Dynamics of disordered interacting quantum systems

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Outline

We provide a full description of the dynamics of isolated interacting many-body systems—i.e., from $t = 0$ to $t \to \infty$.

- **Models:** Full random matrices (FRM) and one-dimensional interacting spin-$1/2$ chain with on-site disorder.
- **Static Properties:** Level statistics and eigenstates components.
- **Dynamics:** Survival probability.
  - **Very short times:** Universal quadratic decay.
  - **Short times:** Exponential, Gaussian decays.
  - **Intermediate times:** Power-law decays.
  - **Long times:** Correlation hole.
  - **Even longer times:** Saturation.

Models

$$H = H_0 + V$$

- **Spin-$1/2$ model**: A system of L spin-$1/2$ particles in a lattice with two-body interactions and on-site magnetic field.

$$H_0 = \sum_{k=1}^{L} h_k S_k^z, \quad V = \sum_{k=1} \left(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y \right) + \sum_{k=1} S_k^z S_{k+1}^z.$$

  - $S_k^{x,y,z} = \frac{1}{2}\sigma_k^{x,y,z}$ are spin-$1/2$ operators.
  - h_k are random numbers in $[-h, h]$. Depending on h the system can be taken from an ergodic to a many-body localized (MBL) phase.
  - This is the"standard model" for studying the MBL transition.

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- **Random matrices (GOE)**: For time-reversal invariant systems with rotational symmetry: Matrices filled with uncorrelated random numbers with

$$H_0 : \text{ diagonal matrix with } \langle H_{0_{nn}}^2 \rangle = 2.$$

$$V : \text{ off-diagonal matrix with } \langle V_{nm}^2 \rangle = 1.$$

- Flip-flop term

$$\left(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y\right) | \uparrow_k \downarrow_{k+1}\rangle = \frac{1}{2} | \downarrow_k \uparrow_{k+1}\rangle$$

- Ising interaction

$$S_k^z S_{k+1}^z | \uparrow_k \uparrow_{k+1}\rangle = \frac{1}{4} | \uparrow_k \uparrow_{k+1}\rangle$$

$$S_k^z S_{k+1}^z | \uparrow_k \downarrow_{k+1}\rangle = -\frac{1}{4} | \uparrow_k \downarrow_{k+1}\rangle$$

- Total spin in the z-direction, $\mathcal{S}^z = \sum_k^L S_k^z$, is conserved ($[H, \mathcal{S}^z] = 0$).

Hamiltonian matrix structure: Spin-$1/2$ model

$L = 4$ sites

Base
$|\uparrow\downarrow\uparrow\downarrow\rangle$
$|\uparrow\uparrow\downarrow\downarrow\rangle$
$\dots$

$$\begin{pmatrix}
\frac{3J\Delta}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{J\Delta}{4} & \frac{J}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{J}{2} & -\frac{J\Delta}{4} & \frac{J}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{J}{2} & -\frac{J\Delta}{4} & \frac{J}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{J}{2} & \frac{J\Delta}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{J\Delta}{4} & \frac{J}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{J}{2} & -\frac{3J\Delta}{4} & \frac{J}{2} & \frac{J}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{J}{2} & -\frac{J\Delta}{4} & 0 & \frac{J}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{J}{2} & 0 & -\frac{J\Delta}{4} & \frac{J}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{J}{2} & \frac{J}{2} & -\frac{3J\Delta}{4} & \frac{J}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{J}{2} & \frac{J\Delta}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{J\Delta}{4} & \frac{J}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{J}{2} & -\frac{J\Delta}{4} & \frac{J}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{J}{2} & -\frac{J\Delta}{4} & \frac{J}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{J}{2} & \frac{J\Delta}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3J\Delta}{4}
\end{pmatrix}$$

$\mathcal{S}^z = \sum_k^L S_k^z$

$[H, \mathcal{S}^z] = 0$

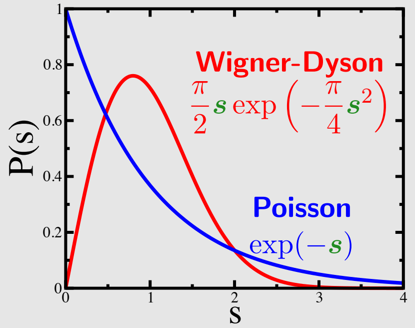
- **We work in the $\mathcal{S}^z = 0$ sector, dimension $N = L!/(L/2!)^2$.**

