Thouless and Relaxation Time Scales in Many-Body Quantum Systems

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TIME SCALES

Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577

PRB(R) **97**, 060303 (2018)
Generic dynamical features of quenched interacting quantum systems: Survival probability, density imbalance and out-of-time-ordered correlator

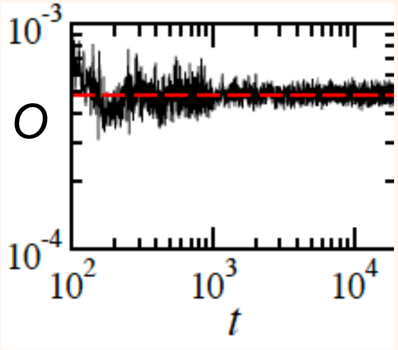
Equilibration

➢ Consensus on what equilibration in isolated finite quantum systems means.

➢ What are the time scales involved in the relaxation process? Contradictory results.

*) equilibration happens at very short times.

*) claim that extremely long times are required.



Time Scales in Real Systems

➢ Consensus on what equilibration in isolated finite quantum systems means..

➢ What are the time scales involved in the relaxation process? Contradictory results.

*) equilibration happens at very short times.

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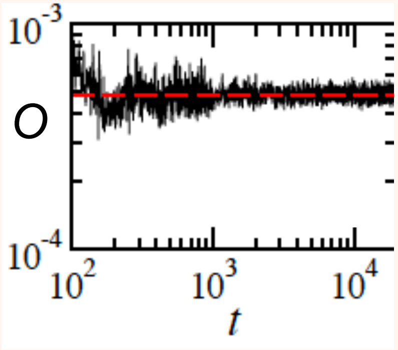


Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577

In realistic isolated many-body quantum systems.

*) Focus: two very long time scales -- Thouless time and Relaxation time.

*) These two increase exponentially with system size.

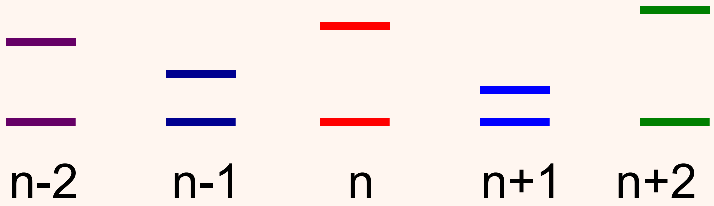
1D Disordered Spin-1/2 Model

$$H = \sum_{n=1}^{L} \frac{h_n}{2} \sigma_n^z + \sum_{n=1}^{L} \frac{J}{4} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z \right)$$

Onsite disorder
h: disorder strength

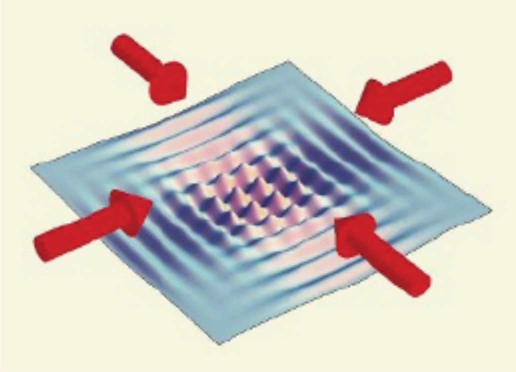
hopping

Ising
interaction



Random numbers

$$h_n \in [-h, h]$$



experimental system

L: number of sites

1D Disordered Spin-1/2 Model

$$H = \sum_{n=1}^{L} \frac{h_n}{2}\sigma_n^z + \sum_{n=1}^{L}\frac{J}{4}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \sigma_n^z\sigma_{n+1}^z\right)$$

Onsite disorder
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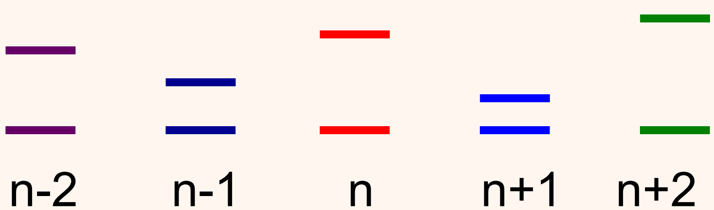


Random numbers

$$h_n \in [-h, h]$$

$$|\Psi(0)\rangle = \uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow$$

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi(0)\rangle$$

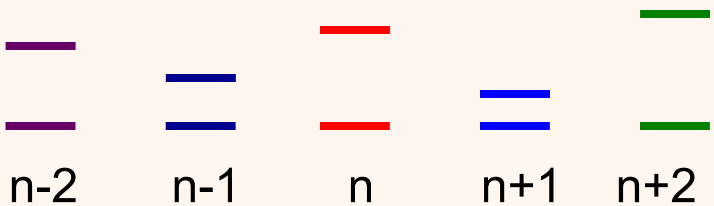
VERY STRONG PERTURBATION

L: number of sites

1D Disordered Spin-1/2 Model

$$H = \sum_{n=1}^{L} \frac{h_n}{2}\sigma_n^z + \sum_{n=1}^{L} \frac{J}{4}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \sigma_n^z\sigma_{n+1}^z\right)$$

Onsite disorder
h: disorder strength



Random numbers

$$h_n \in [-h, h]$$

- **ENTIRE DYNAMICS**
- **ANALYTICAL ESTIMATES**

$$|\Psi(0)\rangle = \uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow$$

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$$

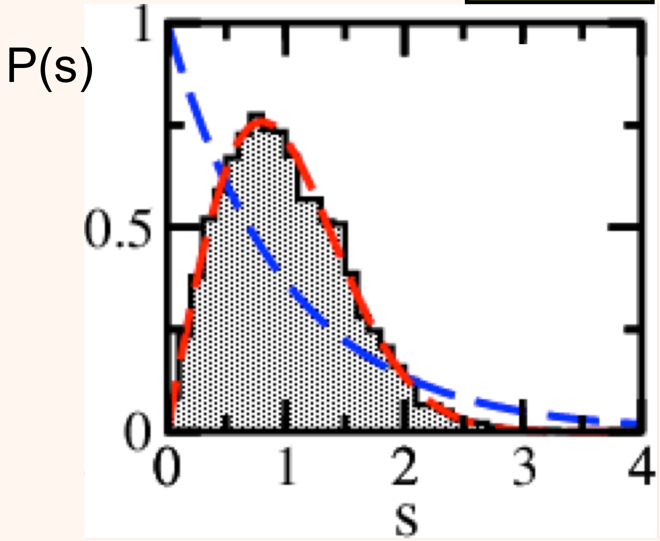
AVERAGES
10^4-10^5 data

L: number of sites

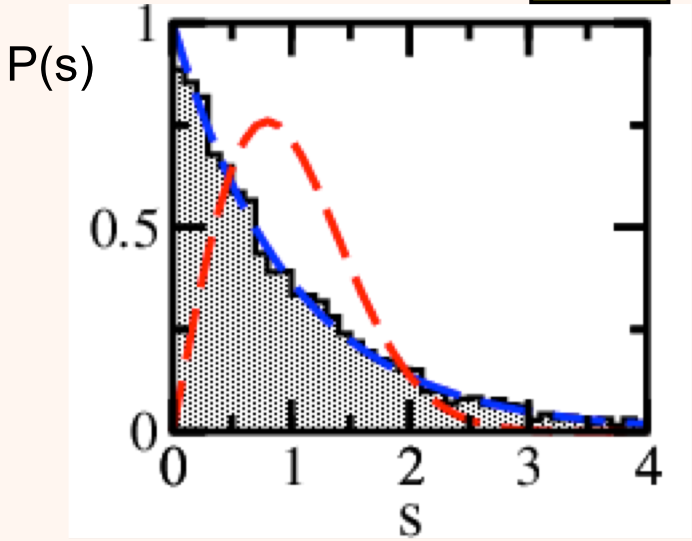
1D Disordered Spin-1/2 Model

$$H = \sum_{n=1}^{L} \frac{h_n}{2} \sigma_n^z + \sum_{n=1}^{L} \frac{J}{4} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z)$$

Chaotic h=0.5J



Integrable $h>h_c$



LFS, Rigolin, Escobar
Entanglement versus chaos in disordered spin chains
PRA **69**, 042304 (2004)

Survival Probability

$$\left|\langle \Psi(0) \mid \Psi(t) \rangle\right|^2$$

$$SP(t) = \left|\langle \Psi(0) \mid \Psi(t) \rangle\right|^2 = \left| \sum_\alpha \left|C_{k_0}^\alpha\right|^2 e^{-iE_\alpha t} \right|^2$$

$$\mid \Psi(t) \rangle = \sum_\alpha C_{k_0}^\alpha e^{-iE_\alpha t} \mid \alpha \rangle$$

Eigenvalues and eigenstates
of the total Hamiltonian

$$C_{k_0}^\alpha = \langle \alpha | \Psi(0) \rangle$$

Other experimental observables

- Spin autocorrelation function
- Participation ratio

Survival Probability

$$\left|\langle \Psi(0) | \Psi(t) \rangle\right|^2$$

$$SP(t) = \left|\langle \Psi(0) | \Psi(t) \rangle\right|^2 = \left| \sum_{\alpha} \left| C_{k_0}^{\alpha} \right|^2 e^{-iE_{\alpha}t} \right|^2$$

$$| \Psi(t) \rangle = \sum_{\alpha} C_{k_0}^{\alpha} e^{-iE_{\alpha}t} | \alpha \rangle$$

