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Abstract

We obtain general continuous-wave (CW) solutions in the model of a spinor positronium condensate in the absence of magnetic field. The CW solutions with both in-phase ($n=0$) and out-of-phase ($n=1$) spin components exist, with their ranges limited by the total particle density, $\rho$. In the limit of negligible population exchange between the spin components, the CW solutions are found to be stable or unstable, depending on the particle density of the para-positronium. Ortho-positronium, in the $F=1$ spinor state, forms a ferromagnetic condensate with stable in-phase CW solutions only. Subsequent examination of the modulational instability is carried out both in the limit case of identical wavenumbers in the spin components, $\Delta k \equiv k_1 - k_{-1} = 0$, and in the more general case of $\Delta k \neq 0$ too. The CW solutions with $n=0$ and 1 solutions, which are stable in the case of $\Delta k = 0$, are unstable for $\Delta k \neq 0$, for the natural repulsive sign of the nonlinearities. The total particle density, $\rho$, in the limit of $\Delta k = 0$ is found to have a significant role in the stability of the condensate, which is determined by the sign of the self-interaction nonlinearity.

Supplementary material for this article is available online

Keywords: continuous-wave solutions, modulational instability, spinor condensates, positronium

1. Introduction

Positronium (Ps) is a commonly known bound state of an electron and a positron, with the total angular momentum $F = 0$ or $F = 1$. The spin configurations with $F = 0$ and $F = 1$, viz., the para- ($^1S_0$) and ortho-Ps ($^3S_0$), are separated by an energy gap, $\Delta E = 8.44 \times 10^{-4}$ eV, and annihilate, respectively, by emitting two or three gamma-quanta [1, 2]. This makes positronium a useful source for 511 keV gamma-ray lasers [3, 4]. The lifetime of the para-Ps is 0.125 ns, while the ortho-Ps lives much longer, 142 ns. Referring to its longer lifetime, Platzmann and Mills [5] proposed, in 1994, the possibility of making ortho-Ps Bose–Einstein condensate (BEC) in a cold silicon cavity. As the condensation temperature is inversely proportional to the mass of the species under consideration, and proportional to its density, for positronium the condensation must occur at a much higher temperature and/or density compared to that for the usual bosonic atoms [6, 7]. Currently, spin-polarized ensembles of Ps atoms are available, but with densities of at least two orders of magnitude less than that required to form the condensate [8]. Nevertheless, a specific method has been recently proposed...
for the realization of the Ps BEC [6]. The method relies on the original cooling of Ps atoms through interaction with a cold silica cavity and Ps–Ps two-body collisions, followed by laser cooling.

Atomic condensates of rubidium ($^8$Rb) [9], sodium ($^{23}$Na) [10], and lithium ($^7$Li) [11] were first created in magnetic traps. In such BECs, the spin degree of freedom is frozen due to its coupling to the field, hence the system is defined by a scalar order parameter. Unlike magnetic traps, optical ones hold all spin components of a given hyperfine state, without forcing the atoms to align their spins in a specific direction. Spinor BECs with $2F + 1$ spin components have been experimentally created in optical traps [12–14], thus providing an opportunity to explore intrinsic spin dynamics in the condensate. Spinor BECs display various phenomena, such as spin domains [15], skyrmions [16], magnetism and its dependence on scattering lengths [17], interaction-dependent ferromagnetic phase transitions [18], modulational instability (MI) in the case of repulsive nonlinearity [19–21], the existence of multicomponent solitons [22], oscillatory coherent spin mixing [23, 24], etc. Apart from these facts, spinor condensates, including nonlinear spin-exchange interactions, find uses in magnetometry [25] and atom interferometry [26].

The objective of the present work is to examine continuous-wave (CW) solutions and their MI in the spinor-BEC model of positronium. This is a relevant aim, as flat CW backgrounds support various dynamical phenomena, including modulational instability, dark and anti-dark solitons, vortices, etc [27]. MI is the exponential growth of the population exchange between different spin states, while the interaction energy $\epsilon_j$ and the internal energy of the spin state $j$. In the mean-field approximation, the interaction Hamiltonian is (see its detailed derivation in [33]):

$$\hat{H}_{\text{int}} = \frac{1}{2} \int d^3r (\hat{g}_0 \rho^2 + \hat{g}_1 |\psi_j(\psi_{-j}^*|)^2).$$

where $\rho = \sum_{j=\pm 1,0} |\psi_j|^2$ is the total atomic density, $\hat{g}_0$ and $\hat{g}_1$ are nonlinearity coefficients, defined by $\hat{g}_0 = 4\pi\hbar^2 a_0/m$ and $\hat{g}_1 = \pi\hbar^2 (a_0 - a_1)/m$, while $a_0$ and $a_1$ are the scattering lengths of the para ($S=0$) and ortho ($S=1$) positronium, respectively (these parameters were theoretically calculated in several theoretical works [35–38]). An independent para-ortho scattering length does not appear in equation (2), as the interaction between the ortho- and para-positronium vanishes for an odd spin channel. This fact follows from the presentation of the atomic spin in the form of $F_{\text{colliding pair}} = F_{\text{ortho}} + F_{\text{para}} \equiv 1$ [17, 34].