Level statistics: Level spacing distribution

$$s = \frac{E_{\alpha+1} - E_{\alpha}}{\Delta}$$

- Integrable systems: Non correlated eigenvalues $\Rightarrow$ Poisson.
- Chaotic systems: Correlated eigenvalues $\Rightarrow$ Wigner-Dyson.
- Shape of Wigner-Dyson depends on system symmetry.

- System with temporal inversion symmetry:
  - Gaussian Orthogonal Ensemble (**GOE**).
  - Real symmetric matrices.



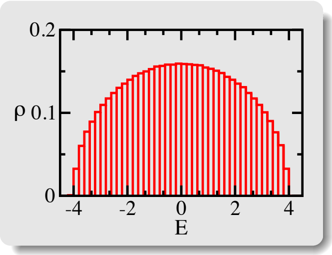
M.V. Berry and M. Tabor, Proc. Roy. Soc. A **356** (1977).
O. Bohigas, M.-J. Giannoni and C. Schmit, Phys. Rev. Lett. **52** (1984).

Global properties are not described by FRM

$$H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle$$

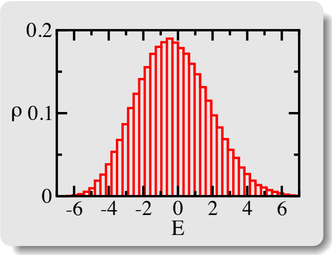
- Energy distribution: $\rho(E) = \sum_\alpha \delta(E - E_\alpha)$.

Full random matrices



Wigner (1957)

Few-body interactions



French & Wong (1970)

E Wigner, Ann. of Math., **67**, 325 (1957).
JB French and SSM Wong. Phys. Lett. B, **33**, 449, (1970).

Structure of eigenstates

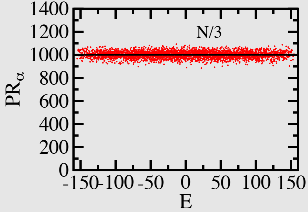
$$|\psi_\alpha\rangle = \sum_n C_n^\alpha |\phi_n\rangle$$

Participation ratio

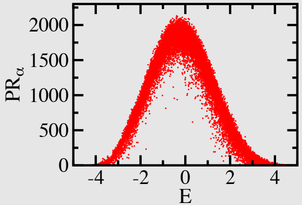
$$\text{PR}_\alpha = 1 / \sum_{n=1}^{N} |C_n^\alpha|^4$$

- **Random matrices**
  - Amplitudes C_n^α are random variables.
  - All eigenstates completely delocalized.
- **Few-body interactions**
  - Only eigenstates close to the middle of the spectrum are delocalized.

Full random matrices



Two-body interactions



Level statistics

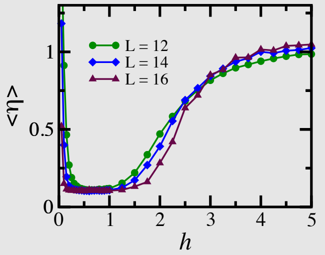
$$H = \sum_{k=1}^{L} \left(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y + S_k^z S_{k+1}^z \right) + \sum_{k=1}^{L} h_k S_k^z$$

Level spacing distribution

We quantify the proximity to the Wigner-Dyson distribution with

$$\eta = \frac{\int_0^{s_0} [P(s) - P_{WD}(s)] ds}{\int_0^{s_0} [P_P(s) - P_{WD}(s)] ds}$$

- $< \eta > \approx 1 \Rightarrow$ Poisson.
- $< \eta > \approx 0 \Rightarrow$ Wigner-Dyson.



- $h \ll 1 \Rightarrow$ Poisson.
- $h \lesssim 1 \Rightarrow$ Wigner-Dyson (ergodic).
- $1 \lesssim h \lesssim 3 \Rightarrow$ Intermediate statistics.
- $h \gtrsim 3 \Rightarrow$ Many-body localization.

