LOCALISATION AND TRANSPORT IN TWO-DIMENSIONAL RANDOM MODELS WITH SEPARABLE HAMILTONIANS

G. Corona-Patricio[1], U. Kuhl[2],
F. Mortessagne[2], L. Tessieri[1], P. Vignolo[2]

[1] *Instituto de Física y Matemáticas*
Universidad Michoacana de San Nicolás de Hidalgo
Morelia, Mexico

[2] *Institut de Physique de Nice*
Université de Nice
Nice, France

International Workshop:
"Disordered Systems: From Localisation to Thermalisation and Topology"
PCS-IBS
Daejeon, South Korea
3rd-7th September 2018

OUTLINE OF THE WORK

- Key properties of the 1D aperiodic Kronig-Penney model.

- Definition of two bidimensional random models with separable Hamiltonians.

- Connections between 2D separable models and 1D K-P model. Structure of the electronic states.

1D APERIODIC KRONIG-PENNEY MODEL

Schrödinger equation

$$\psi''(x) + \left[E - \sum_{n=-\infty}^{\infty} (U + u_n)\, \delta\,(x - n\alpha - \alpha_n) \right] \psi(x) = 0$$

in units such that $\hbar^2/2m = 1$

The K-P model can be analysed in terms of a *parametric oscillator*

$$\ddot{x} + \left[q^2 - \sum_{n=-\infty}^{\infty} (U + u_n)\, \delta\,(t - n\alpha - \alpha_n) \right] x = 0$$

Localisation of the electronic states $\Longleftrightarrow$ Instability of the orbits of the oscillator

A dynamical analysis of the parametric oscillator makes possible to derive the localisation length for the Kronig-Penney model *in the weak-disorder case*

HAMILTONIAN MAP APPROACH

Parametric oscillator

$$\ddot{x} + \left[q^2 - \sum_{n=-\infty}^{\infty} (U + u_n)\, \delta\,(t - n\alpha - \alpha_n) \right] x = 0$$

Integrating the dynamical equation of the parametric oscillator over the time interval between two kicks gives the map

$$\begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = \mathbf{T}_n \begin{pmatrix} x_n \\ p_n \end{pmatrix}$$

with

$$\mathbf{T}_n = \begin{pmatrix} \cos\left[q\,(\alpha + \Delta_n)\right] + (U + u_n)\,\frac{1}{q}\sin\left[q\,(\alpha + \Delta_n)\right] & \frac{1}{q}\sin\left[q\,(\alpha + \Delta_n)\right] \\ -q\sin\left[q\,(\alpha + \Delta_n)\right] + (U + u_n)\cos\left[q\,(\alpha + \Delta_n)\right] & \cos\left[q\,(\alpha + \Delta_n)\right] \end{pmatrix}$$

where

$$\Delta_n = \alpha_{n+1} - \alpha_n$$

Hamiltonian map $\Longleftrightarrow$ Transfer matrix

THE INVERSE LOCALISATION LENGTH

LYAPUNOV EXPONENT

$$\lambda = \frac{1}{8a}\Big[\langle\tilde{u}_n^2\rangle W_1(k\alpha) + \langle\tilde{\Delta}_n^2\rangle W_2(k\alpha) - 2\langle\tilde{u}_n\tilde{\Delta}_n\rangle \cos(k\alpha)\, W_3(k\alpha)\Big]$$

POWER SPECTRA

$$W_1(k\alpha) = 1 + 2\sum_{l=1}^{\infty} \frac{\langle u_n u_{n+l}\rangle}{\langle u_n^2\rangle}\cos(2k\alpha l)$$

$$W_2(k\alpha) = 1 + 2\sum_{l=1}^{\infty} \frac{\langle \Delta_n \Delta_{n+l}\rangle}{\langle \Delta_n^2\rangle}\cos(2k\alpha l)$$

$$W_3(k\alpha) = 1 + 2\sum_{l=1}^{\infty} \frac{\langle u_n \Delta_{n+l}\rangle}{\langle u_n \Delta_n\rangle}\cos(2k\alpha l)$$

RESCALED DISORDER

$$\tilde{u}_n = \frac{\sin(q\alpha)}{q\sin(k\alpha)}u_n \qquad \tilde{\Delta}_n = \frac{U}{\sin(k\alpha)}\Delta_n$$

with

$$\Delta_n = \alpha_{n+1} - \alpha_n$$

2D MODEL WITH RANDOM ANGULAR POTENTIAL

Quantum particle in a "ring cake tin"

Cavity bounded by two truncated coaxial cylinders with radii R_1 and R_2; top and bottom horizontal surfaces separated by distance d_{max}.