For the BEC confined in a cigar-shaped optical trap, the corresponding Hamiltonian $\hat{H}$ with $\epsilon_j = \epsilon_o$ for $j = 0, \pm 1$ produces a set of coupled one-dimensional GP equations, following the usual procedure of the dimensional reduction [33, 39, 40]:

$$i\hbar \dot{\psi}_j = (H_0 + \epsilon_o + 2g_1 |\psi_j|^2) \psi_j + g_1 \psi_j^* (\psi_j^2 - \psi_0^2), \quad (3a)$$

$$i\hbar \dot{\psi}_0 = (H_0 + \epsilon_o + g_0 |\psi_j|^2) \psi_0 - g_1 \psi_0^* (2\psi_j \psi_{-j}^* + \psi_0^2), \quad (3b)$$

$$i\hbar \dot{\psi}_{-j} = (H_0 + \epsilon_o + 2g_1 |\psi_j|^2) \psi_{-j} + g_1 \psi_0^* (\psi_j^2 - \psi_0^2), \quad (3c)$$

$$i\hbar \dot{\psi}_p = (H_0 + \epsilon_p + g_1 |\psi_p|^2) \psi_p + g_1 \psi_0^* (2\psi_j \psi_{-j}^* - \psi_0^2)$$

where the overdot stands for $\partial/\partial t$, and

$$H_0 = -\frac{\hbar^2}{2m_0} \frac{\partial^2}{\partial z^2} + V_{\text{ext}}(z) + g_0 \rho. \quad (4)$$

The annihilation of the positronium is disregarded here, hence the GP equations are valid on a limited time scale. The nonlinear coefficients are now defined by $g_0 = \frac{4\pi}{m_0 \omega^2} a_0$ and $g_1 = \frac{\hbar^2}{2m_0} (a_0 - a_1)$ with $a_0 = \frac{\hbar^2}{m_0 \omega^2}$, and $\omega$ is the transverse trap frequency. The terms $g_1$ in equation (3) govern the population exchange between different spin states, while the total density remains constant. Equations (3) may be normalized to a dimensionless form by means of rescaling:

$$t' = t \omega. \quad (5a)$$

2. The model

We consider a uniform spinor BEC in an optical trap, without external magnetic fields. Accordingly, positronium is free to realize any of its four spin states, labeled as $|p\rangle$ and $|1\rangle$, $|0\rangle$, $|-1\rangle$, with $|p\rangle$ representing the para state and $|-1\rangle$, $|0\rangle$, $|1\rangle$ standing for three values of the magnetic quantum number $M$ in the ortho-state. Thus, different states $|F, M\rangle$ of the ortho positronium are designated by $|1, M\rangle$, while the para state corresponds to $|0, 0\rangle$. The Hamiltonian of this system is a combination of the non-interacting single-particle Hamiltonians and the interaction energy $\hat{H}_{\text{int}}$ [33, 34]:

$$\hat{H} = \int d^3r \sum_{j=0,\pm 1, p} \psi_j^* \left[ \frac{p^2}{2m} + V_{\text{ext}}(r) + \epsilon_j \right] \psi_j + \hat{H}_{\text{int}}, \quad (1)$$

where $\rho$ is the momentum operator, $V_{\text{ext}}(r)$ the trapping potential, and $\epsilon_j$ the internal energy of the spin state $j$. In the mean-field approximation, the interaction Hamiltonian is (see its detailed derivation in [33]):

$$\hat{H}_{\text{int}} = \frac{1}{2} \int d^3r (\hat{g}_0 \rho^2 + \hat{g}_1 |\psi_j(\psi_{-j}^*|)^2).$$

The presentation in the paper is organized as follows. Section 2 introduces the theoretical model, based on the Gross–Pitaevskii (GP) equations governing the mean-field dynamics of the spinor condensate, ignoring effects produced by the finite lifetime (spontaneous annihilation) of the ortho-Ps. Section 3 addresses CW solutions, focusing on conditions for their existence and stability. Section 4 discusses the MI and dispersion relations in different cases. Section 5 summarizes the results of the analysis of the spinor positronium, and section 6 concludes the work. Section 6 also includes a brief discussion of possible manifestations of the obtained results in experimental studies of the MI.
The resulting dimensionless GP equations have the same form as above, but with $\zeta_0 = 0$ and $\zeta_1 = 1$ and new dimensionless nonlinear coefficients $\zeta_0 = a_1 / a_1$ and $\zeta_1' = (a_0 - a_1) / a_1$, while the internal energy of the $j$th state is measured in units of $\zeta_0$. In the rest of the paper, the energies are shifted to $\zeta_0 = 0$ and $\zeta_1 = 1$. These scaled variables are used in figures displayed below, while equations are written in dimensional units.

3. CW solutions

We begin by considering the following general CW solutions [41]:

$$\psi_j = A_j \exp[i(k_j z + \theta_j - \omega_j t)], \quad (6)$$

where $j = 0, \pm 1, p$ represents, as above, the respective spin components. Wavenumbers, phase shifts, and frequencies (chemical potentials) $k_j$, $\theta_j$ and $\omega_j$ are real, while amplitudes $A_j$ are positive.