EJT-H and LF Santos, Ann. Phys. (Berlin) **529**, 1600284 (2017).

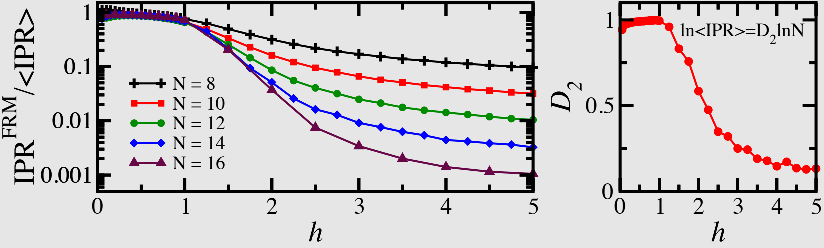
Structure of eigenstates

Inverse participation ratio: $IPR_\alpha = \sum_n |C_n^\alpha|^4$.

Ergodic extended state: $IPR_\alpha \propto N^{-1}$

Non-ergodic extended state: $IPR_\alpha \propto N^{-D_2}, D_2 < 1$

Localized state: $IPR_\alpha \propto \mathcal{O}(1)$



EJT-H and LF Santos, Ann. Phys. (Berlin) **529**, 1600284 (2017).

Quench dynamics

- System is prepared in some initial state $|\Psi(0)\rangle$, eigenstate of H_0.

- A sudden change of some parameter in H_0,

$$H_0 \longrightarrow H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle$$

- Then the evolution is dictated by H,

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle = \sum_\alpha C_\alpha^{\text{ini}} e^{-iE_\alpha t}|\psi_\alpha\rangle$$

$$C_\alpha^{\text{ini}} = \langle\psi_\alpha|\Psi(0)\rangle$$

Observables

- **Survival probability: Probability to find the system in its initial state at time t.**

$$F(t) = |\langle \Psi(0) | \Psi(t) \rangle|^2 = \left| \langle \Psi(0) | e^{-iHt} | \Psi(0) \rangle \right|^2$$

It is a simple quantity to characterize the dynamics of the system.

- **An arbitrary observable O that commutes with H_0, is written as**

$$O(t) = F(t) O(0) + \sum_{n,m \neq \textbf{ini}} \langle n_{\textbf{ini}} | e^{iHt} | n \rangle \langle n | O | m \rangle \langle m | e^{-iHt} | n_{\textbf{ini}} \rangle$$

Survival probability

$$F(t) = \left|\langle\Psi(0)|e^{-iHt}|\Psi(0)\rangle\right|^2 = \left|\sum_{\alpha}|C_{\alpha}^{\mathbf{ini}}|^2 e^{-iE_{\alpha}t}\right|^2 = \left|\int P(E)e^{-iEt}dE\right|^2$$

Local density of states (LDOS)

$$P(E) = \sum_{\alpha}|C_{\alpha}^{\text{ini}}|^2\delta(E - E_{\alpha})$$

$F(t)$ is the Fourier transform of the LDOS

Survival probability

$$F(t) = \left|\langle\Psi(0)|e^{-iHt}|\Psi(0)\rangle\right|^2 = \left|\sum_\alpha |C_\alpha^{\mathsf{ini}}|^2 e^{-iE_\alpha t}\right|^2 = \left|\int P(E)e^{-iEt}dE\right|^2$$

- Evolution at **very short times**

$$F(t) \overset{t\ll\sigma_{\mathsf{ini}}^{-1}}{\approx} 1 - \sigma_{\mathsf{ini}}^2 t^2$$

$$\sigma_{\mathsf{ini}}^2 = \sum_\alpha |C_\alpha^{\mathsf{ini}}|^2 (E - E_{\mathsf{ini}})^2 \qquad \text{and} \qquad E_{\mathsf{ini}} = \langle\Psi(0)|H_{\mathsf{F}}|\Psi(0)\rangle$$