Separable potential

$$U(r, \theta, z) = U_t(r, \theta) U_l(z)$$

with

$$U_t(r, \theta) = \begin{cases} \infty & \text{if} \quad r < R_1 \\ \dfrac{1}{r^2} V(\theta) & \text{if} \quad R_1 \leq r \leq R_2 \\ \infty & \text{if} \quad R_2 < r \end{cases}$$

and

$$U_l(z) = \begin{cases} \infty & \text{if} \quad z < 0 \\ \text{const.} & \text{if} \quad 0 \leq z \leq d_{\mathrm{max}} \\ \infty & \text{if} \quad d_{\mathrm{max}} < z \end{cases}$$

RANDOM POTENTIAL

$$V(\theta) = \sum_{n=1}^{N} (U + u_n)\delta\left(\theta - \frac{2\pi}{N}n - \alpha_n\right)$$

RANDOM VARIABLES

$u_n \quad \rightarrow$ compositional disorder

$\Delta_n = \alpha_{n+1} - \alpha_n \quad \rightarrow$ structural disorder

STATISTICAL PROPERTIES

$$\langle u_n \rangle = 0 \qquad \langle \Delta_n \rangle = 0$$

$$\left.\begin{array}{c} \langle u_n^2 \rangle \ll U^2 \\ \langle \Delta_n^2 \rangle q^2 \ll 1 \\ \langle \Delta_n^2 \rangle U \ll 1 \end{array}\right\} \text{WEAK DISORDER}$$

$$\chi_1(k) = \frac{\langle u_n u_{n+k} \rangle}{\langle u_n^2 \rangle}$$

$$\chi_2(k) = \frac{\langle \Delta_n \Delta_{n+k} \rangle}{\langle \Delta_n^2 \rangle}$$

$$\chi_3(k) = \frac{\langle u_n \Delta_{n+k} \rangle}{\langle u_n \Delta_n \rangle}$$

BINARY CORRELATORS

SCHRÖDINGER EQUATION

Separation of the z variable

$$\psi(r,\theta,z) = \psi_t(r,\theta)\,\mathrm{sin}\,(k_z z)$$

Quantum number k_z

$$\psi(r,\theta,z=0) = \psi(r,\theta,z=d_{\mathrm{max}}) = 0$$
$$\Downarrow$$
$$k_z = \frac{\pi m}{d_{\mathrm{max}}} \quad \text{with} \quad m = 1,2,3,\ldots$$

Schrödinger equation $(\hbar^2/2m = 1)$

$$\left[-\nabla_t^2 + V(\theta)/r^2\right]\psi_t(r,\theta) = \left(E - k_z^2\right)\psi_t(r,\theta)$$

with

$$\nabla_t^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

APPLICATION: MICROWAVE CAVITY (1)

The mathematical identity of the Schrödinger and Helmholtz equations allows one to apply the analysis to the electric field of a TM mode in a microwave cavity

$$\mathbf{E}_{TM}(r,\theta,z) = E_z(r,\theta)\,\mathsf{sin}\left(\frac{m\pi}{d_0}z\right)\hat{\mathbf{z}}$$

Helmholtz equation for the electric field of the m-th TM mode

$$\left[-\nabla_t^2 + \left(\frac{m\pi}{d_0}\right)^2\right] E_z(r,\theta) = \frac{\omega^2}{c^2} E_z(r,\theta)$$