The substitution of the CW ansatz (6) for $j = 0, \pm 1, p$ into governing equations (3) yields

$$\omega_1 = - \frac{\hbar^2 k_1^2}{2m} + g_0 \rho + \epsilon + 2g_1 A_0^2$$
$$+ (1)^n g_1 \frac{A_{-1}}{A_1} (A_p^2 - A_0^2), \quad (7a)$$
$$\omega_0 = - \frac{\hbar^2 k_0^2}{2m} + g_0 \rho + \epsilon + g_1 A_0^2 - g_1 (1)^n A_{-1} A_1 + A_p^2, \quad (7b)$$

$$\omega_{n-1} = \frac{\hbar^2 k_n^2}{2m} + g_0 \rho + \epsilon + 2g_1 A_0^2$$
$$+ (1)^n g_1 \frac{A_{-1}}{A_1} (A_p^2 - A_0^2), \quad (7c)$$
$$\omega_n = \frac{\hbar^2 k_n^2}{2m} + g_0 \rho + g_1 A_0^2 + g_1 (1)^n A_{-1} A_1 - A_p^2, \quad (7d)$$

where $n$ and $s$ are integers, which are defined below in equation (8c), and the following relations between the wavenumbers, frequencies, and phase shifts of the different components must hold:

$$k_p = k_0 = \frac{1}{2}(k_1 + k_{-1}), \quad (8a)$$
$$\omega_p = \omega_0 = \frac{1}{2}(\omega_1 + \omega_{-1}), \quad (8b)$$
$$\theta_p + s \pi = \theta_0 = \frac{1}{2}(\theta_1 + \theta_{-1} + n \pi) \quad (8c)$$

For the compatibility of equation (8) with the other equations, amplitude $A_0$ of the $\psi_0$ component must satisfy the condition

$$A_0^2 = \frac{1}{2} \left[ \rho - (A_1 - (1)^n A_{-1})^2 + \frac{2g_1^2 (1)^n A_{-1} A_1}{A_1^2 + A_{-1}^2} \right], \quad (9)$$

where

$$\gamma = \frac{\hbar^2 (k_1 - k_{-1})^2}{8m g_1^2} + \frac{\epsilon}{2g_1} \quad (10)$$

The second term on the right hand side of equation (10) displays the competition between the internal energy difference $\epsilon$ and the spin-mixing interaction with the effective 1D strength $g_1$. The ortho- to para- interconversion is substantial for $g_1 \gg \epsilon$.

From the CW ansatz it follows that the condition of $A_0$ being positive makes the left hand side of equation (9) real and non-negative, thus giving a criterion for the existence of...
the CW solutions. Thus, for the ground state of positronium with \( g_1 > 0 \), CW solutions for both even \( n \) and odd \( n \) exist, but with different existence ranges, as shown in Figure 1. These ranges for different CW solutions are found to depend on the total number density, \( \rho \), in addition to the magnitude and sign of \( g_1 \), which, in turn, can be tuned by means of the Feshbach-resonance techniques [42]. In particular, the CW solutions with even \( n \) (represented by \( n = 0 \)) exist if
\[
\gamma \geq \frac{(A^2_1 + A^2_{-1})}{2A_1A_{-1}}[(A_1 - A_{-1})^2 - \rho]. \tag{11}
\]
There are CW solutions with odd \( n \) (represented by \( n = 1 \)) if
\[
\gamma \leq \frac{(A^2_1 + A^2_{-1})}{2A_1A_{-1}}[\rho - (A_1 + A_{-1})^2]. \tag{12}
\]
Clearly, values which \( A_1 \) and \( A_{-1} \) must assume for the fulfillment of the conditions for the existence of the respective CW solutions are limited by the total density, \( \rho \).

The case of \( g_1 < 0 \), which may be realized, as mentioned above, with the help of the Feshbach resonance, modifies the existence ranges of the CW solutions, as shown in Figure 2. There are CW solutions for even \( n \) when
\[
\gamma \leq \frac{(A^2_1 + A^2_{-1})}{2A_1A_{-1}}[(A_1 - A_{-1})^2 - \rho]. \tag{13}
\]
However, for odd \( n \), CW solutions exist at
\[
\gamma \geq \frac{(A^2_1 + A^2_{-1})}{2A_1A_{-1}}[(A_1 + A_{-1})^2 - \rho]. \tag{14}
\]

In accordance with equation (8b), and combining equations (7b) and (7d) in the limit of \( g_1 \gg \epsilon \), we arrive at an equation similar to the one obtained in [33], which relates the ortho- and para-populations of the condensate:
\[
A^p_j = A^o_j - 2(-1)^nA^o_jA^2_{-1}. \tag{15}
\]
The ground state is found to have the maximum para-population for odd \( n = 1 \) CW solutions, which, for equal amplitudes of the \( M = \pm 1 \) components, becomes equal to the density of the ortho-component:
\[
\rho_p = \rho_o - 2(A_1 + (-1)^nA_{-1})^2 \tag{16}
\]
where \( \rho_p = A^2_p \) and \( \rho_o = A^2_o + A^2_{-1} \) are densities of the para- and ortho-components, respectively. The ortho-sector of the condensate for equal densities of ortho- and para-components is equivalent to a polar state of the \( F = 1 \) condensate [17, 33].

