IWDS: From Localization to Thermalization and Topology
PCS IBS, Daejeon, 3 September 2018

Many-body Multifractality throughout the Superfluid to Mott Insulator Phase Transition

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Motivation/Question

Statistical nature of many-body wavefunctions in Hilbert space

- Dynamical/transport properties (e.g. equilibration, localisation)
  Borgonovi et al., Phys. Rep. (2016), MBL special issue, Ann. Phys. (2017)

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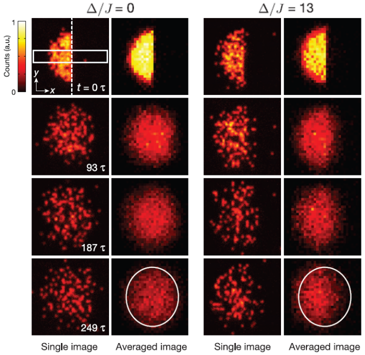

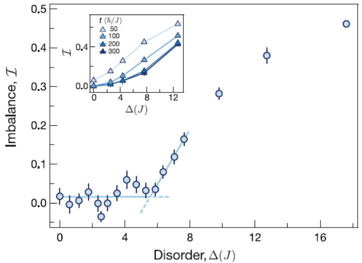

Choi et al., Science (2016)

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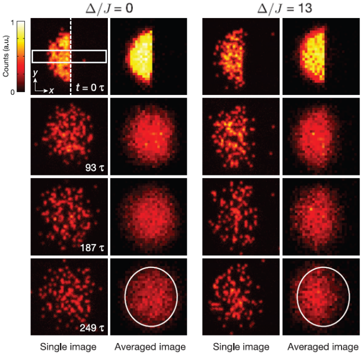
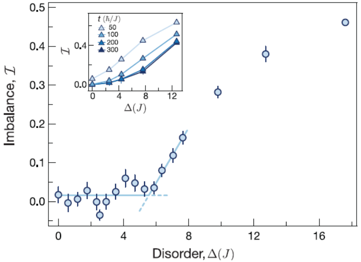




Choi et al., Science (2016)

Tools: Participation ratios, Rényi entropies, autocorrelation, . . .

Multifractality

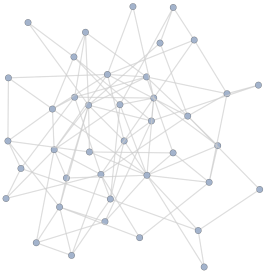
Motivation/Question

Significance of multifractality in Hilbert space

Motivation/Question

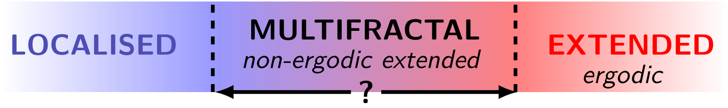
Significance of multifractality in Hilbert space

- Localisation in disordered graphs/Bethe-Lattice



Kravtsov, Ioffe, Altshuler, Ann. Phys. (2018)
Altshuler et al., PRL (2016)

Tikhonov, Mirlin, Skvortsov, PRB (2016)
García-Mata et al., PRL (2017)



Observed in random matrix models

Kravtsov, Khaymovich, Cuevas, Amini, New J. Phys. (2015)

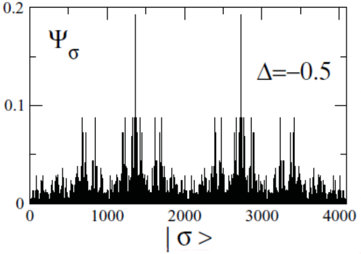
Motivation/Question

Significance of multifractality in Hilbert space

- Generic feature of 'clean' many-body spin Hamiltonians

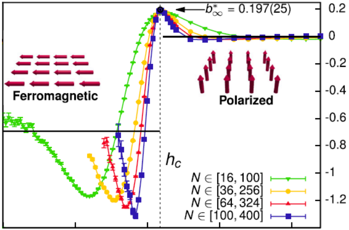
Ground state *multifractality*

Atas, Bogomolny, PRE (2012)



QPTs revealed by *multifractal scaling*

Luitz et al., PRL (2014)



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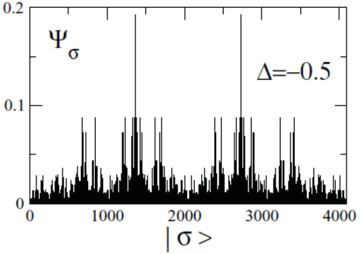

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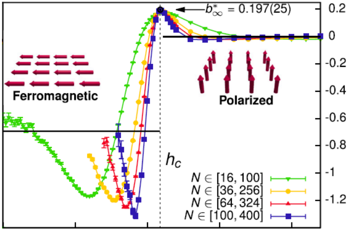


Is multifractality an efficient probe of many-body systems?