- For $t \to \infty$, there is saturation

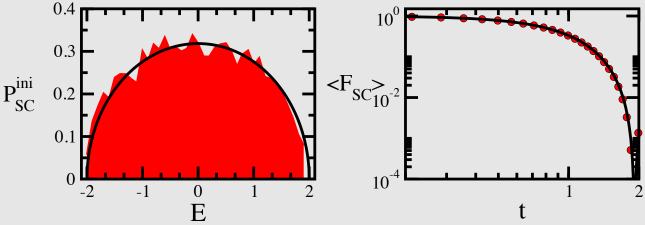
$$\overline{F(t)} = \sum_\alpha |C_\alpha^{\mathsf{ini}}|^4 \equiv \mathsf{IPR}_{\mathsf{ini}}$$

Full random matrices (GOE)

- Initial states: eigenstates of H_0.
- Energy of initial states is in the middle of the spectrum of the total Hamiltonian, $E_{\text{ini}} = \langle \Psi(0)|H|\Psi(0)\rangle \approx 0$.

Short time: Fast decay

$$P_{SC}^{\text{ini}}(E) = \frac{1}{2\pi\sigma_{\text{ini}}^2}\sqrt{4\sigma_{\text{ini}}^2 - E^2} \quad \Rightarrow \quad F_{SC}(t) = \frac{\mathcal{J}_1^2(2\sigma_{\text{ini}}t)}{\sigma_{\text{ini}}^2 t^2}$$



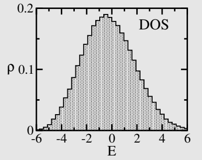
The envelope of the LDOS determines the decay.

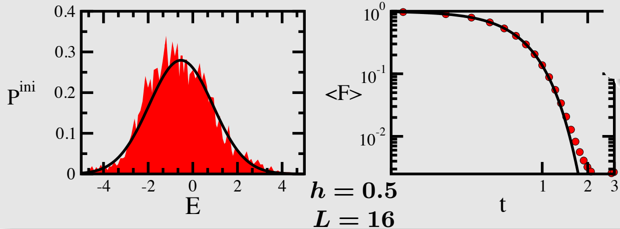
EJT-H, M Vyas, and LF Santos, New J. Phys. **16**, 063010 (2014).

Disordered spin-$1/2$ model: Weak disorder

$$H_0 = \sum_{k=1}^{L} h_k S_k^z, \qquad V = \sum_{k=1}^{L} \left(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y + S_k^z S_{k+1}^z \right)$$

- Initial states: eigenstates of H_0; $|\uparrow\downarrow\uparrow\downarrow\ldots\rangle$, $|\uparrow\uparrow\downarrow\downarrow\ldots\rangle$, ...
- $E_{\text{ini}} = \langle \Psi(0)|H|\Psi(0)\rangle \approx 0$.





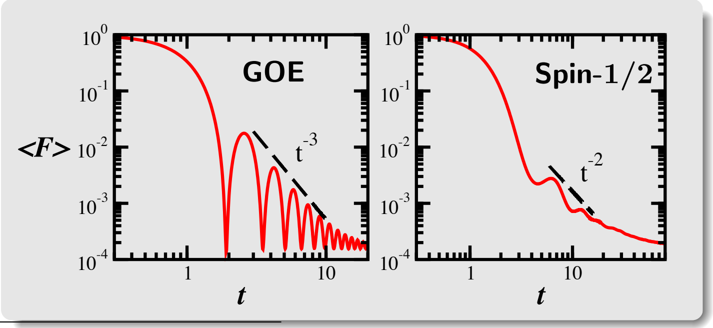
$$P_G^{\text{ini}}(E) = \frac{1}{\sqrt{2\pi\sigma_{\text{ini}}^2}} \exp\left[-\frac{(E-E_{\text{ini}})^2}{2\sigma_{\text{ini}}^2} \right] \quad \Rightarrow \quad F(t) = \exp\left(-\sigma_{\text{ini}}^2 t^2\right)$$

FM Izrailev and A Castañeda-Mendoza, Phys. Lett. A **350**, 355 (2006).
EJT-H and L. F. Santos, Phys. Rev. A 89, 043620 (2014).
EJT-H and LF Santos, Phys. Rev. A **90**, 033623 (2014).