Eigenvalues and eigenstates of the total Hamiltonian

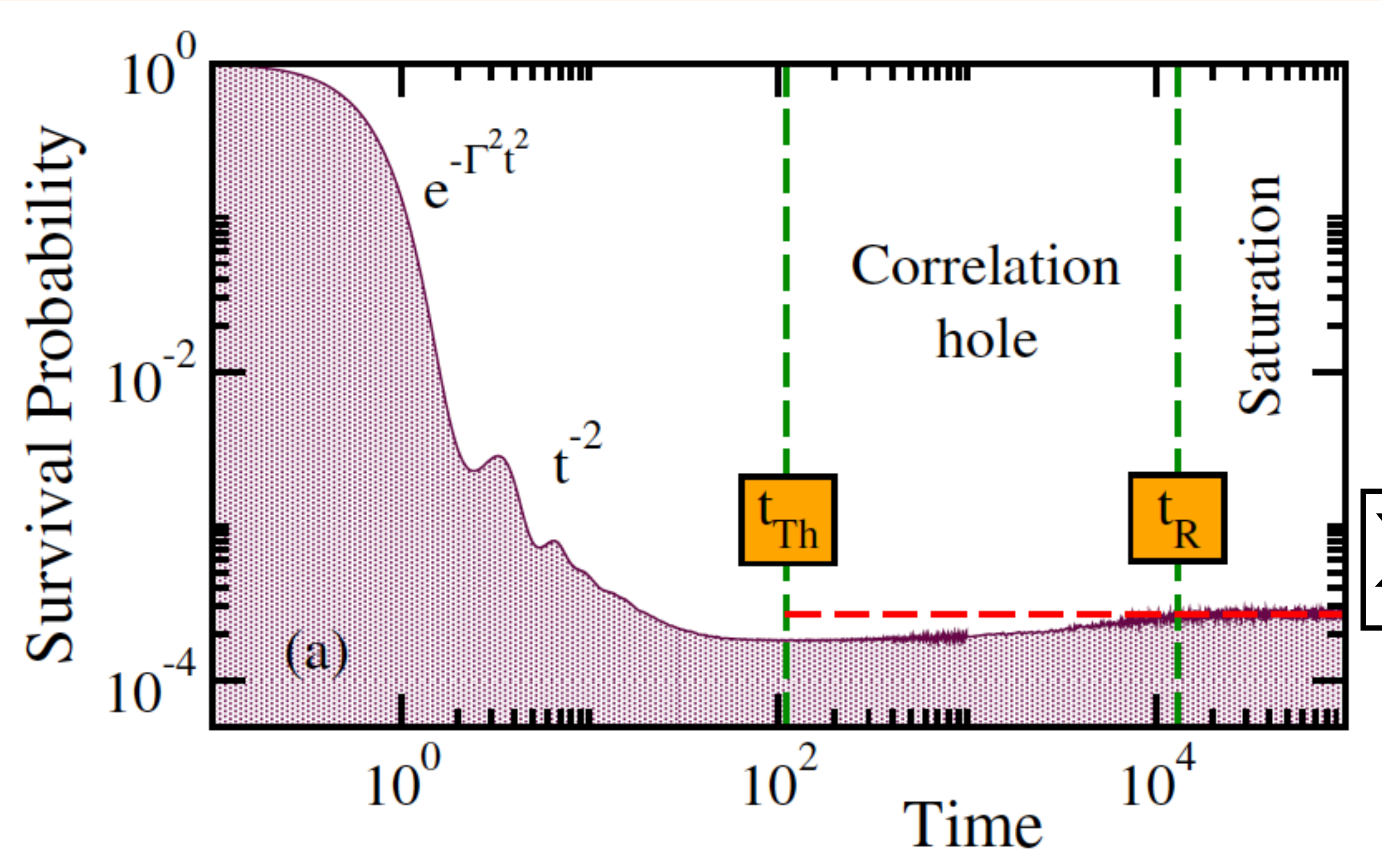
$$= \sum_{\alpha \neq \beta} \left| C_{\alpha}^{ini} \right|^2 \left| C_{\beta}^{ini} \right|^2 e^{-i(E_{\alpha} - E_{\beta})t} + \sum_{\alpha} \left| C_{\alpha}^{ini} \right|^4$$

Infinite time average

$$C_{k_0}^{\alpha} = \langle \alpha | \Psi(0) \rangle$$

Survival Probability

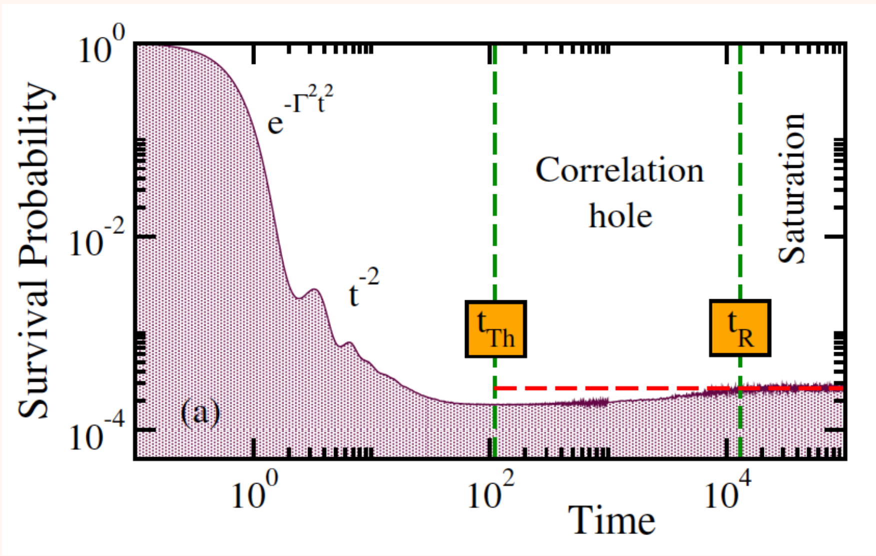
Chaotic h=0.5J



$$\overline{SP} = \sum_\alpha \left| C_\alpha^{ini} \right|^4 \propto \frac{1}{D}$$

Survival Probability

Chaotic h=0.5J



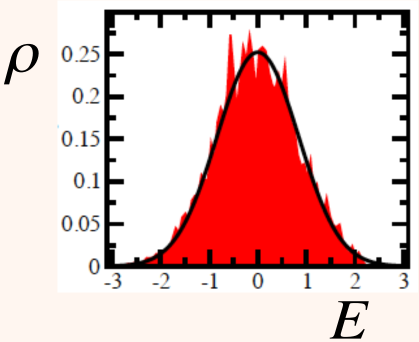
$$\overline{SP}$$

$$= \sum_{\alpha \neq \beta} \left| C_\alpha^{ini} \right|^2 \left| C_\beta^{ini} \right|^2 e^{-i(E_\alpha - E_\beta)t} + \sum_\alpha \left| C_\alpha^{ini} \right|^4$$

shape correlations within

$$\left| \int_{Elow}^{Eup} \rho(E) e^{-iEt} \, dE \right|^2$$

Γ Width of the distribution
Decay rate



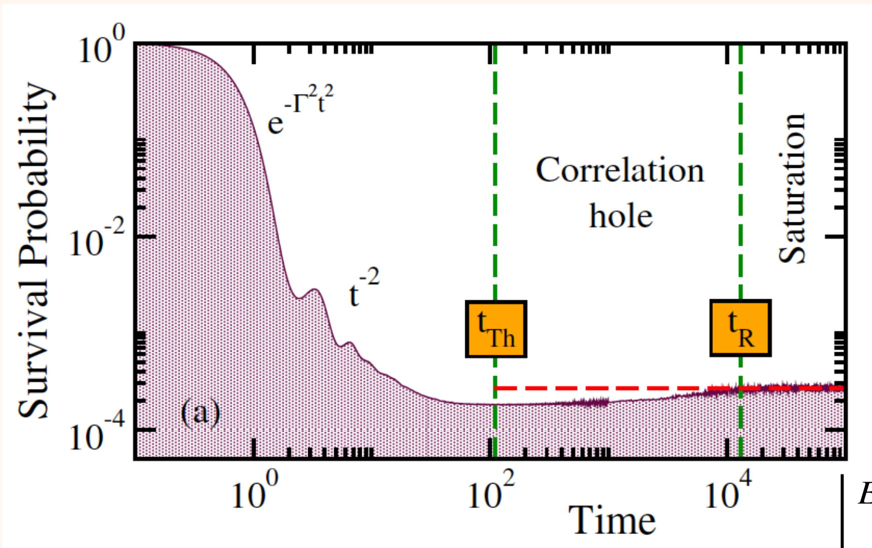
Comment: survival collapse

Survival Probability

Chaotic **h=0.5J**

$$\overline{SP}$$

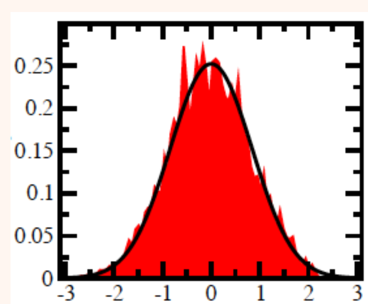
$$= \sum_{\alpha\neq\beta} \left|C_\alpha^{ini}\right|^2 \left|C_\beta^{ini}\right|^2 e^{-i(E_\alpha - E_\beta)t} + \sum_\alpha \left|C_\alpha^{ini}\right|^4$$



shape correlations within

$$\left| \int_{Elow}^{Eup} \rho(E) e^{-iEt}\, dE \right|^2 = \frac{e^{-\Gamma^2 t^2}}{4\mathcal{N}^2} \left| \left[erf\left(\frac{E_0 - E_{low} + i\Gamma^2 t}{\sqrt{2}\Gamma} \right) - erf\left(\frac{E_0 - E_{up} + i\Gamma^2 t}{\sqrt{2}\Gamma} \right) \right] \right|^2$$