For the stability analysis of the CW solutions, we neglect the ortho-to-para interconversion, assuming \( g_1 \ll \epsilon \). Except for the wavenumbers \( k_j \) with \( j = 0, \pm 1, \rho \), CW parameters, such as the amplitudes \( A_j \) and the phases \( \theta_j \), are made time-dependent. The linearization of equations (3), taking into account the conservation of total density, \( \rho = \sum_{j=0,\pm 1}\rho^2_j \), and magnetization, \( \rho^2_j - A^2_j \), and subsequent linearization with respect to amplitude and phase perturbations (the latter reduce to \( \partial \theta(t) = (\theta_1(t) + \theta_{-1}(t) - 2\theta_0(t)) \)) shows that the CW solutions are either stable or unstable, depending on the density of the para-component relative to that of the one with \( M = 0 \) in the ortho-component. Oscillation eigenfrequencies of infinitesimal perturbations are then given by
\[
\omega^2 = -\frac{8\rho_o A_0}{\hbar} \left( A_1 + (-1)^nA_{-1} \right)^2 + \frac{(A^2_1 - A^2_{-1})^2(A^2_o - A^2_p)}{4A^2_1A^2_{-1}} \tag{17}
\]
These frequencies are real, and hence the CW solutions are stable, if \( A^2_o > A^2_p \). If \( A^2_o < A^2_p \), then, for \( n = 0 \), the CW solutions are stable if
\[
A^2_p < A^2_o + \left( \frac{2A_1A_{-1}}{A_{-1} - A_1} \right)^2. \tag{18}
\]
Likewise, for \( n = 1 \) and \( A_1 \neq A_{-1} \), the CW solutions are stable if
\[
A^2_p < A^2_o + \left( \frac{2A_1A_{-1}}{A_{-1} - A_1} \right)^2. \tag{19}
\]
The respective spin oscillations with real frequency $\omega$ represent the exchange of populations between the different spin states [43, 44].

In the limit of $\psi_p = 0$, we are left with the ortho-Ps condensate only, the system being equivalent to the $F = 1$ spinor condensate. The spinor ortho-Ps condensate with $g_1 > 0$ supports solely even ($n = 0$) CW solutions for all values of $A_1$ and $A_{-1}$ with

$$A_0^2 = 2(1)^n A_1 A_{-1} \left(1 + \frac{h^2}{2m g_1} \frac{(\Delta k)^2}{2} \right).$$

(20)

where $\Delta k \equiv k_1 - k_{-1}$. This is a peculiarity of the ferromagnetic states of the $F = 1$ spinor condensate [41]. Thus, the CW states of the ortho-Ps only represent a ferromagnetic condensate. The conditions for the existence of the possible CW solutions in spinor Ps are summarized in Table 1.

Table 1. Possible CW solutions with respective conditions for their existence in the spinor condensate of positronium for different values of $\gamma = \frac{1}{\gamma \text{m}} [\hbar^2 (k_1 - k_{-1})^2 + 4m \epsilon]$. $A_1$ and $A_{-1}$ are allowed amplitudes of the respective spin components, and $\rho$ is the total density.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\psi_p$</th>
<th>CW (n = 0)</th>
<th>CW (n = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 0$</td>
<td>0</td>
<td>$A_1, A_{-1} \in R \geq 0$</td>
<td>No Solutions</td>
</tr>
<tr>
<td>$= 0$</td>
<td>$\gamma \leq \frac{(\Delta k)^2}{2m g_1} \frac{(A_1 - A_{-1})^2}{2} - \rho$</td>
<td>$\gamma \leq \frac{(\Delta k)^2}{2m g_1} \frac{[\rho - (A_1 + A_{-1})^2]}{2}$</td>
<td></td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td>$= 0$</td>
<td>$\gamma \leq \frac{(\Delta k)^2}{2m g_1} \frac{(A_1 - A_{-1})^2}{2} - \rho$</td>
<td></td>
</tr>
</tbody>
</table>

The stability analysis of the CW solutions by means of the linearization with respect to the amplitude and the phase perturbations show that the CW solutions are stable against the infinitesimal perturbations, with real perturbation eigenfrequencies

$$\omega^2_{n=0} = \left(\frac{g_1 A_0^2}{h} \right)^2 \left[ (A_1^2 - A_{-1}^2)^2 + \frac{8(A_1 + A_{-1})^2}{A_0^2} \right].$$

(21)

These are frequencies of the coherent spin mixing, which account for the exchange of populations between the different spin states of the ortho-Ps.

4. Modulational instability (MI)

To investigate MI in the Ps BEC, described by the dynamical equations (3), we start by addressing small perturbations $\delta \phi(z, t)$ added to the CW solutions [20, 21, 28, 41, 45, 46]:

$$\phi_j = [A_j + b_j \delta \phi_j(z, t)] \exp[i(k_j z + \theta_j - \omega t)]$$

(22)

where $j$ defines the spin index 0, $\pm 1$ and $p$.

Assuming that the system size is greater than the healing length, which determines the characteristic length scale for MI, we assume the perturbations to be in the form of plane waves,

$$\delta \phi_j(z, t) = \lambda_j \cos(k_j z - \omega t) + i \eta_j \sin(k_j z - \omega t),$$

(23)

where $\lambda_j$ and $\eta_j$ are perturbation amplitudes, while $k$ and $\omega$ are the wavenumber and the (generally complex) frequency, respectively.

The substitution of equations (22) and (23) into equation (3), for the four spin indices, gives a set of eight homogeneous equations, with respect to $\lambda_j$ and $\eta_j$, in the matrix form:

$$\mathbf{M} \Psi = \left( [ - h \omega + \frac{\hbar^2 k}{2m} (k_1 - k_{-1}) ] \mathbf{I} + \mathbf{X} + \mathbf{Y} + \mathbf{Z} \right) \Psi = 0$$