Power-law decay

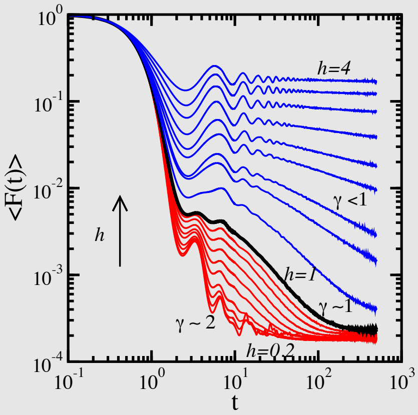
Bounds in the energy spectrum lead to a power-law decay of the survival probability.

$$F(t) = \left| \int_{E_{\text{low}}}^{E_{\text{up}}} P^{\text{ini}}(E) e^{-iEt} dE \right|^2 \overset{t \gg \sigma^{-1}}{=} \begin{cases} t^{-3}, & \text{semicircle} \\ t^{-2}, & \text{Gaussian} \end{cases}$$



EJT-H and LF Santos, Phys. Rev. B **92**, 014208 (2015).
M Távora, EJT-H, LF Santos Phys. Rev. A **94**, 041603(R) (2017).
M Távora, EJT-H, LF Santos Phys. Rev. A **95**, 013604 (2017).

Effects of stronger disorder



$$\langle F(t) \rangle \propto t^{-\mu}$$

Components C_α^{ini} are correlated

$$\text{IPR}_{\text{ini}} \propto N^{-D_2}$$

$$F(t) \propto t^{-D_2}, \quad D_2 < 1$$

Bounds of the energy spectrum

$$\text{IPR}_{\text{ini}} \propto N^{-1}$$

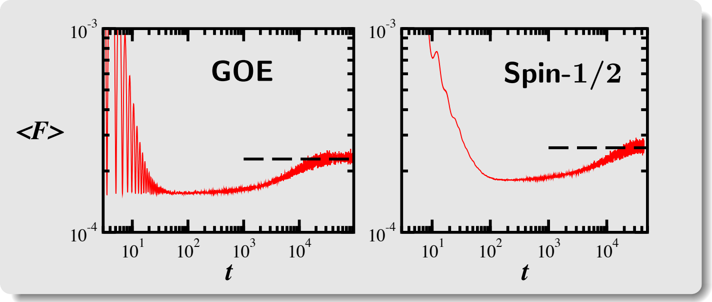
$$F(t) \propto t^{-\gamma}, \quad 1 < \gamma < 2$$

Power law is not exclusive of criticality

EJT-H and LF Santos, Phys. Rev. B **92**, 014208 (2015).
EJT-H and LF Santos, Ann. Phys. (Berlin) **529**, 1600284 (2017).

Correlation hole

After the power-law decay a "hole" appears in the dynamics



Correlations between eigenvalues manifest themselves in the dynamics

L Leviandier *et al.*, Phys. Rev. Lett. **56**, 2449 (1986).
Y Alhassid and RD Levine, Phys. Rev. A **46**, 4650 (1992).
EJT-H and LF Santos, Philos. Trans. Royal Soc. A **375**, 20160434 (2017).

Correlation hole: GOE

$$F(t) = \sum_{\alpha_1,\alpha_2} |C_{n_0}^{\alpha_1}|^2 |C_{n_0}^{\alpha_2}|^2 e^{iE_{\alpha_1}t} e^{-iE_{\alpha_2}t} = \int G(E) e^{-iEt} dE + \overline{F}.$$

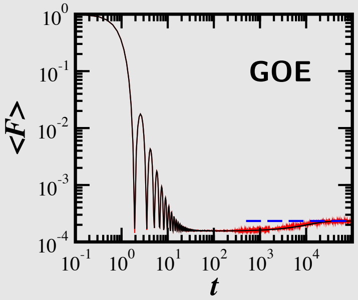
Spectral autocorrelation function

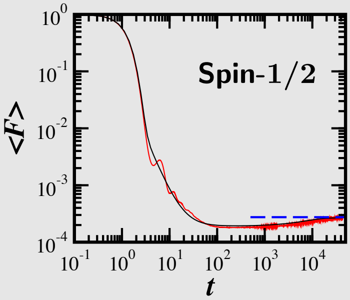
$$G(E) = \sum_{\alpha_1 \neq \alpha_2} |C_{n_0}^{\alpha_1}|^2 |C_{n_0}^{\alpha_2}|^2 \delta(E - E_{\alpha_1} + E_{\alpha_2}).$$