Γ Width of the distribution
Decay rate



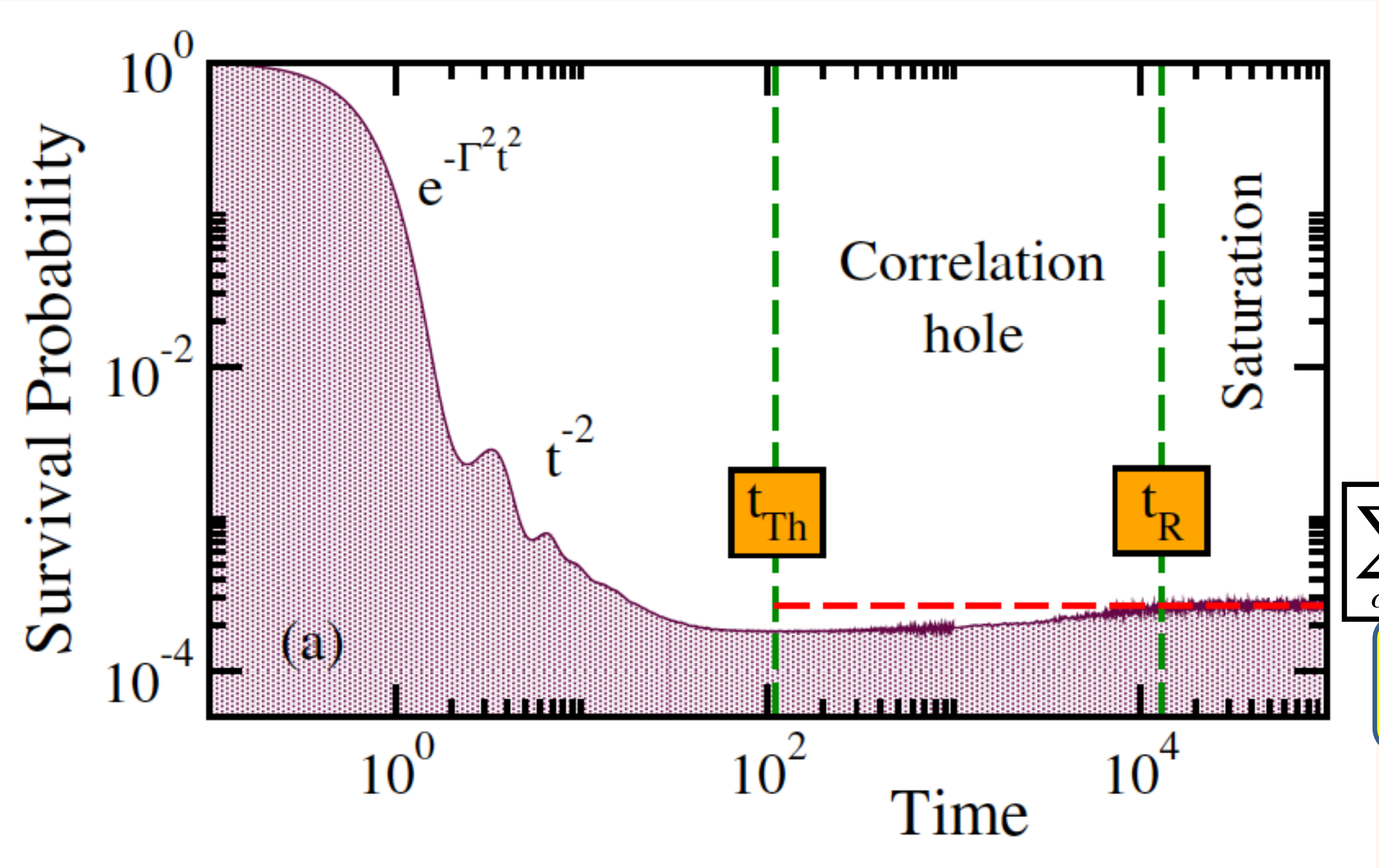
$$\frac{e^{-\Gamma^2 t^2}}{(\Gamma t)^2}$$

Comment: survival collapse

Lea F. Santos, Yeshiva University

Survival Probability

Chaotic h=0.5J

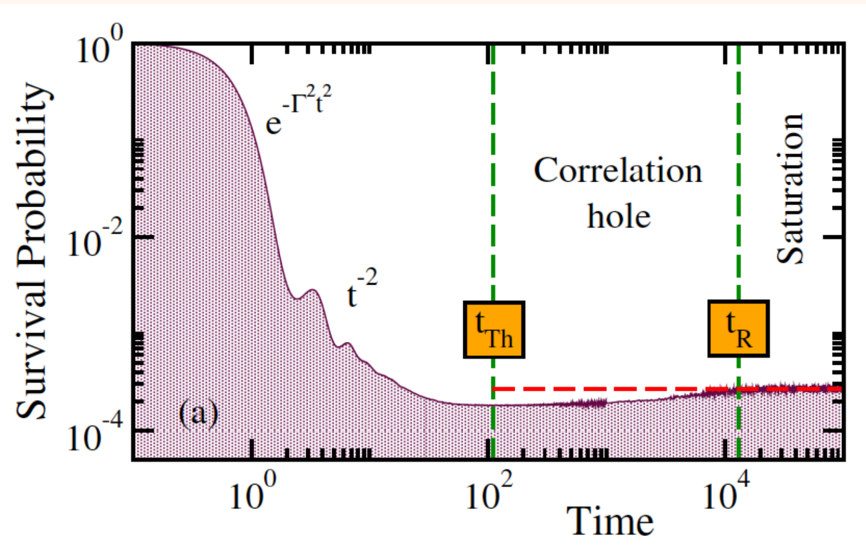


$$\overline{SP} = \sum_\alpha \left| C_\alpha^{ini} \right|^4 \propto \frac{1}{D}$$

Survival Probability

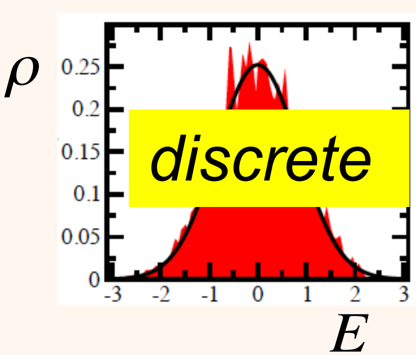
Chaotic h=0.5J

$$\overline{SP} = \sum_{\alpha\neq\beta} \left|C_\alpha^{ini}\right|^2 \left|C_\beta^{ini}\right|^2 e^{-i(E_\alpha - E_\beta)t} + \sum_\alpha \left|C_\alpha^{ini}\right|^4$$