If the depth d_0 of the cavity is not constant, but varies slowly with r, the Helmholtz eq. takes the form

$$\left\{-\nabla_t^2 + \left[\left(\frac{m\pi}{d(r,\theta)}\right)^2 - \left(\frac{m\pi}{d_{\mathsf{max}}}\right)^2\right]\right\} E_z(r,\theta) = \left[\frac{\omega^2}{c^2} - \left(\frac{m\pi}{d_{\mathsf{max}}}\right)^2\right] E_z(r,\theta)$$

APPLICATION: MICROWAVE CAVITY (2)

Identity of Schrödinger and Helmholtz equations requires

$$\frac{V(\theta)}{r^2} = m^2\pi^2 \left[\frac{1}{d^2(r,\theta)} - \frac{1}{d^2_{\text{max}}}\right]$$

$$\Downarrow$$

$$d(r,\theta) = \frac{m\pi d_{\text{max}} r}{\sqrt{d^2_{\text{max}} V(\theta) + (m\pi r)^2}}$$

The desired potential can be obtained by appropriately shaping the distance between the top and bottom plates of the cavity. Insertion of radially directed trapezoidal slabs mimicks the potential

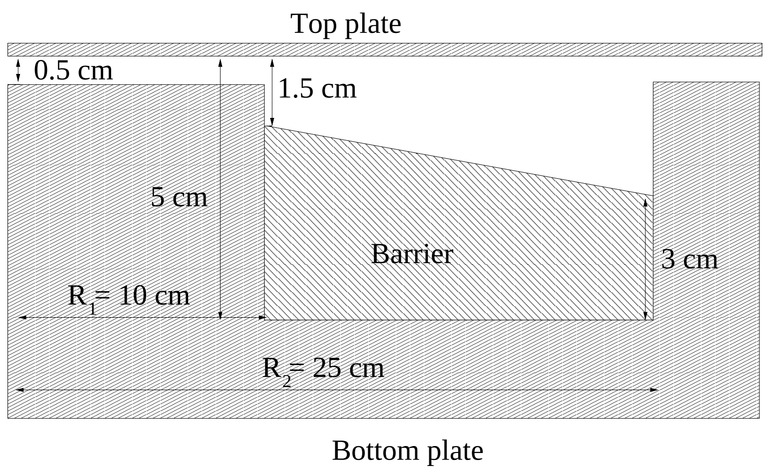
$$V_\varepsilon(\theta) = \sum_{n=1}^{N} U_n \eta_\varepsilon(\theta - \theta_n)$$

with

$$\eta_\varepsilon(\theta) = \begin{cases} \dfrac{1}{2\varepsilon} & \text{if} \quad \theta \in [-\varepsilon, \varepsilon] \\ 0 & \text{if} \quad \theta \notin [-\varepsilon, \varepsilon] \end{cases}$$

APPLICATION: MICROWAVE CAVITY (3)

Schematic section of a microwave cavity (different length scales used along the x- and y-axis)



Actual cavity

VARIABLE SEPARATION

Wavefunction (cylindrical coordinates)

$$\psi(r,\theta,z) = R(r)\Theta(\theta)\sin(k_z z)$$

Quantum number k_z

$$\psi(r,\theta,z=0) = \psi(r,\theta,z=d_{\text{max}}) = 0$$
$$\Downarrow$$
$$k_z = \frac{\pi m}{d_{\text{max}}} \quad \text{with} \quad m = 1,2,3,\ldots$$

Schrödinger equation ($\hbar^2/2m = 1$)

$$-\frac{d^2\Theta}{d\theta^2} + \sum_{n=1}^{N} (U + u_n)\,\delta\left(\theta - \frac{2\pi}{N}n - \alpha_n\right)\Theta = q^2\Theta$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[E - k_z^2 - \frac{q^2}{r^2}\right]R = 0$$

K.-P. MODEL FOR THE ANGULAR VARIABLE $\Theta(\theta)$ (I)