For GOE full random matrices

$$\langle G(E) \rangle_{\text{FRM}} = \left\langle \sum_{\alpha_1 \neq \alpha_2} |C_{n_0}^{\alpha_1}|^2 |C_{n_0}^{\alpha_2}|^2 \right\rangle_{\text{FRM}} \langle \delta(E - E_{\alpha_1} + E_{\alpha_2}) \rangle_{\text{FRM}}.$$

$$\langle F(t) \rangle_{\text{FRM}} = \frac{1 - \langle \overline{F} \rangle_{\text{FRM}}}{N-1} \left[N \frac{\mathcal{J}_1^2(2\sigma_{\textbf{ini}} t)}{(\sigma_{\textbf{ini}} t)^2} - b_2 \left(\frac{\sigma_{\textbf{ini}} t}{2N} \right) \right] + \langle \overline{F} \rangle_{\text{FRM}}.$$





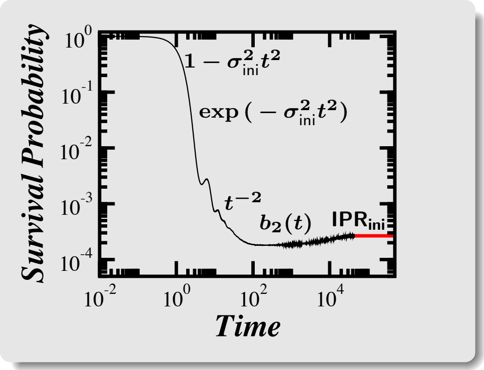
$$\langle F(t)\rangle = \begin{cases} \frac{1-\langle \overline{F}\rangle_{\mathrm{FRM}}}{N-1}\left[N\frac{\mathcal{J}_1^2(2\sigma_{\mathsf{ini}}t)}{(\sigma_{\mathsf{ini}}t)^2} - b_2\left(\frac{\sigma_{\mathsf{ini}}t}{2N}\right)\right] + \langle\overline{F}\rangle_{\mathrm{FRM}}, \\ \frac{1-\langle \overline{F}\rangle}{N-1}\left[N\frac{e^{-\sigma_{\mathsf{ini}}^2 t^2}}{\sigma_{\mathsf{ini}}^2 t^2} g(t) - b_2\left(\frac{\sigma_{\mathsf{ini}}t}{2N}\right)\right] + \langle\overline{F}\rangle. \end{cases}$$

EJT-H and LF Santos, Philos. Trans. Royal Soc. A **375**, 20160434 (2017).
EJT-H, MG-G, and LF Santos, Phys. Rev. B **97**, 060303(R) (2018).

Summary

We presented a complete description of the dynamics of isolated interacting many-body systems—i.e., from $t = 0$ to $t \to \infty$.

- **Slightly disordered systems show generic behavior.**



M Schiulaz, EJT-H, LF Santos, arXiv preprint arXiv:1807.07577.

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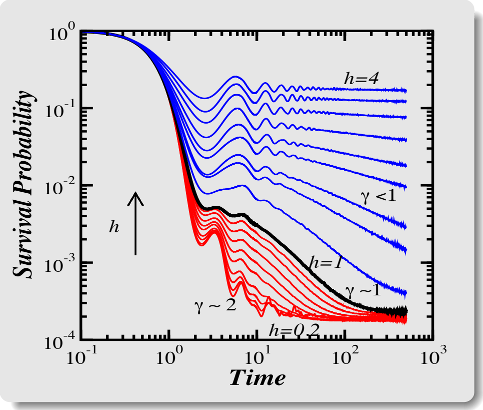
We presented a complete description of the dynamics of isolated interacting many-body systems—i.e., from $t = 0$ to $t \to \infty$.