shape

correlations within

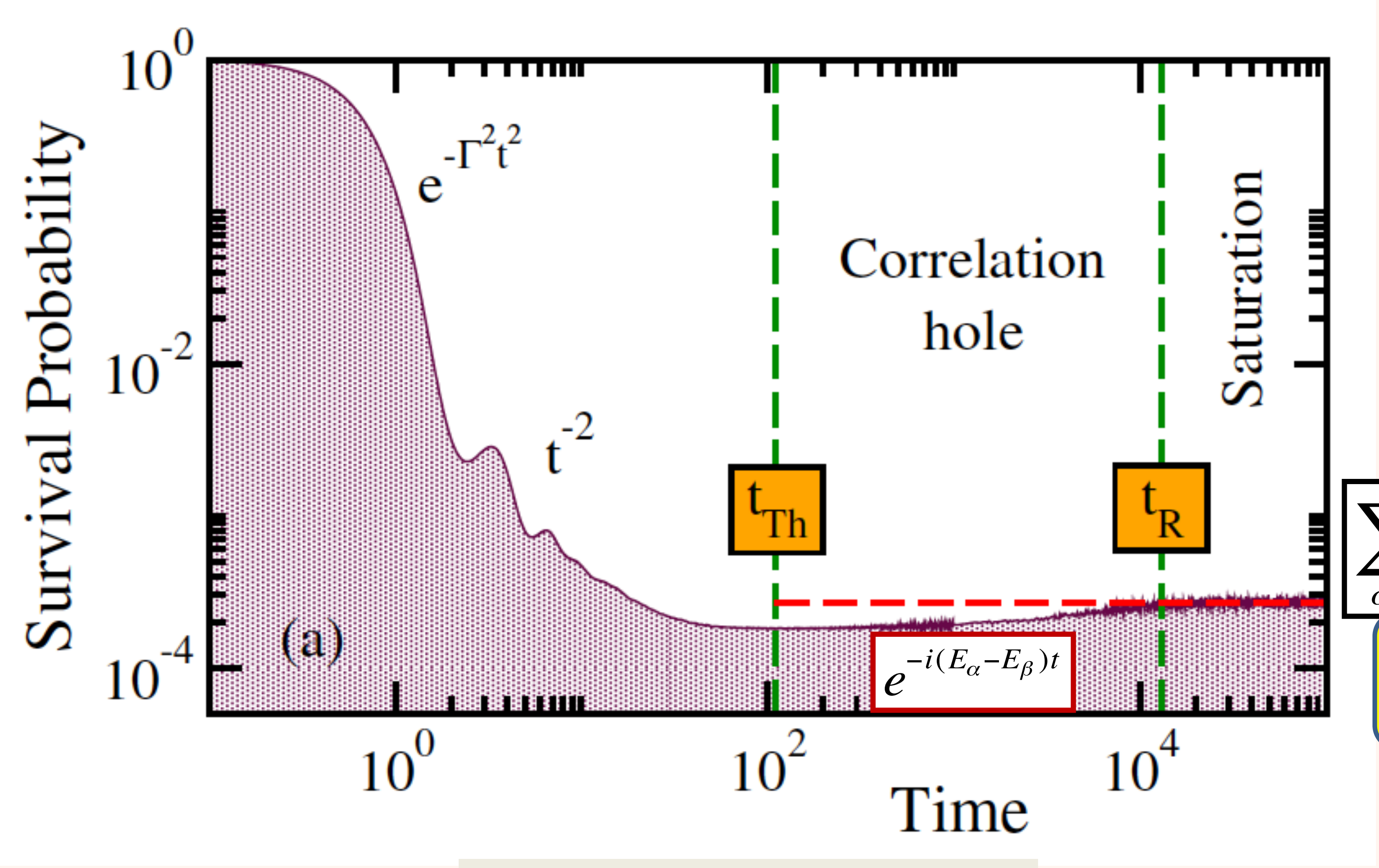


$$\int \left(\int \delta\left(E - (E_1 - E_2)\right) T_2(E_1, E_2)\, dE_1 dE_2 \right) e^{-iEt} dE \Rightarrow b_2\left(\frac{\Gamma t}{2D}\right)$$

where $b_2(\bar{t}) = [1 - 2\bar{t} + \bar{t}\ln(1 + 2\bar{t})]\Theta(1 - \bar{t}) + \{-1 + \bar{t}\ln[(2\bar{t} + 1)/(2\bar{t} - 1)]\}\Theta(\bar{t} - 1)$ and Θ is the Heaviside step function.

Survival Probability

Chaotic h=0.5J



$$\overline{SP} = \boxed{\sum_\alpha \left|C_\alpha^{ini}\right|^4} \propto \frac{1}{D}$$

Why Thouless time?

Spread in the many-body space

Borgonovi, Izrailev & LFS,
arXiv: 1802.08265

$$|\Psi(0)\rangle = |\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\rangle$$

$$|\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\rangle \qquad |\uparrow\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\rangle$$

$$|\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\rangle$$

dimension of Hilbert space

$$D = \frac{L!}{(L/2)!^2}$$

L: number of sites

$$V = \sum_{n=1}^{L} \frac{J}{4} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z)$$

Survival Probability

Chaotic h=0.5J

$$SP(t) = \frac{1-\overline{SP}}{D-1}\left[D\frac{e^{-\Gamma^2 t^2}}{(\Gamma t)^2} g(t) - b_2\left(\frac{\Gamma t}{2D}\right)\right] + \overline{SP}$$

$$g(t) = [(\Gamma t)^2 + A(e^{\Gamma^2 t^2} - 1)]/(1 + A)$$

Survival Probability

Chaotic h=0.5J

$$SP(t) = \frac{1-\overline{SP}}{D-1}\left[D\frac{e^{-\Gamma^2 t^2}}{(\Gamma t)^2}g(t) - b_2\left(\frac{\Gamma t}{2D}\right)\right] + \overline{SP}$$

Direct analogy with the analytical result for GOE full random matrices

$$SP(t) = \frac{1-\overline{SP}}{D-1}\left[D\frac{\mathcal{J}_1^2(2\Gamma t)}{(\Gamma t)^2} - b_2\left(\frac{\Gamma t}{2D}\right)\right] + \overline{SP}$$

D: dimension of Hilbert space

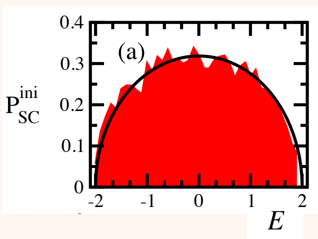
$$D = \frac{L!}{(L/2)!^2}$$

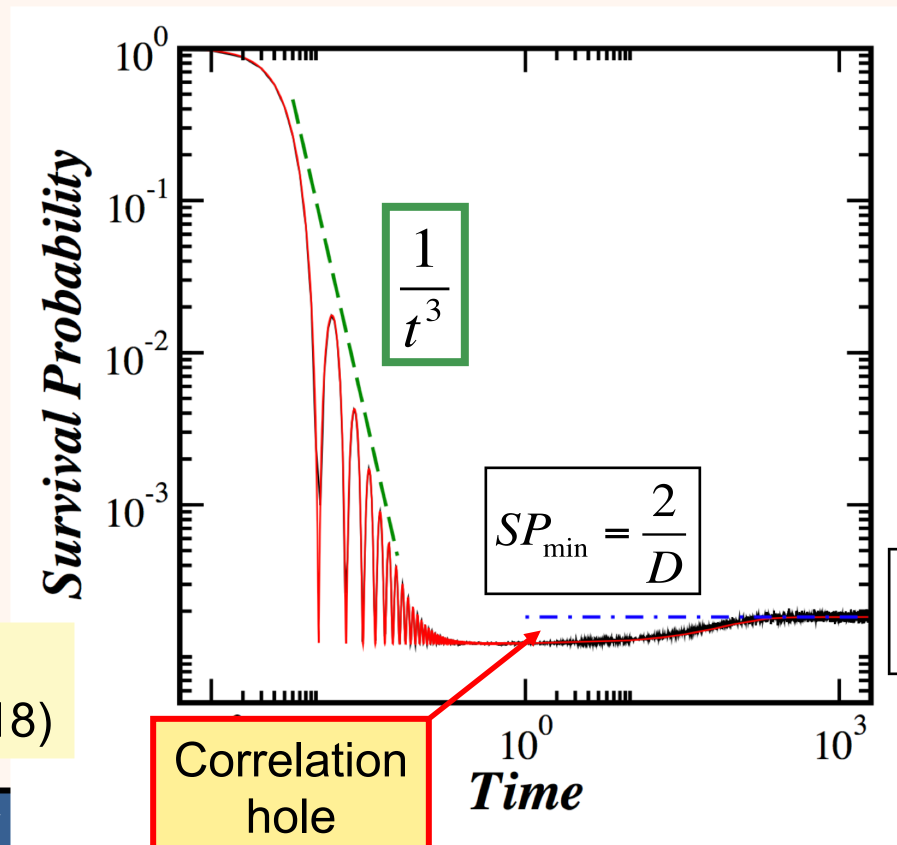
L: number of sites

Survival Probability

GOE Full Random Matrix

$$SP(t) = \frac{1-\overline{SP}}{D-1}\left[D\,\frac{\mathcal{J}_1^2(2\Gamma t)}{(\Gamma t)^2} - b_2\left(\frac{\Gamma t}{2D}\right)\right] + \overline{SP}$$





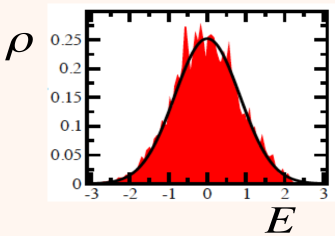
$\overline{SP} = \frac{3}{D}$

Torres et al
PRB **97(R)**, 060303 (2018)

Survival Probability

Chaotic h=0.5J

$$SP(t) = \frac{1-\overline{SP}}{D-1}\left[D\frac{e^{-\Gamma^2 t^2}}{(\Gamma t)^2}g(t) - b_2\left(\frac{\Gamma t}{2D}\right)\right] + \overline{SP}$$