(24)

where $\Psi = (\eta_0, \lambda_0, \eta_0, \lambda_0, \eta_1, \lambda_1, \eta_{-1}, \lambda_{-1})^T$, $\mathbf{I}$ is the usual unit matrix and

$$X = \frac{\hbar^2}{2m} k$$

(25a)

$$Y = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_0 A_0^2 & 0 & g_0 A_0 A_1 & 0 & (g_0 + 2g_1) A_1 A_{-1} & 0 & g_0 A_1 A_p & 0 \\
0 & 0 & 0 & g_0 A_p^2 & 0 & 0 & 0 & -g_1 A_0 A_p \\
g_0 A_1 A_0 & 0 & (g_0 + g_1) A_0^2 & 0 & g_0 A_0 A_{-1} & 0 & (g_0 - g_1) A_0 A_p & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(g_0 + 2g_1) A_1 A_{-1} & 0 & g_0 A_1 A_{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -g_1 A_0 A_p & 0 & 0 & 0 & g_1 A_p^2 \\
g_0 A_1 A_p & 0 & (g_0 - g_1) A_0 A_p & 0 & g_0 A_1 A_p & 0 & (g_0 + g_1) A_p^2 & 0
\end{bmatrix}$$

(25b)
Frequency $\omega$ of the perturbations, the average wavenumber, $k_0 = k_p = \frac{(k_0 + k_p)}{2}$, and the wavenumber difference, $\Delta k$, are diagonal entries of the stability matrix. In the most general case, the wavenumber difference assumes a nonzero value, $\Delta k \neq 0$, while $\Delta k = 0$ corresponds to the limit case, in which all the wavenumbers are equal. The off diagonal elements depend upon various CW parameters, either explicitly or implicitly via $A_0$ (see equation (9)) and $A_p$ (see equation (15)).

Equation (24) represents an eigenvalue problem for matrix $M$. The solution to the problem aims to identify all possible eigenvalues $\hbar \omega$ and the corresponding eigenvectors $(\eta_j, \lambda_j; j = 1, 2, \ldots, 8)$. In this study, we produce solutions for eigenvalues $\hbar \omega$ for both cases of $\Delta k = 0$ and $\Delta k \neq 0$. Complex eigenvalues $\hbar \omega$, if obtained by solving the dispersion equation, $\det(M) = 0$, for some positive values of $k^2$, render the spinor condensate modulationally unstable. In the present case, this is the eighth-order equation with respect to $\hbar \omega$, with the dependence on the CW parameters implied in the coefficients, $C_{n,0,1,0}^{0,1}$, while $\alpha$ and $\beta$ are integer numbers:

$$
(\hbar \omega)^8 + (\hbar \omega)^3 C_{k,\omega}^{0,0,1} k^1 + (\hbar \omega)^3 \sum_{j=1}^{3} C_{k,\omega}^{n,0,0,1} k^{2j-2} + (\hbar \omega)^4 \sum_{j=1}^{3} C_{k,\omega}^{0,0,1,1} k^{2j-2} + (\hbar \omega)^4 \sum_{j=1}^{4} C_{k,\omega}^{n,0,1,1} k^{2j-2} + (\hbar \omega)^5 \sum_{j=1}^{5} C_{k,\omega}^{0,0,1,0,1} k^{2j-2} + (\hbar \omega)^6 \sum_{j=1}^{6} C_{k,\omega}^{n,0,1,0,1} k^{2j-2} = 0.
$$

(26)

The coefficients $C_{k,\omega}^{n,0,1,0}$ represent a blend of nonlinearity coefficients and various CW parameters, being too cumbersome to be included in the main text. They are explicitly displayed in Supplement 1, which is available online at stacks.iop.org/JPB/51/045006/mmedia.

Equation (26) can be solved analytically only in a few limit cases, as shown below.

For the ortho-Ps condensate ($\psi_p = 0$), for which only the in-phase CW solutions ($n = 0$) exist, the sixth-order dispersion equation is

$$
(\hbar \omega)^6 + (\hbar \omega)^3 C_{k,\omega}^{0,0,1} k^1 + (\hbar \omega)^3 \sum_{j=1}^{3} C_{k,\omega}^{n,0,0,1} k^{2j-2} + (\hbar \omega)^4 \sum_{j=1}^{4} C_{k,\omega}^{n,0,1,1} k^{2j-2} + (\hbar \omega)^5 \sum_{j=1}^{5} C_{k,\omega}^{n,0,1,0,1} k^{2j-2} = 0.
$$

Expression for coefficients, $C_{k,\omega}^{n,0,1,0}$, are again too cumbersome for the main text and can be found in the online supporting information, Supplement 2.

In the case of zero fields with $M = \pm 1$, the dynamics of the system is governed by a pair of coupled GP equations, and the resulting characteristic polynomial of the fourth order is analytically solvable:

$$
(\hbar \omega)^4 + (\hbar \omega)^3 C_{k,\omega}^{1,0,1} k^1 + (\hbar \omega)^3 \sum_{j=1}^{3} C_{k,\omega}^{n,1,0,1} k^{2j-2} + (\hbar \omega)^4 \sum_{j=1}^{4} C_{k,\omega}^{n,1,1,1} k^{2j-2} + (\hbar \omega)^5 \sum_{j=1}^{5} C_{k,\omega}^{n,1,1,0,1} k^{2j-2} = 0.
$$

Here coefficients $C_{k,\omega}^{n,1,0,1}$, being functions of the nonlinearity coefficients $g_0$ and $g_1$, and can be found in the online supporting information, Supplement 3. These coupled GP equations have been extensively studied in terms of optics [47, 48] and BEC [49, 50].

5. Results and discussions

Below we discuss cases for which the dispersion relations of different orders obtained above are solved for the perturbation
frequency in the limiting case of equal wavenumbers, \( \Delta k = 0 \), when simple analytical solutions are obtained, as well as in the general case when wavenumbers of different components are not equal, and related solutions are not available. In the latter case, results are plotted for difference \( \Delta k = 1 \) between scaled wavenumbers of the spin components with \( M = \pm 1 \).