- **Effects of stronger disorder.**



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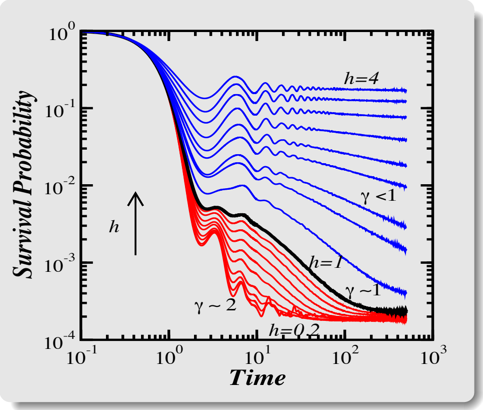
Summary

We presented a complete description of the dynamics of isolated interacting many-body systems—i.e., from $t = 0$ to $t \to \infty$.

- **Effects of stronger disorder.**

Thank you

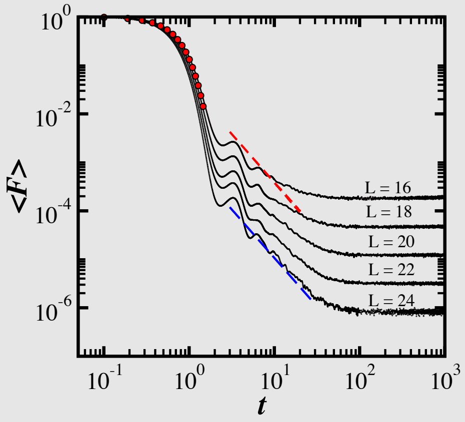


Thank you

M Schiulaz, EJT-H, LF Santos, arXiv preprint arXiv:1807.07577.

Power-law decay

The power-law decay is clearer for larger system sizes



EJT-H, LF Santos, arXiv preprint arXiv:1804.06401.

How to determine the exponent γ?

- **Dynamics**

$$F(t) = \int_{-\infty}^{\infty} e^{-i\omega t} C(\omega) d\omega, \quad C(\omega) \equiv \sum_{\alpha,\beta} |C_\alpha^{\text{ini}}|^2 |C_\beta^{\text{ini}}|^2 \delta(E_\alpha - E_\beta - \omega)$$

$$C(\omega) \xrightarrow{\omega \to 0} \omega^{\gamma-1} \;\Rightarrow\; F(t) \to t^{-\gamma}, \qquad \gamma \le 1$$

- **Statical properties**

$$\left\langle \sum_\alpha |C_\alpha^{\text{ini}}|^{2q} \right\rangle \propto N^{-D_q(q-1)}, \;\; \overset{q=2}{\Longrightarrow} \;\; \left\langle \sum_\alpha |C_\alpha^{\text{ini}}|^4 \right\rangle \equiv \langle \text{IPR}_{\text{ini}} \rangle \propto N^{-D_2}$$

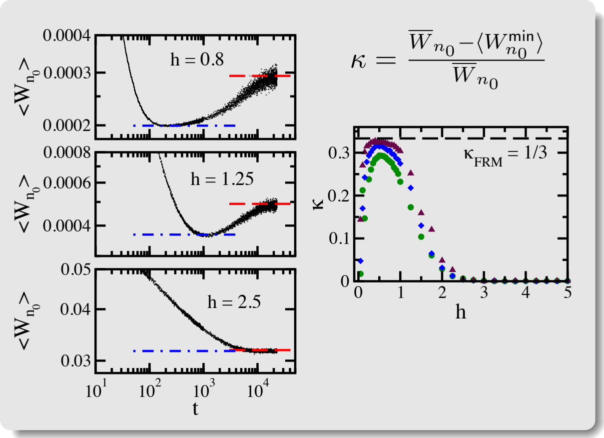
D_q : Fractal dimensions

$$\boldsymbol{\gamma = D_2}$$

- **The approach is also valid for interacting systems!**

B Huckestein and L Schweitzer, PRL **72**, 713 (1994).
EJT-H and LF Santos, Phys. Rev. B **92**, 014208 (2015).

Correlation hole: Many-body localization



- **Strong disorder → No correlation hole.**
- **Small disorder → Deeper and wider correlation hole.**

EJT-H and LF Santos, Ann. Phys. (Berlin) **529**, 1600284 (2017).