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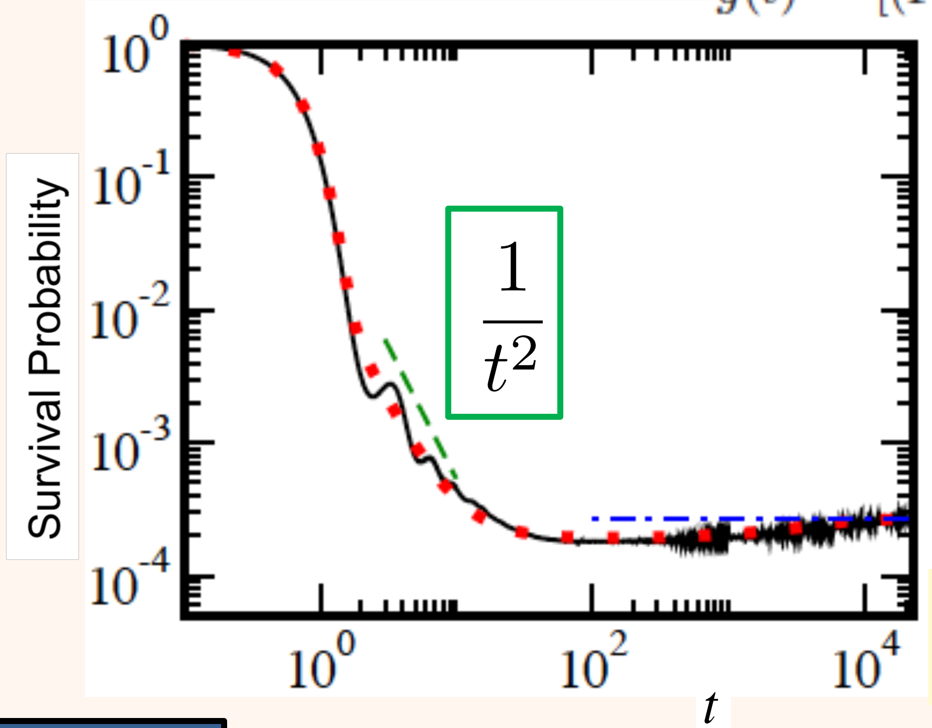


dimension of Hilbert space

$$D = \frac{L!}{(L/2)!^2}$$

L: number of sites

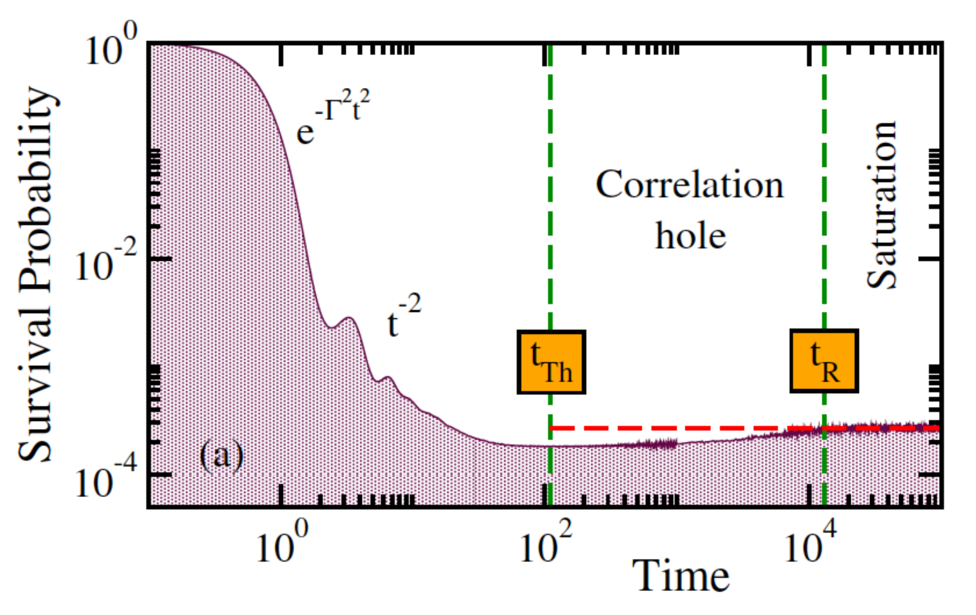
Torres et al
PRB **97(R)**, 060303 (2018)

Thouless Time

Chaotic h=0.5J

$$SP(t) = \frac{1-\overline{SP}}{D-1}\left[D\frac{e^{-\Gamma^2 t^2}}{(\Gamma t)^2} g(t) - b_2\left(\frac{\Gamma t}{2D}\right)\right] + \overline{SP}$$

$$\underbrace{\frac{D}{(\Gamma t)^2} \quad (t \to \infty) \qquad 1 - \frac{\Gamma t}{D} \quad (t \to 0)}$$



$$D = \frac{L!}{(L/2)!^2}$$

$$\boxed{t_{Th} \propto \frac{D^{2/3}}{\Gamma}}$$

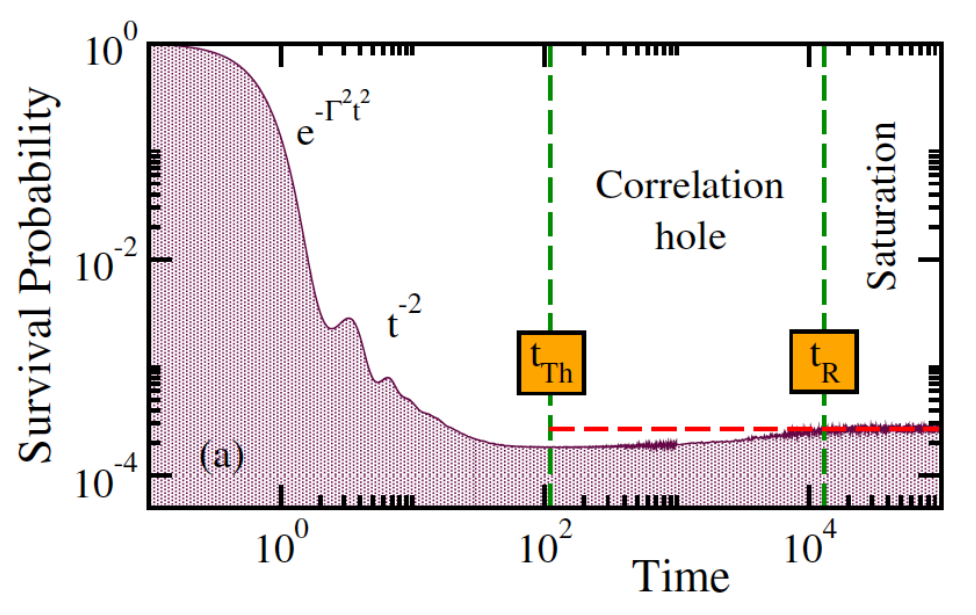
Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577

D: dimension of Hilbert space
L: number of sites

Relaxation Time

Chaotic h=0.5J

$$SP(t) = \frac{1-\overline{SP}}{D-1}\left[D\frac{e^{-\Gamma^2 t^2}}{(\Gamma t)^2} g(t) - b_2\left(\frac{\Gamma t}{2D}\right)\right] + \overline{SP}$$



$t \to \infty$

$$\frac{SP(t)-\overline{SP}}{\overline{SP}} \sim \delta \qquad \frac{D^2}{3(\Gamma t)^2}$$

Inverse of mean level spacing

$$\boxed{t_R \propto \frac{D}{\Gamma\sqrt{\delta}}}$$

Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577

Thouless and Relaxation Time

Realistic CHAOTIC spin-1/2 model

$$t^{-2} \Rightarrow \boxed{t_{Th} \propto \frac{D^{2/3}}{\Gamma}}$$

$$D = \frac{L!}{(L/2)!^2}$$

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Thouless Time

Realistic CHAOTIC spin-1/2 model

$$t^{-2} \Rightarrow \boxed{t_{Th} \propto \frac{D^{2/3}}{\Gamma}}$$

GOE full random matrix

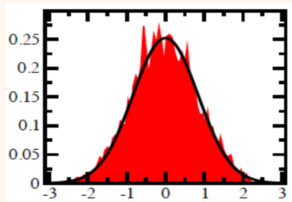
$$t^{-3} \Rightarrow \boxed{t_{Th} = \left(\frac{3}{\pi}\right)^{1/4} \frac{\sqrt{D}}{\Gamma}}$$

Schiulaz, Torres-Herrera & LFS,
arXiv: 1807.07577

Thouless Time

Realistic CHAOTIC spin-1/2 model

$$\boxed{t_{Th} \propto \frac{D^{2/3}}{\Gamma}}$$



$$\Gamma \propto \sqrt{L}$$

$\Downarrow$

$$\boxed{t_{Th} \propto \frac{e^{(2L\ln 2)/3}}{\sqrt{L}}}$$

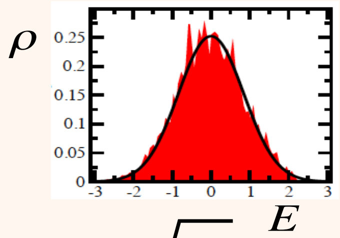
GOE full random matrix

$$\boxed{t_{Th} = \left(\frac{3}{\pi}\right)^{1/4} \frac{\sqrt{D}}{\Gamma}}$$

Thouless Time

Realistic CHAOTIC spin-1/2 model

$$\boxed{t_{Th} \propto \frac{D^{2/3}}{\Gamma}}$$



$$\Downarrow \qquad \Gamma \propto \sqrt{L}$$

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GOE full random matrix

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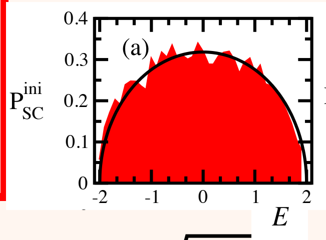

$$\Downarrow \qquad \Gamma \propto \sqrt{D}$$

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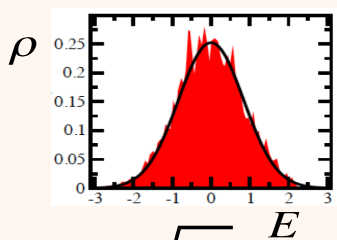
Thouless Time