5.1. Limit cases with simple analytic solutions

The simplest and the most generic case corresponds to the single-component model with three of the four spin components equal to zero. In this case, the CW frequency (chemical potential) is

\[
\hbar \omega = \frac{\hbar^2 k^2}{2m} + \epsilon + g A_j^2,
\]

and the eigenfrequency of the perturbations is given by

\[
\hbar \omega = \frac{\hbar^2 k_j^2}{m} \pm \sqrt{\frac{\hbar^2 k_j^2}{2m} \left( \frac{\hbar^2 k_j^2}{2m} + 2 g A_j^2 \right)},
\]

with \( g = g_0 \) for spins \( j = \pm 1 \), \( g = g_0 + g_1 \) for spins \( j = 0, \rho \), and \( \epsilon = 0 \) for the para component. Equation (30) is the Bogoliubov dispersion relation [51] for the propagation of small perturbations (sound waves) on top of CW solutions (6). The results obtained here are found to be in complete agreement with the previously reported ones [52–54]. Namely, for the attractive nonlinearity (\( g < 0 \)), there is MI against the perturbations in the wavenumber range \( 0 < k < 2 \sqrt{|g| m A_j / \hbar} \), characterized by the MI gain, \( \xi = \text{Im} (\omega) \). The maximum MI gain,

\[
\xi_{\text{max}} = |g| A_j / \hbar,
\]

is attained at

\[
k_{\text{max}} = \sqrt{2 |g| m A_j} / \hbar.
\]

In the case of zero components with \( M = \pm 1 \), the BEC consists of a mixture of \( |1, 0 \rangle \) and \( |0, 0 \rangle \) spin components with vanishing interspecific interaction. The frequency of the perturbations is in fact solutions of equation (28) for the same wavenumbers \( (k_0 = k_p = k) \) and the Bogoliubov dispersion relation with \( \rho_2 = A_0^2 + A_p^2 \) and \( \alpha = 8 A_0^2 A_p^2 g_1^2 \) modifies as

\[
\hbar \omega = \frac{\hbar^2 k^2}{m} \pm \left[ \frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + \rho_2 (g_0 + 2 g_1) \right) + \alpha \right]
\]

\[
\pm \left[ \frac{\hbar^2 k^2}{2m} \left( \rho_2 g_0^2 + 2 \alpha \left( 1 - \frac{g_0}{g_1} \right) \right) \right.
\]

\[
+ 2 \rho_2 \alpha \left( 2 - \frac{g_0}{g_1} \right) + \alpha^2 \right]^{1/2},
\]

From the above equation (33), it is found that MI in the system for a proper choice of \( A_0 \) and \( A_p \), as shown in figure 3, is possible if either of the nonlinearities, \( g_0 \) or \( g_1 \), is attractive \( (g_0 < 0 \) or \( g_1 < 0) \), which in fact does not hold for the Ps BEC, hence the system is modulationally stable. The computation of relations for \( k_{\text{max}} \), and maximum growth rate \( \xi \) in terms of the nonlinearities and the respective wavenumbers, is too complex to be performed analytically.

In accordance with the above result, and aiming to properly choose \( A_0 \) and \( A_p \), we studied the variation of the largest MI gain for a range of values of \( A_0 \) and \( A_p \). In the limit case of equal wavenumbers, figure 4 shows the peak MI gain versus amplitudes of the two spin components for \( g_0 < 0 \) and \( g_0 > 0 \), respectively. In the latter case, the system is modulationally unstable if and only if condition \( g_1 < 0 \) holds, and amplitudes of the spin components are different, as seen in figure 4(b).

The analytic solutions for the perturbation frequency around the \( n = 0 \) CW background in the case of the ortho Ps condensate, i.e., the three-component system with identical
The wavenumbers are
\[ \hbar \omega = (2 \pm 1) \frac{\hbar^2 k^2}{2m}, \]  
\[ \hbar \omega = \left[ (2 \pm 1) \frac{\hbar^2 k^2}{2m} \pm 2 g_1 (A_1 + A_{-1})^2 \right], \]  
\[ \hbar \omega = \frac{\hbar^2 k^2}{m} \pm \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2 g_0 (A_1 + A_{-1})^2 \right)}. \]  

The results given by equation (34) are in exact agreement with the previously known results for the ferromagnetic \( F = 1 \) spinor condensates [41]. Thus, the \( n = 0 \) CW background of the ferromagnetic ortho Ps with \( g_0 > 0 \) is stable against the exponential growth of perturbations when the wavenumbers are equal, and it is modulationally unstable if \( g_0 < 0 \), so that the largest MI gain, \( \xi_{\text{max}} = -g_0 \frac{(A_1 + A_{-1})^2}{h} \), occurs at \( k_{\text{max}} = \sqrt{-2m g_0 \frac{(A_1 + A_{-1})^2}{h}} \).