Schrödinger equation

$$-\frac{d^2\Theta}{d\theta^2} + \sum_{n=1}^{N} U_n \delta\left(\theta - \alpha n - \alpha_n\right)\Theta = q^2\Theta$$

with $\alpha = 2\pi/N$ (lattice step)
Periodic boundary condition

$$\Theta(0) = \Theta(2\pi)$$

Hamiltonian map

$$\begin{pmatrix} \Theta_{n+1} \\ \Theta'_{n+1} \end{pmatrix} = \mathbf{T}_n \begin{pmatrix} \Theta_n \\ \Theta'_n \end{pmatrix}$$

with

$$\Theta_n = \Theta((n\alpha + \alpha_n)^-) \qquad \Theta'_n = \Theta'((n\alpha + \alpha_n)^-)$$

$$\mathbf{T}_n = \begin{pmatrix} \cos\left[q\left(\alpha + \Delta_n\right)\right] + \left(U + u_n\right)\frac{1}{q}\sin\left[q\left(\alpha + \Delta_n\right)\right] & \frac{1}{q}\sin\left[q\left(\alpha + \Delta_n\right)\right] \\ -q\sin\left[q\left(\alpha + \Delta_n\right)\right] + \left(U + u_n\right)\cos\left[q\left(\alpha + \Delta_n\right)\right] & \cos\left[q\left(\alpha + \Delta_n\right)\right] \end{pmatrix}$$

where

$$\Delta_n = \alpha_{n+1} - \alpha_n$$

K.-P. MODEL FOR THE ANGULAR VARIABLE (II)

Corresponding tight-binding model

$$\frac{1}{\sin\left[q\left(\alpha+\Delta_n\right)\right]}\Theta_{n+1}+\frac{1}{\sin\left[q\left(\alpha+\Delta_{n-1}\right)\right]}\Theta_{n-1}$$
$$=\left\{\cot\left[q\left(\alpha+\Delta_n\right)\right]+\cot\left[q\left(\alpha+\Delta_{n-1}\right)\right]+\frac{U+u_n}{q}\right\}\Theta_n$$

Tight-binding model with diagonal and off-diagonal disorder

VANISHING DISORDER

Tight-binding model

$$\Theta_{n+1} + \Theta_{n-1} = 2\left[\cos(q\alpha) + \frac{U}{2q}\sin(q\alpha)\right]\Theta_n$$

Eigenstates: Bloch waves

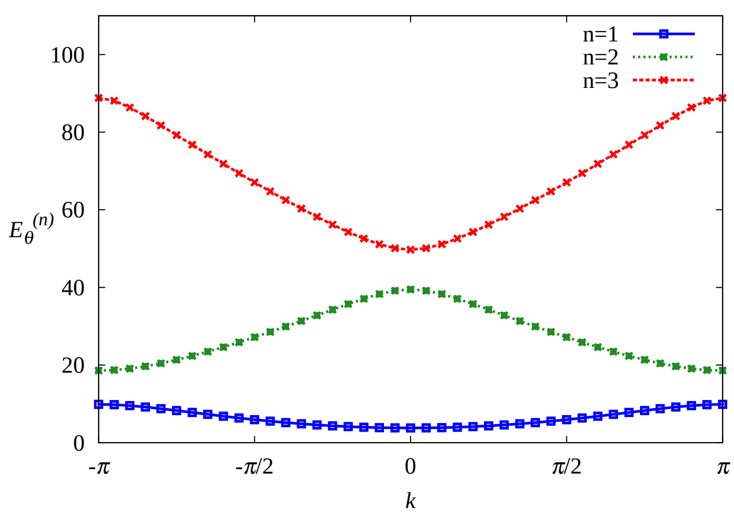
$$\Theta^{(k)}(\theta) = e^{ik\alpha n}\left\{\cos\left[q(\theta-\alpha n)\right] + \frac{\sin\left[q(\theta-\alpha n)\right]}{\sin(q\alpha)}\left[e^{ik\alpha} - \cos(q\alpha)\right]\right\}$$

with $\theta \in [\alpha n, \alpha(n+1)]$.