Realistic CHAOTIC spin-1/2 model

$$\boxed{t_{Th} \propto \frac{D^{2/3}}{\Gamma}}$$



$$\Gamma \propto \sqrt{L}$$

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GOE full random matrix

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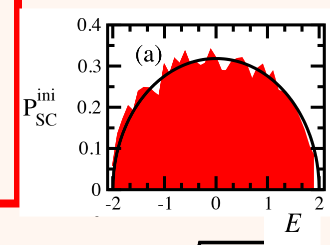


$$\Gamma \propto \sqrt{D}$$

$$\boxed{t_{Th} = \left(\frac{3}{\pi}\right)^{1/4}}$$

Relaxation time diverges for both models

$$\boxed{t_R \propto \frac{D}{\Gamma\sqrt{\delta}}}$$

Inverse of the mean level spacing

Realistic spin-1/2 model

Chaotic h=0.5J

Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577

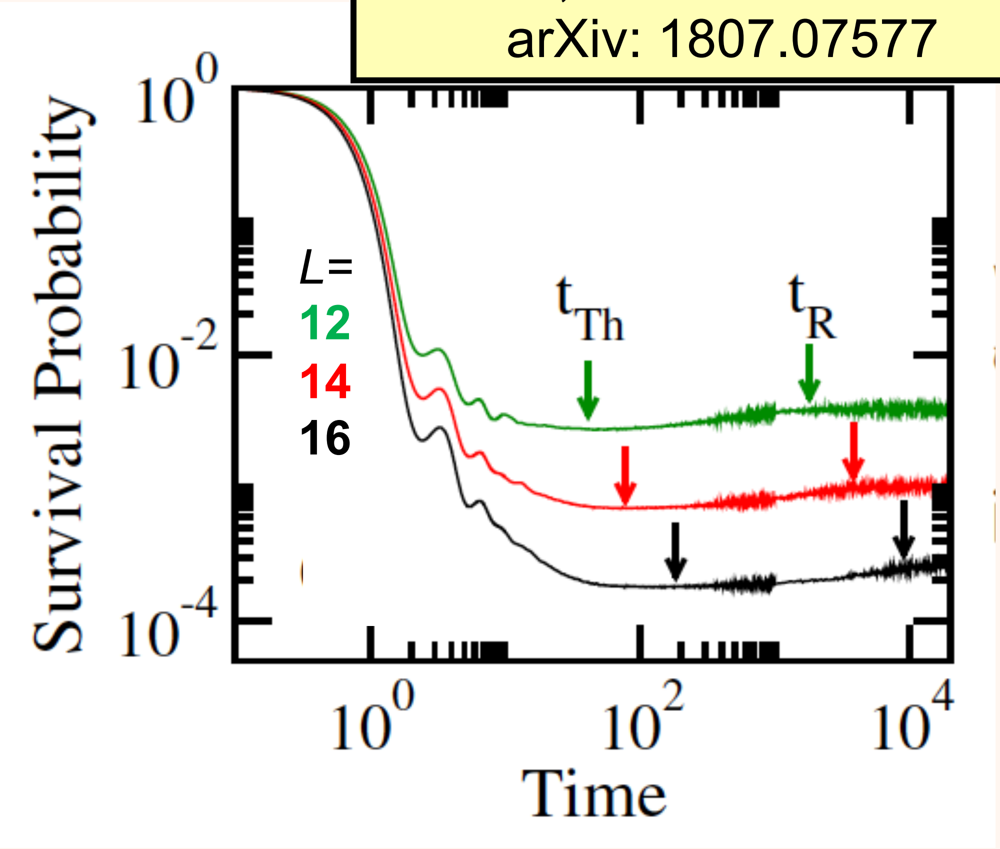
Thouless time

$$\boxed{t_{Th} \propto \frac{D^{2/3}}{\Gamma}}$$

$$\Gamma \propto \sqrt{L}$$

Relaxation time

$$\boxed{t_R \propto \frac{D}{\Gamma\sqrt{\delta}}}$$



$$\boxed{t_R \propto D^{1/3} t_{Th}}$$

hole stretches with L

Realistic spin-1/2 model

Chaotic h=0.5J

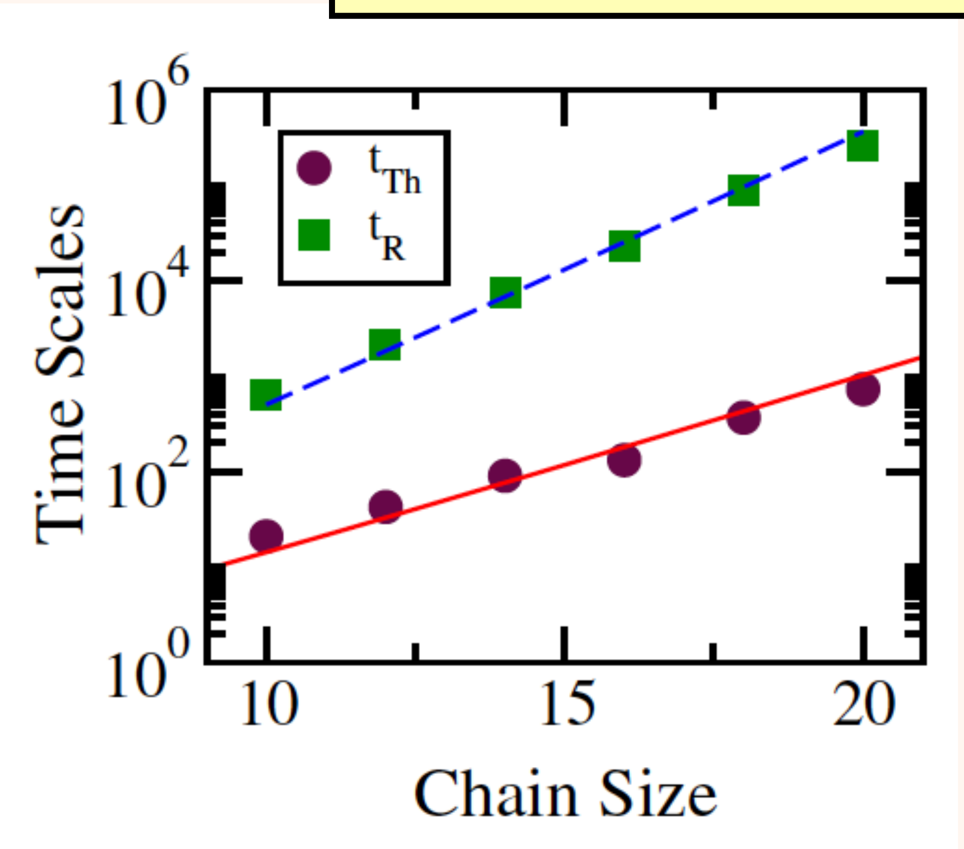
Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577

Thouless time

$$\boxed{t_{Th} \propto \frac{D^{2/3}}{\Gamma}} \qquad \Gamma \propto \sqrt{L}$$

Relaxation time

$$\boxed{t_R \propto \frac{D}{\Gamma\sqrt{\delta}}}$$



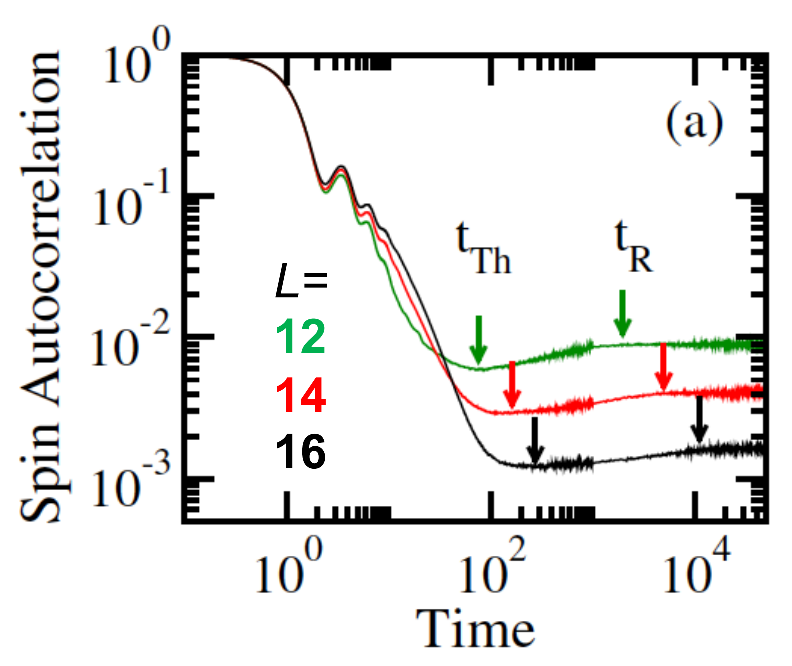
Divergence: consistent with the correspondence principle