Outline

Multifractality in a nutshell

One-dimensional Bose-Hubbard system

- Multifractality of the ground state in Fock space

Probing of the SF-MI transition through multifractality

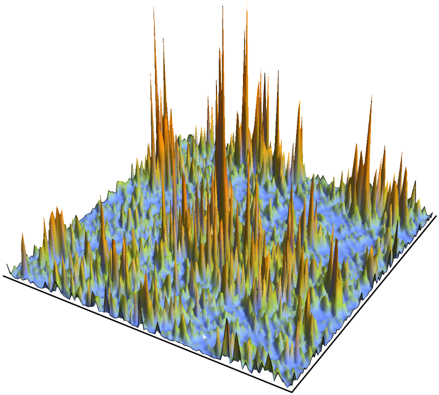
Multifractality

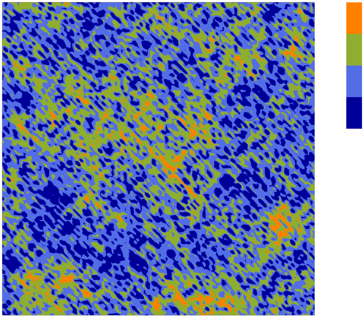
Multifractal state $\psi(\boldsymbol{r})$ in space of volume $\mathcal{N}$:

- Sets of common intensity $\alpha(\boldsymbol{r}) = -\log_{\mathcal{N}} |\psi(\boldsymbol{r})|^2$ are fractals:

$$\text{Volume of set} \sim \mathcal{N}^{f(\alpha)}$$

- Multiple intensity sets exhibiting different scaling behaviour





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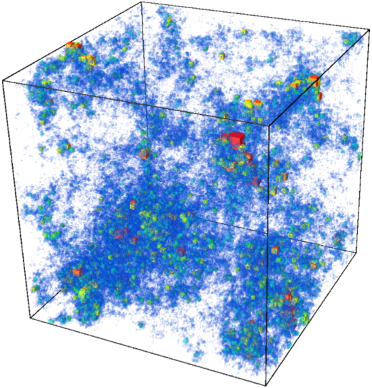

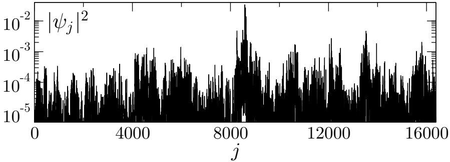

Multifractality

$$|\psi\rangle = \sum_{j=1}^{\mathcal{N}} \psi_j \, |j\rangle \longrightarrow R_q \equiv \sum_j |\psi_j|^{2q}$$

Generalised inverse participation ratios

Scaling as $\mathcal{N} \to \infty$

$$\boldsymbol{R_q \sim \mathcal{N}^{-(q-1)D_q}}$$

Generalised fractal dimensions

$$D_q = \lim_{\mathcal{N}\to\infty} \frac{1}{1-q} \frac{\ln R_q}{\ln \mathcal{N}} = \lim_{\mathcal{N}\to\infty} \widetilde{D}_q$$

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Information dimension: $D_1 = \lim_{\mathcal{N}\to\infty} \frac{1}{\ln \mathcal{N}} \left[-\sum_j |\psi_j|^2 \ln |\psi_j|^2 \right]$

Correlation dimension: D_2 Decay of spatial correlations and survival probability

Minimum dimension: $D_\infty = -\frac{\ln \max\{|\psi_j|^2\}}{\ln \mathcal{N}}$

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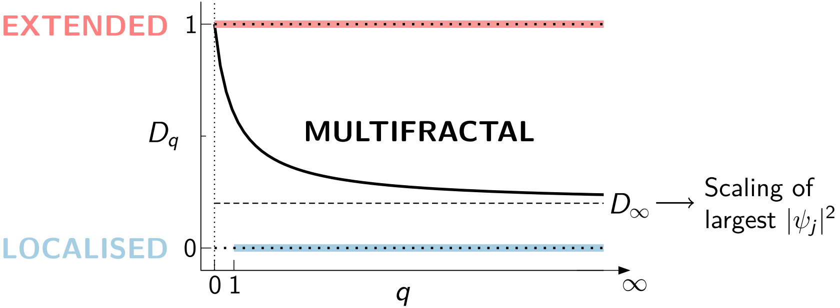