Bloch vector

$$k = \frac{2\pi}{N}n \quad \text{with} \quad n = 0, 1, \ldots, N-1.$$

Band structure ($U = 5.7$ and $N = 40$)



WEAK DISORDER

Weak disorder

$$\langle u_n^2 \rangle \ll U^2, \quad \langle \Delta_n^2 \rangle E_\theta \ll 1, \quad \text{and} \quad \langle \Delta_n^2 \rangle U \ll 1.$$

Inverse localisation length

$$l_{\text{loc}}^{-1} = \frac{1}{8 \sin^2(k\alpha)} \left[\frac{\sin^2(q\alpha)}{(q\alpha)^2} \langle u_n^2 \rangle W_1(k\alpha) + U^2 \langle \Delta_n^2 \rangle W_2(k\alpha) \right.$$
$$\left. - \quad 2U \frac{\sin(q\alpha)}{q\alpha} \langle u_n \Delta_n \rangle \cos(k) W_3(k\alpha) \right]$$

Power spectra

$$W_1(k\alpha) = 1 + 2 \sum_{l=1}^{\infty} \frac{\langle u_n u_{n+l} \rangle}{\langle u_n^2 \rangle} \cos(2k\alpha l)$$
$$W_2(k\alpha) = 1 + 2 \sum_{l=1}^{\infty} \frac{\langle \Delta_n \Delta_{n+l} \rangle}{\langle \Delta_n^2 \rangle} \cos(2k\alpha l)$$
$$W_3(k\alpha) = 1 + 2 \sum_{l=1}^{\infty} \frac{\langle u_n \Delta_{n+l} \rangle}{\langle u_n \Delta_n \rangle} \cos(2k\alpha l)$$

CORRELATED DISORDER

A few points to remember:

- Power spectra wich vanish over continuum energy intervals create *effective* localisation-delocalisation transitions

- Disorder correlations can suppress localisation in certain energy windows and enhace it in the complementary intervals

- For any given power spectrum, a corresponding random sequence can be generated (inverse problem)

CORRELATED DISORDER: EXAMPLE

Compositional and structural disorder with the same spectrum:

$$W_1(k) = W_2(k) = \begin{cases} \dfrac{\pi}{2\,(k_2 - k_1)} & \text{if} \quad k \in [k_1, k_2] \\ 0 & \text{if} \quad k \in [0, k_1] \cup \left[k_2, \frac{\pi}{2}\right] \end{cases}$$