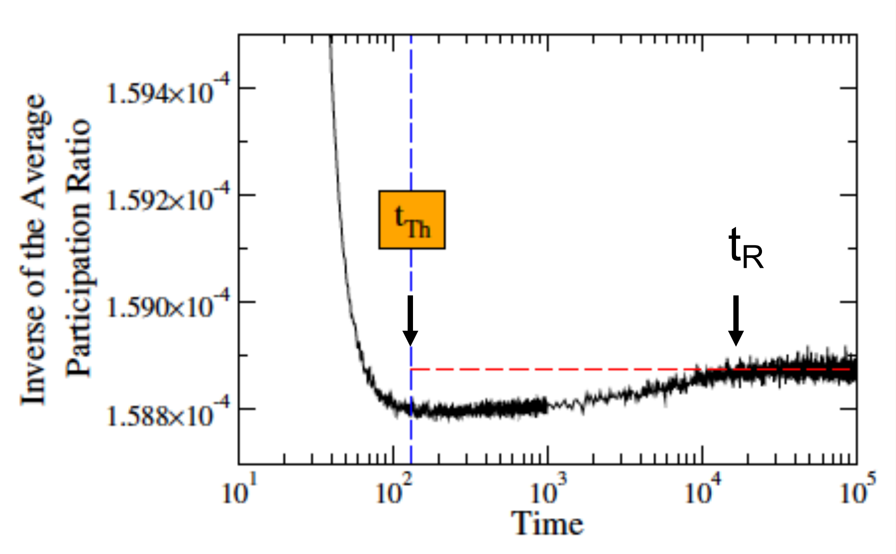
Realistic spin-1/2 model

Chaotic h=0.5J

$$I(t) = \frac{1}{L}\sum_{k=1}^{L}\langle\Psi(0)|\sigma_k^z e^{iHt}\sigma_k^z e^{-iHt}|\Psi(0)\rangle$$

$$IPR(t) = \frac{1}{L}\sum_{k=1}^{L}|\langle k|e^{-iHt}|\Psi(0)\rangle|^4$$



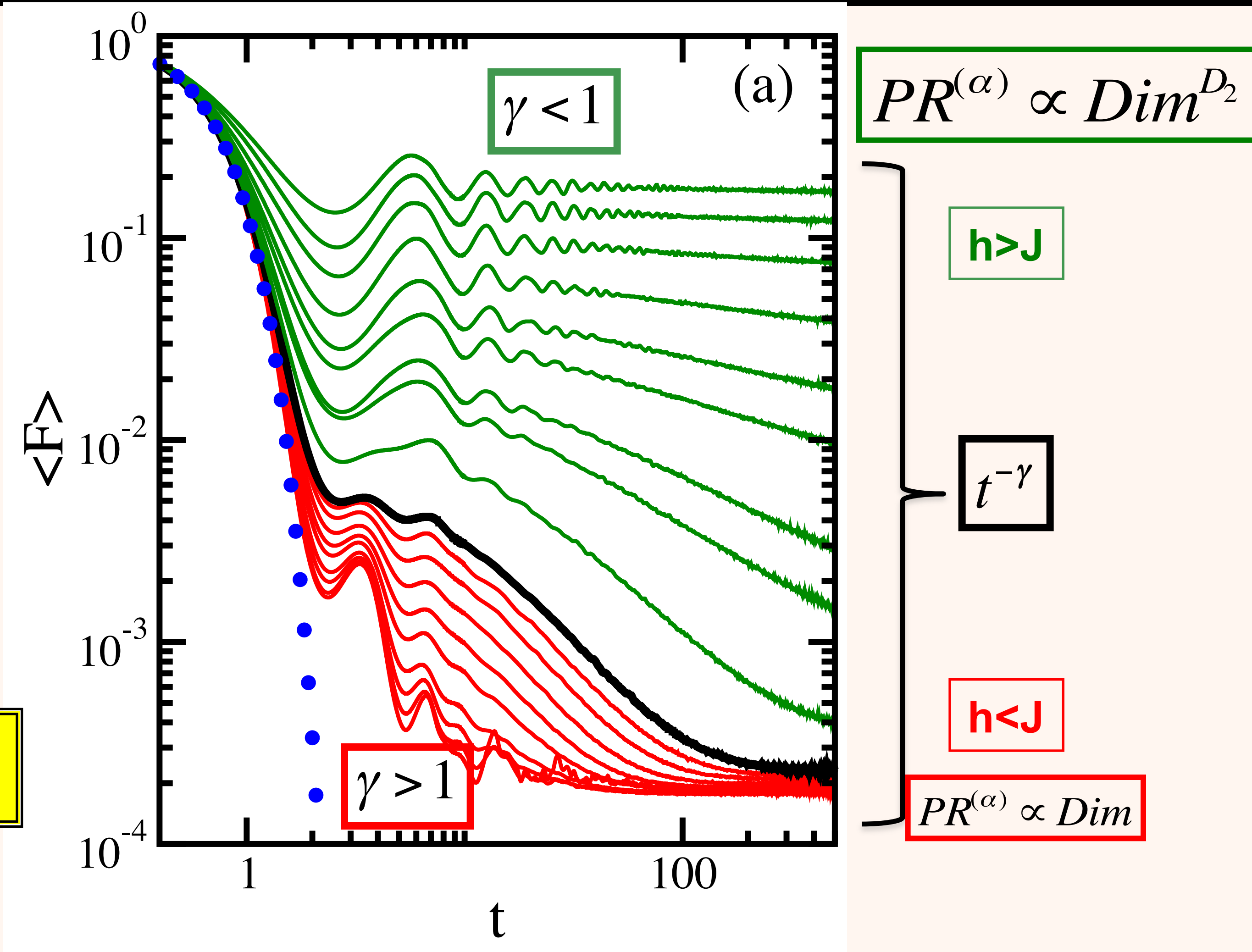


Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577

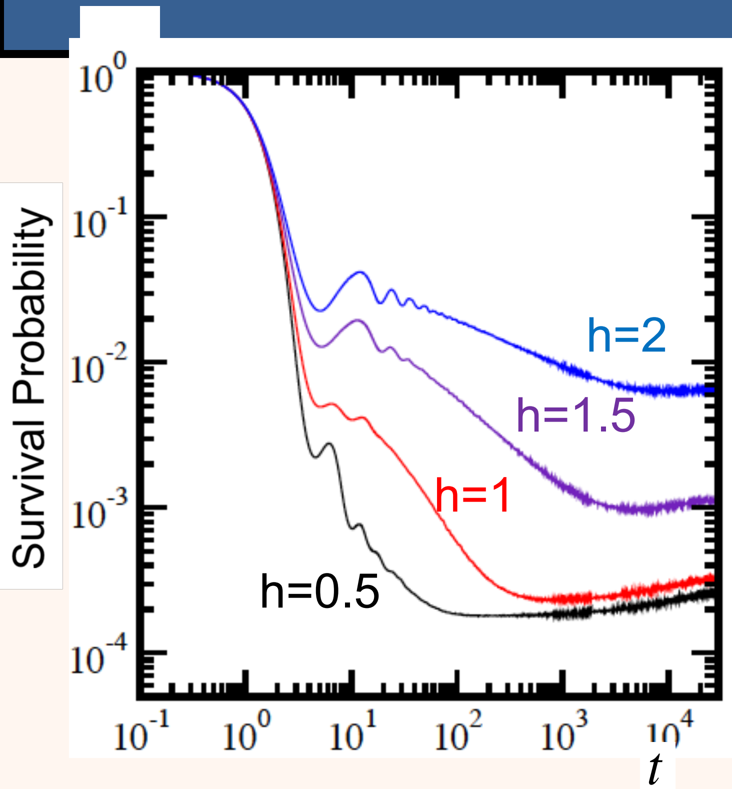
Power-law exponent: energy bounds

$$\boxed{t^{-\gamma}}$$

Torres & LFS
PRB **92**, 01420 (2015)