One-dimensional Bose-Hubbard system

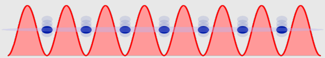
Interacting bosons in a 1-D optical lattice

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1)$$

Ground state phase transition at fixed $\nu \equiv N/L$ ($L \to \infty$, $N \to \infty$)

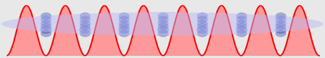
Mott insulator (MI)

- Gapped & Incompressible
- Superfluid fraction $= 0$



Superfluid (SF)

- Gapless & Compressible
- Superfluid fraction $\neq 0$



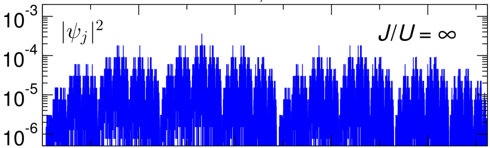
0 $\quad$ $\boldsymbol{(J/U)_c \simeq 0.3}$ $\quad$ $\boldsymbol{\infty}$

Boéris et al., Phys. Rev. A 93, 011601(R) (2016)

Ground state in Fock space at integer filling ν

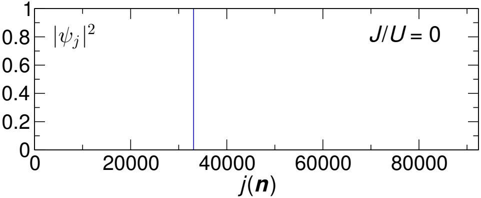
$$|\psi(J/U)\rangle = \sum_{\boldsymbol{n}} \psi_{\boldsymbol{n}}^{(J/U)} |\boldsymbol{n}\rangle, \qquad |\boldsymbol{n}\rangle \equiv |n_1, n_2, \ldots, n_L\rangle$$

$L = 10,\ \nu = 1$



$J/U = \infty$

$$\left|\psi_{\boldsymbol{n}}^{(\infty)}\right|^2 = \frac{N!}{L^N n_1! n_2! \ldots n_L!}$$



$J/U = 0$

$$|\psi(0)\rangle = |\nu, \nu, \ldots, \nu\rangle$$

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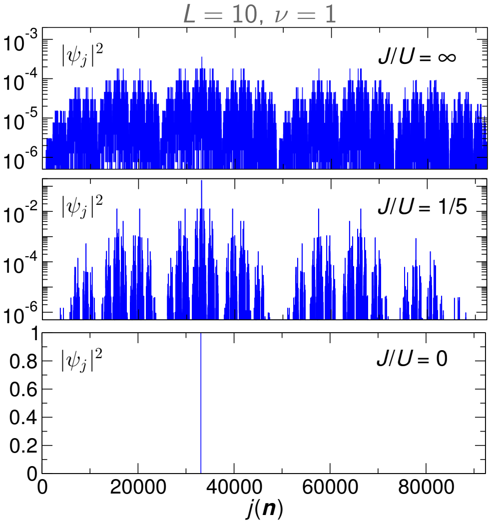

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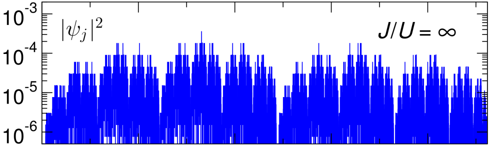

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$\boldsymbol{J/U = \infty}$

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→ **MULTIFRACTAL** *[Bogomolny, 2012]*
Exact D_q available
e.g., $\boldsymbol{D_2 = 0.91}$, $\boldsymbol{D_\infty = 0.72}$