Analytical solutions of equation (26) in the form of Bogoliubov dispersion relations are obtained for the perturbations propagating on top of the CWs with \( n = 0 \) and 1 CW, in the case of identical wavenumbers:
\[ \hbar \omega = (2 \pm 1) \frac{\hbar^2 k^2}{2m}, \]  
\[ \hbar \omega = (2 \pm 1) \frac{\hbar^2 k^2}{2m}, \]  
\[ \hbar \omega = (2 \pm 1) \frac{\hbar^2 k^2}{2m} \pm 2 (A_1 + A_{-1})^2 \]  
\[ \hbar \omega = \frac{\hbar^2 k^2}{m} \pm \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2 g_0 (A_1 + A_{-1})^2 \right)}. \]  

From the above equations it is clear that, in the limit case of identical wavenumbers, the CW backgrounds with \( n = 0 \) and 1 give rise to MI and, thereby, formation of solitons if and only if condition \( g_0 < 0 \) holds, with the largest growth...
The dependence of the MI gain on $g_1$ for the CW solutions with $n = 0$ and $1$ for $\rho = 1$, $A_1 = \frac{1}{2}$, $A_{-1} = \frac{1}{3}$ and $k_1 - k_{-1} = 0$. Both CW solutions with $n = 0$ and $1$ are modulationally unstable for $g_0' < 0$. (a) The dependence of the MI gain on $g_0'$ for $n = 0$ and $g_0' = -1$. (b) The dependence of the MI gain on $g_0'$ for $n = 1$ and $g_0' = -1$.

The dependence of the maximum MI gain on amplitudes of two-component condensate with the wavenumbers difference $k_0 - k_0 = 1$. (a) The dependence of maximum MI gain on $A_0$ and $A_0'$ for $g_0' = -1$ and $g_0' = 1$. (b) The dependence of maximum MI gain on $A_0$ and $A_0'$ for $g_0' = 1$ and $g_0' = -1$.

rate, $\xi_{\text{max}} = -\frac{g}{\pi} \left( \rho + (-1)^n \frac{c_{A_1}}{g_0'(A_0' + A_0')^2} \right)$, corresponding to $k_{\text{max}} = \frac{1}{2} \sqrt{2mg_0 \left( \rho + (-1)^n \frac{c_{A_1}}{g_0'(A_0' + A_0')^2} \right)}$. Thus, the PS condensate with repulsive nonlinearities is modulationally stable in such a case. The total particle density, $\rho$, together with the densities of the components with $M = \pm 1$, have a significant impact on the stability of the CW solutions, while the order ($n$) of the CW solution does not affect the stability. However, the structure of the CW solution and the nonlinearity coefficient $g_1$ affect the instability strength for $g_0 < 0$, with a larger MI gain for $n = 0$, $g_1 > 0$ and $n = 1$, $g_1 < 0$, as shown in figure 5.

For $g_0 < 0$, figure 6 shows the dependence of the gain on the $g_1$ nonlinearity coefficient in the limit case of $k_1 = k_{-1}$. It is evident that the CW backgrounds with both $n = 0$ and $1$ exhibit similar MI regions for opposite signs of nonlinearity $g_1$. The MI gain is larger at $g_1 > 0$ for the CW solutions with $n = 0$, and at $g_1 < 0$ for the CW with $n = 1$, respectively, for a fixed attractive $g_0$ nonlinearity. Note that, on either side of $g_1 = 0$ at fixed $g_0$, the gain attains constant value after an initial change.

5.2. General case with no simple analytic solutions

We have derived the dispersion relations for different cases in the limit case of identical wavenumbers in different spin components. However, in the general case, when the wavenumbers of the components with $M = \pm 1$ are different, such simple analytic results cannot be obtained. Note that even if $k_1$ and $k_{-1}$ assume different values, wavenumbers $k_0$ and $k_0'$ are constrained to be equal, according to equation (8a), unless the field components with $M = \pm 1$ vanish. The effect of the CW order $n$, and of the nonlinearity coefficients $g_0$ and $g_1$, on the condensate’s MI, for both zero and nonzero differences in the wavenumbers of the $M = \pm 1$ components, were analyzed by plotting the gain, $\xi = \text{Im}(\omega)$, versus the perturbation wavenumber $k$ for suitable values of the parameters, including $A_1$ and $A_{-1}$. The possibility of having attractive nonlinearities, achievable via the Feshbach resonance, and their effect on the stability of the condensate, are also discussed in the respective cases.

We chose $\rho = 1$, $A_1 = \frac{1}{2}$ and $A_{-1} = \frac{1}{3}$ for the case when the wavenumbers are identical, and $\rho = 1$ and $A_1 = A_{-1} = \frac{1}{2}$,
as the total particle density and the dimensionless amplitudes of the field components with $M = \pm 1$, respectively, in the case of the wavenumber difference $\Delta k = 1$. The case of $\Delta k \neq 0$ with different amplitudes $A_1 = A_{-1}$ is not analytically solvable.

In accordance with the above set parameters, for a CW solution with zero $M = \pm 1$ components, we chose $A_p = \frac{1}{2}$ and $A_0 = \frac{1}{3}$ in the case when the wavenumbers are identical, the MI sets in for $g_0 < 0$ if and only if the amplitudes of the spin components $|1, 0\rangle$ and $|0, 0\rangle$ are different. However, when the wavenumbers of the two spin components are different, MI occurs even if $A_0$ and $A_p$ share the same values for repulsive $g_0$ and attractive $g_1$ nonlinearities, as shown in figure 7.

Comparison of figures 4(a) and 7(a) makes it obvious that, for $g_0 < 0$, the CW states with the zero $M = \pm 1$ component and nonzero difference between the wavenumbers of the other components are less vulnerable to the MI in a larger domain of values of the $A_0$ and $A_p$ amplitudes, than in the limit case of identical wavenumbers.