Thouless Time and Disorder Strength

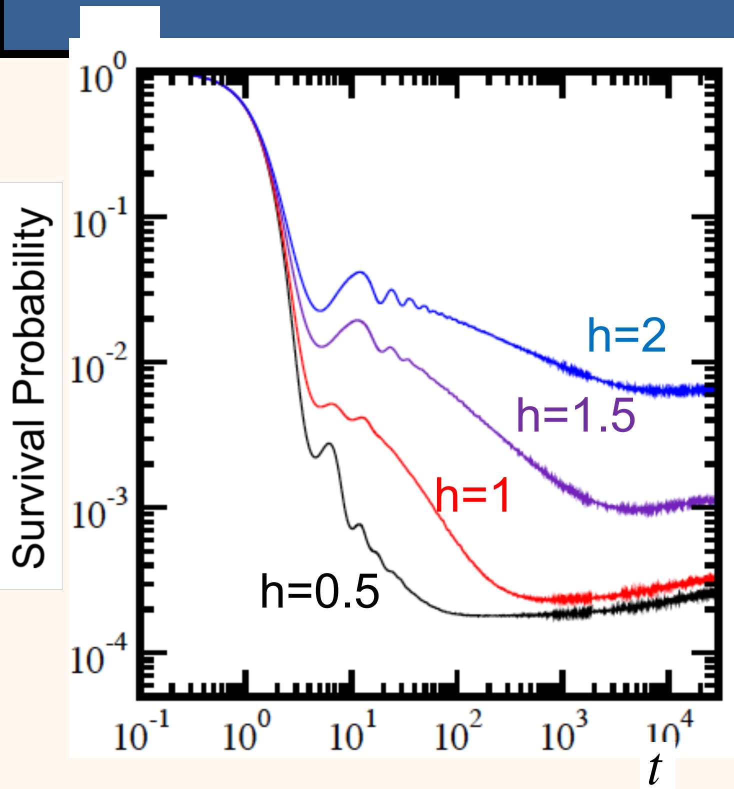


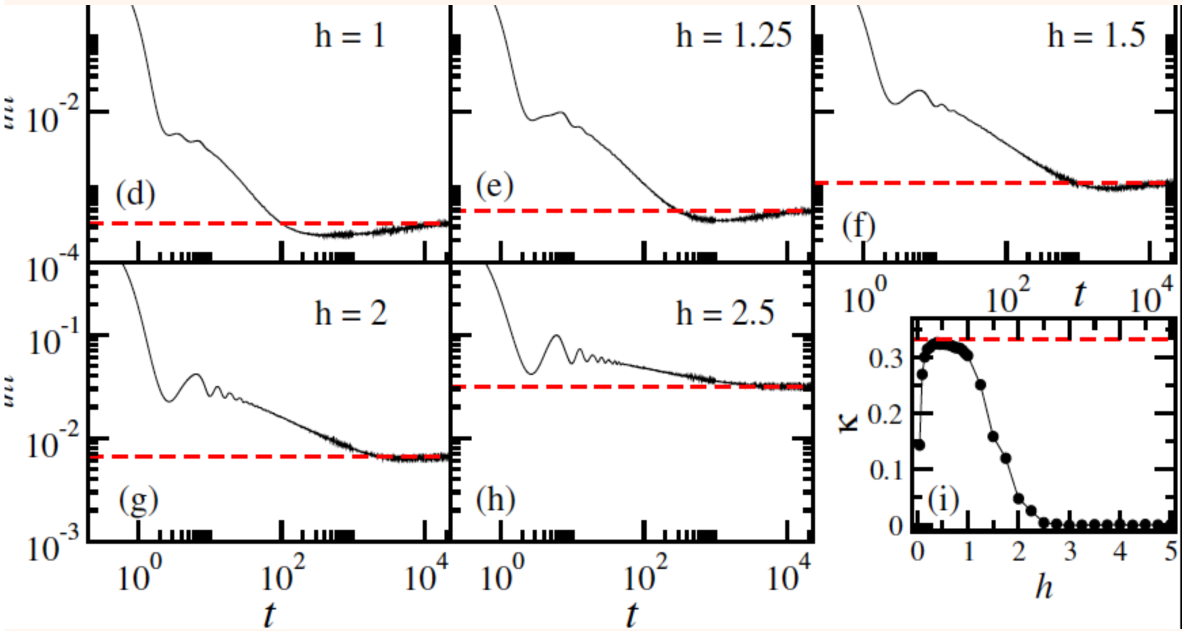
Time for the minimum of the hole (T_{Th}) gets postponed as the disorder (h) increases.

The hole becomes narrower and shallower.

Fixed system size L=16

Thouless Time and Disorder Strength





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Thouless
Dimensionless Conductance

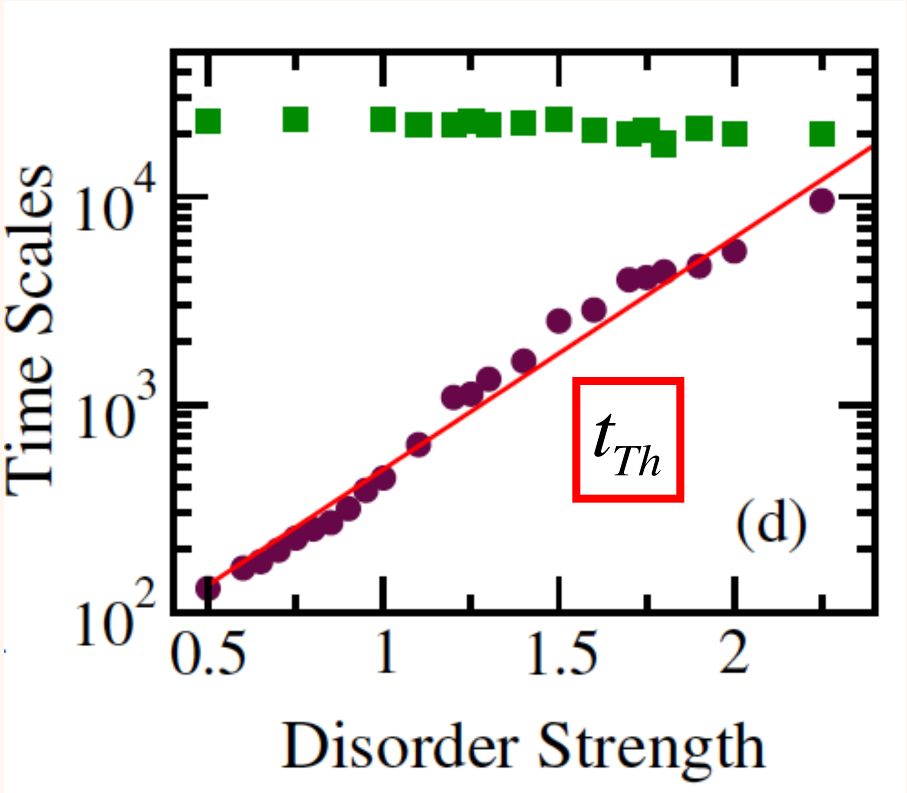
Thouless dimensionless conductance $\dfrac{t_R}{t_{Th}}$

Delocalized (chaotic)

$$\frac{t_R}{t_{Th}} \propto e^{(L\ln 2)/3}$$

Toward localization

$$\frac{t_R}{t_{Th}} \to 1$$

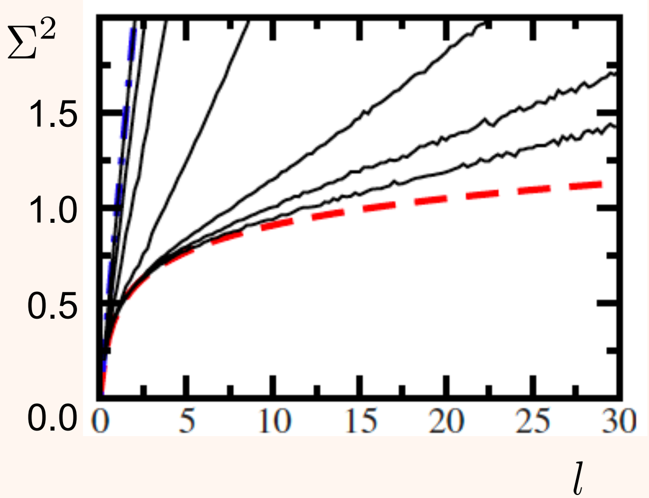


$$t_R \propto \frac{D}{\Gamma\sqrt{\delta}}$$

Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577

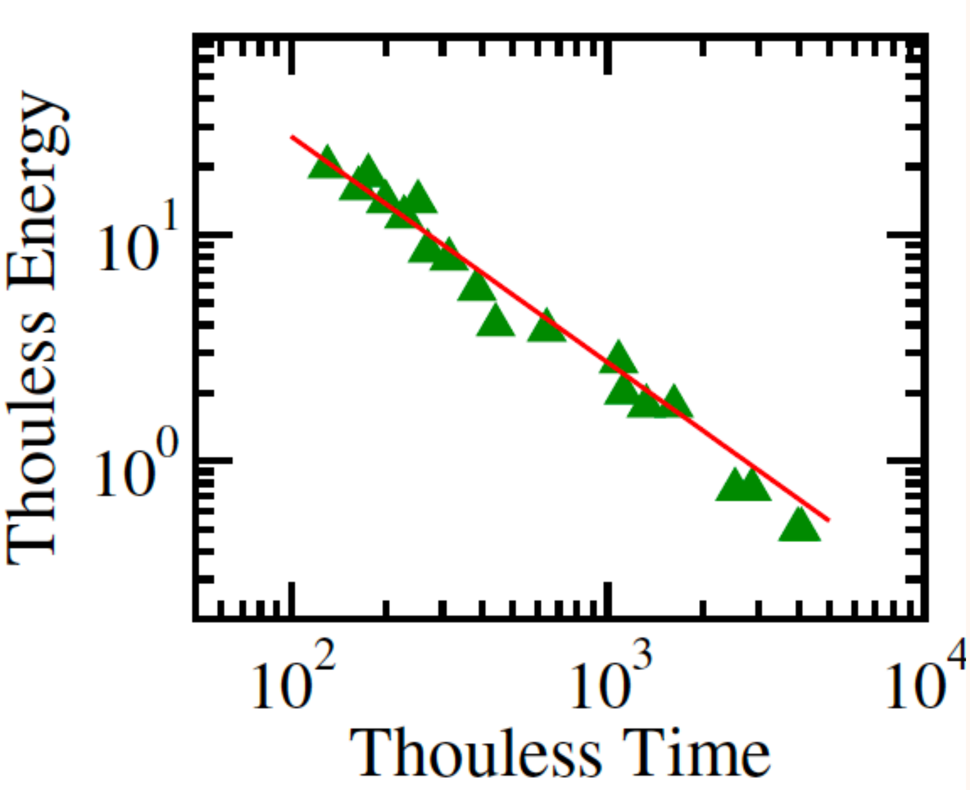
Why Thouless?
Relation with the Thouless Energy

Level number variance



The level statistics of disordered systems follows that of full random matrices up to the Thouless energy

$$E_{Th} \propto \frac{1}{t_{Th}}$$



Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577

Summary

- **Time scales:**

Thouless time – minimum of the correlation hole. (signature of **level repulsion**)

$$\boxed{t_{Th} \propto \frac{D^{2/3}}{\Gamma}}$$

$$\boxed{t_R \propto D^{1/3} t_{Th}}$$

Relaxation time – end of the hole, fluctuations

$$\boxed{t_R \propto \frac{D}{\Gamma\sqrt{\delta}}}$$

- **Analytical expressions** from full random matrices serve as bounds and references for the analysis of realistic models.

- **Correlation hole** is an indicator of the delocalized-localized transition.

Toward localization

$$\boxed{\frac{t_R}{t_{Th}} \to 1}$$

Schiulaz, Torres-Herrera & LFS, arXiv: 1807.07577