↑

$\boldsymbol{J/U = 0}$

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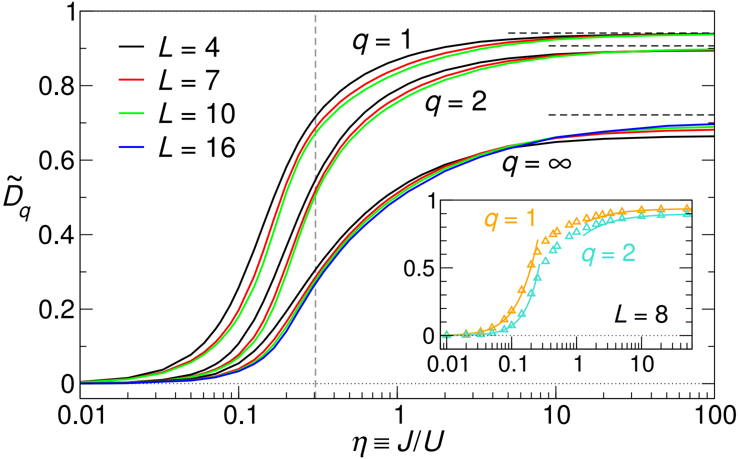
→ $\boldsymbol{D_q = 0}$ for $q > 0$

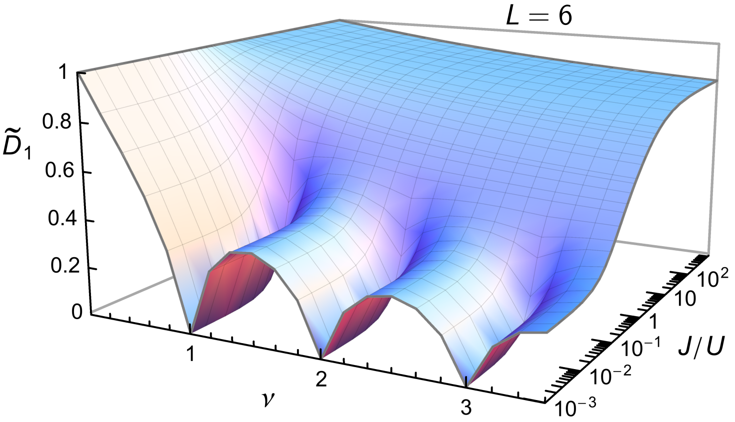
Evolution of $\widetilde{D}_q$ versus J/U at unit filling factor

- Finite-size fractal dimensions increase monotonously
- Seemingly pronounced increase around transition region



Qualitative fingerprint of SF-MI phase transition

- Mott phase signalled by sharp decrease of fractal dimensions at integer filling



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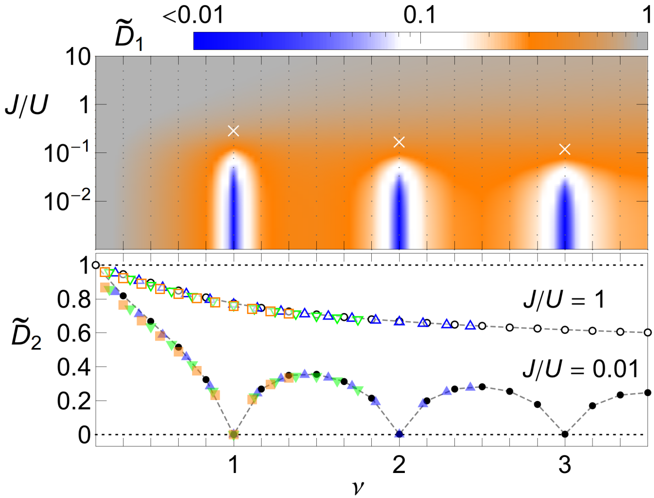

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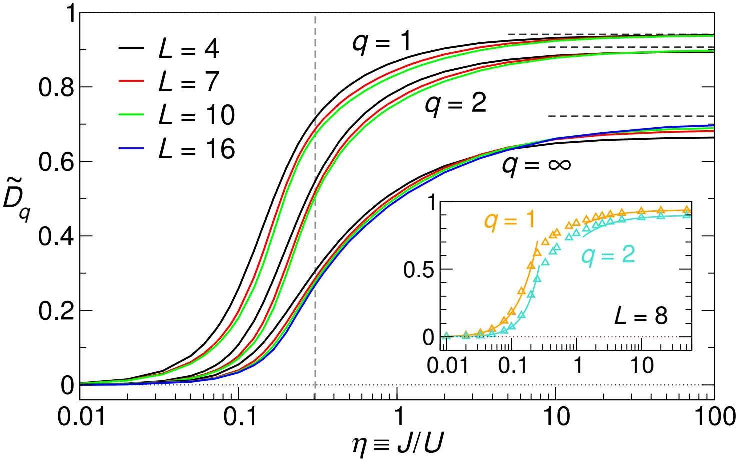
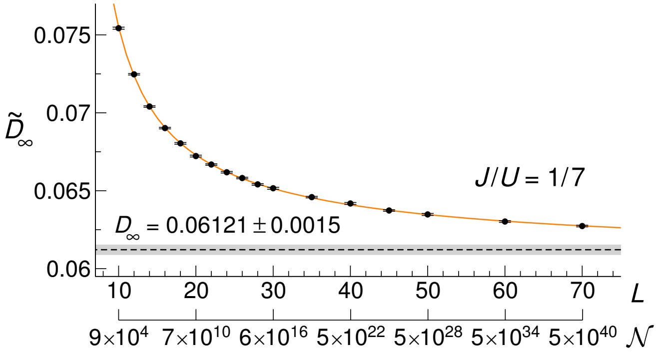