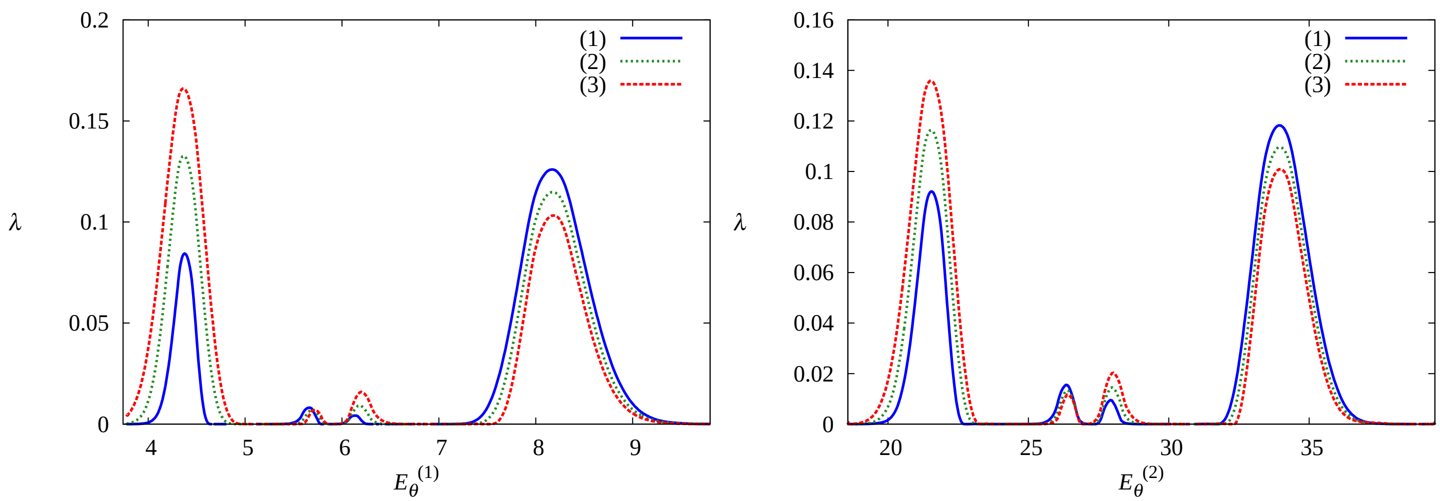
Binary correlators

$$\frac{\langle u_{l+n} u_l \rangle}{\langle u_l^2 \rangle} = \frac{\langle \Delta_{l+n} \Delta_l \rangle}{\langle \Delta_l^2 \rangle} = \frac{1}{2\,(k_2 - k_1)\,n} \left[\sin(2k_2 n) - \sin(2k_1 n)\right]$$

Inverse localisation length

$$\begin{aligned} l_{\text{loc}}^{-1} &= \frac{1}{8 \sin^2 k} \left[\left(\frac{\sin q}{q} \right)^2 \langle u_n^2 \rangle W_1(k) + U^2 \langle \Delta_n^2 \rangle W_2(k) \right. \\ &- \left. 2 \left| \frac{\sin q}{q} \right| U \sqrt{\langle u_n^2 \rangle \langle \Delta_n^2 \rangle W_1(k) W_2(k)} \, \cos k \sin(2\eta) \right] \end{aligned}$$

LYAPUNOV EXPONENT



1st band 2nd band

Data obtained for $U = 5.7$, $\sqrt{\langle \Delta_n^2 \rangle} = 0.05$, $\sqrt{\langle u_n^2 \rangle} = 0.4$. Mobility edges set at $k_1 = 0.26\pi$ and $k_2 = 0.27\pi$. Blue line: positive cross-correlations; green line: no cross-correlations; red line: negative cross-correlations.

TWO NATURAL QUESTIONS

The analysis carried out so far leaves open two questions:

- The theoretical results for the localisation length are obtained for a K.P. model with infinite barriers. Can one apply them to the present case, with a finite number of barriers?

- How does the 2D nature of the model manifest itself?

LOCALISATION IN A K.P. MODEL WITH A FINITE NUMBER OF BARRIERS

The finite domain of the angle variable, and the limited number of barriers that it entails, do not matter much if disorder is strong enough for the localisation length to satisfy the condition

$$l_{\text{loc}} \ll 2\pi$$

For K.-P. models with N wells, the degree of localisation can be measured via the entropic localisation length

$$l_N = \exp\left[S_N\right]$$

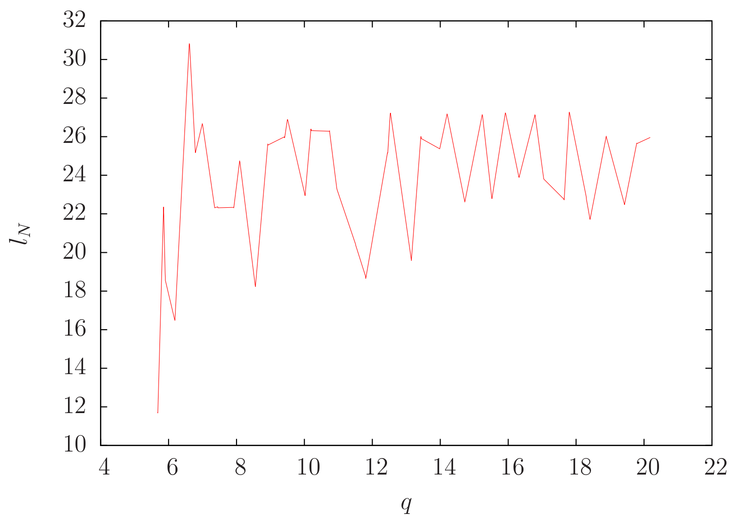
with S_N being the Shannon entropy of the angular eigenstate $\Theta^{(q)}$ associated to the momentum q