As demonstrated in the previous subsection, the CW states of spinor Ps with both $n = 0$ and 1, $g_0 > 0$, and identical wavenumbers are modulationally stable. Nevertheless, when the wavenumbers of the $M = \pm 1$ components are different, the $n = 1$ CW solution is modulationally unstable for the natural repulsive signs of the $g_0$ and $g_1$ nonlinearities, as shown in figure 8. The gain is found to increase with the increase in magnitude of $g_0$ for fixed $g_1 > 0$.

The effect of the CW parameter $n$ and the nature of the nonlinearity on the stability of the Ps condensate can be understood by comparing the subplots (a) and (b) in figures 8 and 9, respectively. The CW background with $n = 0$ and the attractive sign of $g_1$ is modulationally unstable for all values of $g_0$, and for the CW solutions with $n = 1$; the same is true for the repulsive sign of $g_1$. For small $\Delta k$, even CW solutions $(n = 0)$ are found to be stable for identical signs of $g_0$ and $g_1$ nonlinearities and $g_1 > 0$, while odd CW backgrounds $(n = 1)$ are stable for opposite signs of the nonlinearities and $g_1 < 0$. Large differences in the wavenumbers of the spin components with $M = \pm 1$ give rise to the MI even in such backgrounds, which might be expected to be stable. The final
Table 2. Summary of the results of the MI analysis for different combinations of the parameters.

<table>
<thead>
<tr>
<th>Δk = (k₁ - k₂)</th>
<th>n</th>
<th>g₁</th>
<th>g₀</th>
<th>Inference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>The n = 0, 1 CW backgrounds are modulationally unstable for the attractive g₀ nonlinearity (g₀ &lt; 0). It is independent of the nature of g₁.</td>
</tr>
<tr>
<td>1</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Δk ≠ 0

| 0               | +  | +  | stable for small values of Δk |
| 1               | +  | +  | unstable for natural values of nonlinearities |
| 1               | +  | -  | stable for small values of Δk |

6. Conclusion

We have obtained the CW solutions in spinor BEC of positronium, composed of para- (F = 0) and ortho- (F = 1) spin fields in the absence of external magnetic fields. For the condensate without the para-component, ψp = 0, the BEC is tantamount to a spinor F = 1 ferromagnetic condensate with CW solutions existing only for even n, which determines the phase shift, πm, between the different components (i.e., for the in-phase components). The CW solutions in such a case are stable.

In the presence of the para-component (ψp ≠ 0), there exist CW solutions with both n = 0 and n = 1, whose existence ranges are limited by the total density, ρ. The ground state of positronium for g₁ ≫ ε is found to have equal densities of ortho- and para-components for n = 1 CW solutions, provided that the amplitudes of the M = ±1 components are equal, and the condensate is a polar one. Since the ortho-to-para interconversion is minimal, if measured in units of the scaled energy difference, for g₁ ≪ ε and can be neglected. In such a case, the CW solutions are stable if the density of the para-Ps obeys either condition Aₚ² < A₀², or the conditions given by equations (18) and (19).

The obtained CW solutions with n = 0 and 1 in spinor Ps were subsequently examined for the MI, using the linear stability analysis for the small perturbations. In the case of zero fields with M = ±1, the stability of the CW background is found to depend on the amplitudes of the respective components, and on the nonlinearity coefficients g₀ and g₁. For g₀ > 0 and g₁ < 0, the CW solutions with identical wavenumbers are modulationally unstable only for different amplitudes of the components. In the limit case of identical wavenumbers of all the spinor components, MI for both CW backgrounds with n = 0 and n = 1 depends on the sign of g₀ alone. The result is that, both for n = 0 and 1, the CW backgrounds are stable for g₀ > 0. The total particle density, ρ, with the help of individual densities of the M = ±1 components, suppresses the influence of the CW parity, n, on the MI. For g₀ < 0, the gain attains a constant value after a rapid initial change with the variation of the g₁ nonlinearity coefficient.

In the general case of the nonzero differences between the wavenumbers of the M = ±1 spin components, the n = 1 (out-of-phase) CW background is unstable for the natural repulsive signs of g₀ and g₁, with the gain maximum large for larger values of g₀ at a fixed value of g₁. The wavenumber
difference \( \Delta k \equiv k_1 - k_{-1} = 0 \), thereby making the CW solutions with \( n = 0 \) and 1 much more vulnerable to MI.

Finally, it is relevant to briefly discuss peculiarities of possible experimental realization of the MI in the positronium BEC. The MI directly applies if its characteristic growth time, \( \sim 1/\zeta_{\text{max}} \) (see equation (31)) does not exceed the (ortho-) positronium lifetime, \( \approx 140 \) ns \[55\]. However, it is possible to realize the MI in a less extreme form if the positronium condensate is permanently replenished from an external source. Then, an essential difference of the expected experimental situation from that typical for usual atomic condensates \[31, 32\] is the fact that the scattering length for the positronium is smaller by a factor of approximately ten \[35–38\], and, most essentially, its mass is smaller than the atomic mass of \( ^7 \)Li by a large factor \( \approx 6300 \). For this reason, equation (32) demonstrates that the same range of spatial scales of the MI as in the experiments with atomic gases (\( \sim 0.1 \) mm) may be achieved for the positronium density exceeding the atomic one by approximately five orders of magnitude, which may be achieved in the experiment. Then, equation (31) suggests that, in this region of the densities, the characteristic growth time of the MI may be approximately four orders of magnitude smaller than in the atomic BEC, i.e., roughly on the microsecond scale, which is closer to the above-mentioned positronium lifetime.

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