Limit $\mathcal{N} \to \infty$ in the MI phase: Localisation?

- Smallest generalized fractal dimension ($q = \infty$) as $\mathcal{N} \to \infty$:

$$\widetilde{D}_\infty = D_\infty + a\frac{\ln L}{L} + b\frac{1}{L} + c\frac{\ln^2 L}{L^2} + O(L^{-2}\ln L)$$

- $D_\infty \neq 0 \Rightarrow$ MI state for $J/U \neq 0$ not localised in Fock space



Location of critical point

- Sensitivity of ground state to changes in $\eta \equiv J/U$:

Fidelity susceptibility	Derivatives of $\mathrm{var}(n_i)$
Sun, Kolezhuk,Vekua, PRB (2015)	*Łącki, Damski, Zakrzewski, Sci. Rep. (2016)*

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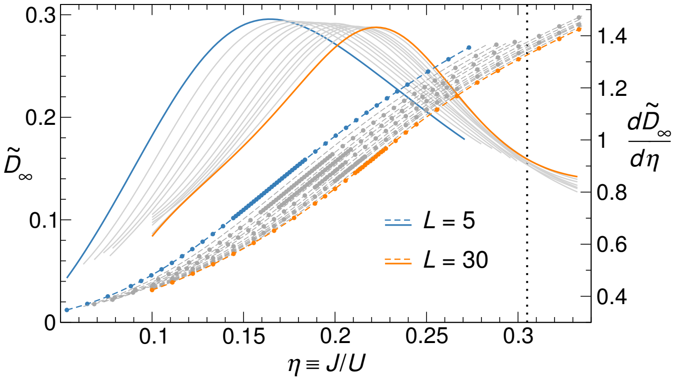
- Derivative of $\widetilde{D}_q$ with respect to η encodes the transition:
  The position $\eta_*(L)$ of the maximum approaches $(J/U)_c$ as $L \to \infty$



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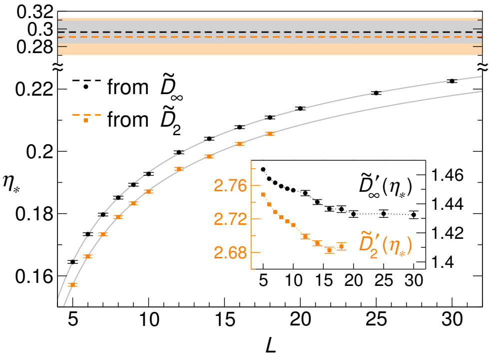


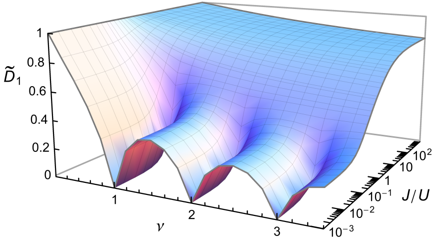
$$\eta_*(L) = \eta_c - \frac{b^2}{\ln^2(L/|\ell_q|)}$$

$$\boldsymbol{\eta_c = 0.296 \pm 0.006}$$

$$\boldsymbol{\eta_c = 0.291 \pm 0.011}$$

Concluding remarks

- Strong evidence of multifractality in Fock space for the ground state of the BHH for arbitrary values of J/U
- SF-MI phase transition is revealed by the generalised fractal dimensions $\widetilde{D}_q$



Lindinger, MSc Thesis (2017)

Lindinger, Rodríguez
Acta Phys. Pol. A 132 (2017)

- Multifractality in excited states
- (Many-body) Fock localisation in the disordered BHH?