$$S_N = -\sum_{n=1}^{N} |\Theta_n^{(q)}|^2 \ln |\Theta_n^{(q)}|^2$$

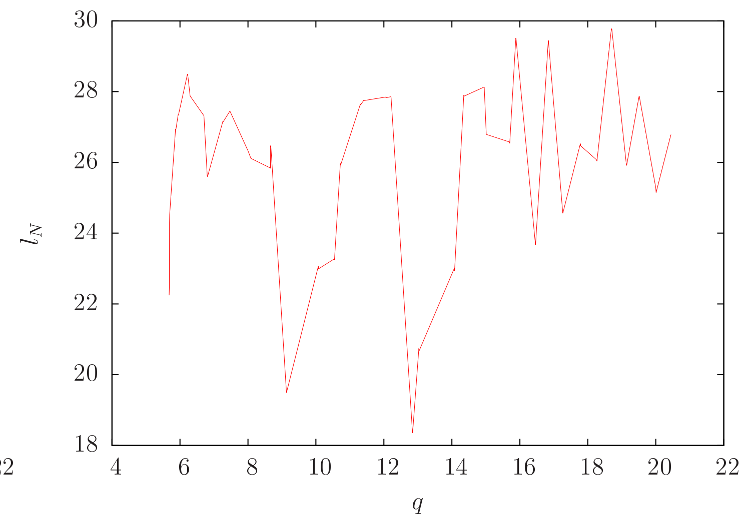
NUMERICAL RESULTS

Case considered $\begin{cases} N = 40 \text{ barriers} \\ \text{Structural disorder with} \\ \sqrt{\langle \Delta_n^2 \rangle} = 0.076 \text{ rad} \end{cases}$

Entropic localisation length *for a single realisation of the disorder:*



Uncorrelated disorder



Correlated disorder

The correlations of disorder enhance localisation and produce angular localisation of specific eigenstates

THE RADIAL SCHRÖDINGER EQUATION

Schrödinger equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[E - k_z^2 - \frac{q^2}{r^2}\right] R = 0$$

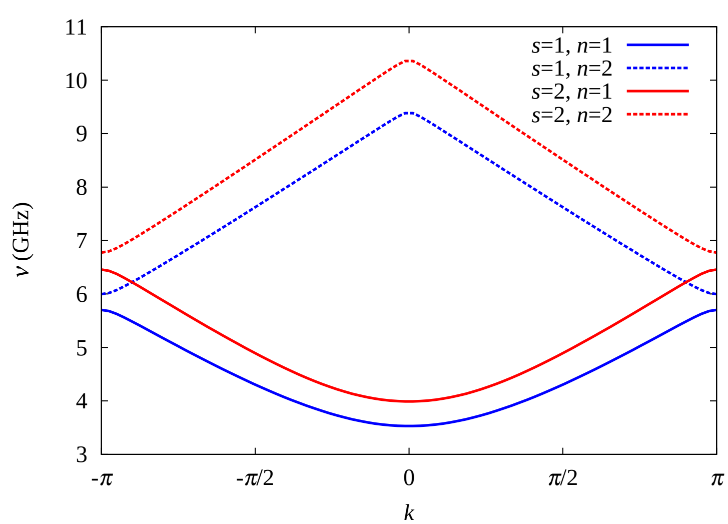
Solution

$$R_q(r) = c_1 J_q(\sqrt{E - k_z^2}r) + c_2 Y_q(\sqrt{E - k_z^2}r)$$

Boundary conditions $R_q(r_1) = R_q(r_2) = 0$ give

$$Y_q(\sqrt{E - k_z^2}r_1)J_q(\sqrt{E - k_z^2}r_2) = J_q(\sqrt{E - k_z^2}r_1)Y_q(\sqrt{E - k_z^2}r_2)$$

