2017 *Quantum Sci. Technol.* **2** 044009
[27] Kevrekidis P G, Frantzeskakis D J and Carretero-González R 2015 *The Defocusing Nonlinear Schrödinger Equation: From Dark Solitons to Vortices and Vortex Rings* (Philadelphia, PA: SIAM)
[28] Agrawal G P 1987 *Phys. Rev. Lett.* **59** 880
[29] Pu H, Law C K, Raghavan S, Eberly J H and Bigelow N P 1999 *Phys. Rev.* A **60** 1463
[30] Mithun T and Porsezian K 2012 *Phys. Rev.* A **85** 013616
[31] Nguyen J H V, Luo D and Hulet R G 2017 *Science* **356** 422
[32] Everitt P J *et al* 2017 *Phys Rev* A **96** 041601(R)

[33] Wang Y H, Anderson B M and Clark C W 2014 *Phys. Rev.* A **89** 043624
[34] Kawaguchi Y and Ueda M 2012 *Phys. Rep.* **520** 253–381
[35] Chaudhuri P, Adhikari S K and Ghosh A S 1997 *J. Phys. B: At. Mol. Opt. Phys.* **30** L81
[36] Oda K, Miyakawa T, Yabu H and Suzuki T 2001 *J. Phys. Soc. Jpn.* **70** 1549
[37] Adhikari S K 2002 *Phys. Lett.* A **294** 308
[38] Morandi O, Hervieux P-A and Manfredi G 2016 *J. Phys. B: At. Mol. Opt. Phys.* **49** 084002
[39] Devassy L, Jisha C P, Alberucci A and Kuriakose V C 2015 *Phys. Rev.* E **92** 022914
[40] Kasamatsu K, Tsubota M and Ueda M 2003 *Phys. Rev.* A **67** 033610
[41] Tasgal R S and Band Y B 2013 *Phys. Rev.* A **87** 023626
Tasgal R S and Band Y B 2015 *Phys. Rev.* A **91** 013615
[42] Blatt S, Nicholson T L, Bloom B J, Williams J R, Thomsen J W, Julienne P S and Ye J 2011 *Phys. Rev. Lett.* **107** 073202
Inouye S, Andrews M R, Stenger J, Miesner H-J, Stamper-Kurn D M and Ketterle W 1998 *Nature (London)* **392** 151
Cornish S L, Claussen N R, Roberts J L, Cornell E A and Wieman C E 2000 *Phys. Rev. Lett.* **85** 1795
[43] Zhang W, Zhou D L, Chang M-S, Chapman M S and You L 2005 *Phys. Rev.* A **72** 013602
[44] Bigelow N 2005 *Nature Phys.* **1** 89–90
[45] Abad M and Recati A 2013 *Eur. Phys. J.* D **67** 148
[46] Ohmi T and Machida K 1998 *J. Phys. Soc. Jpn.* **67** 1822
[47] Yu M, McKinstrie C J and Agrawal G P 1993 *Phys. Rev.* E **48** 2178
[48] Yang J 2010 *Nonlinear Waves in Integrable and Nonintegrable Systems* (Philadelphia, PA: SIAM)
[49] Kasamatsu K and Tsubota M 2004 *Phys. Rev. Lett.* **93** 100402
Kasamatsu K and Tsubota M 2006 *Phys. Rev.* A **74** 013617
[50] Bhat I A, Mithun T, Malomed B A and Porsezian K 2015 *Phys. Rev.* A **92** 063606
[51] Bogolubov N 1947 *J. Phys. (Moscow)* **11** 23–32
[52] Agrawal G P 2013 *Non-linear Fiber Optics* 5th edn (San Diego, CA: Academic)
[53] Theocharis G, Rapti Z, Kevrekidis P G, Frantzeskakis D J and Konotop V V 2003 *Phys. Rev.* A **67** 063610
[54] Salasnich L, Parola A and Reatto L 2003 *Phys. Rev. Lett.* **91** 080405
[55] Adkins G S, Fell R N and Sapirstein J 2000 *Phys. Rev. Lett.* **84** 5086