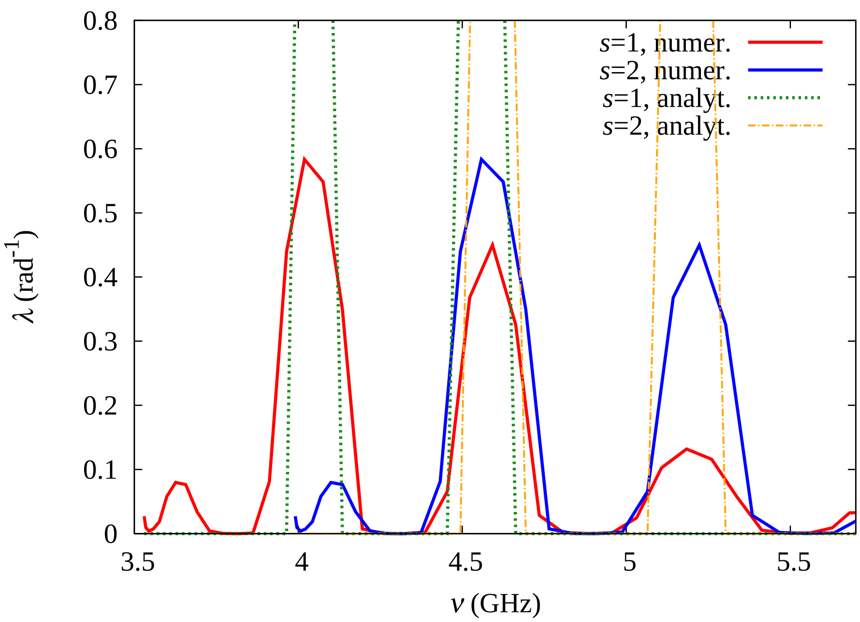
which determines the energy eigenvalues E_{q,s,k_z}



Eigenfrequencies ν of the microwave cavity vs. Bloch wavevector k

2D NATURE OF THE MODEL

The partial overlapping of the energy bands corresponding to different values of the radial quantum number s makes possible the angular localisation within the same energy window of states with different radial quantum number



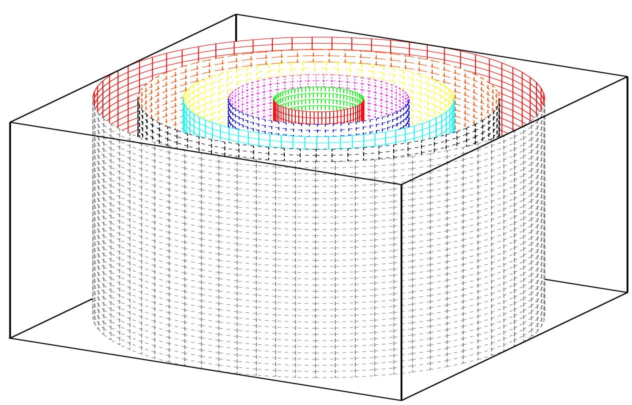
Inverse localisation length λ vs. eigenfrequency ν for the first allowed bands corresponding to $s = 1$ and $s = 2$. Structural self-correlated disorder with strength $\sqrt{\langle \Delta_n^2 \rangle} = 0.076$ rad.

2D MODEL WITH RANDOM RADIAL POTENTIAL

- Problem: quantum particle in a 2D central potential, formed by a sequence of circular delta-barrier of random strengths and positions
- Two related questions:

  1. Localisation of the eigenstates when the sequence of barriers is *infinite*

  2. Transmission through a *finite* number of random barriers



SCHRÖDINGER EQUATION

Schrödinger equation $(\hbar^2/2m = 1)$

$$\left[-\nabla^2 + V(r)\right]\psi(r,\theta) = E\psi(r,\theta)$$

2d Laplacian

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Random potential $(N = \infty)$

$$V(r) = \sum_{n=1}^{\infty} (U + u_n)\,\delta\,(r - r_n)$$

Random variables

Barrier positions:	$r_n = na + (\delta r)_n$
Barrier strengths:	$U_n = U + u_n$

We focus on the case of *weak disorder*

SEPARATION OF THE VARIABLES

Wavefunction

$$\psi(r,\theta) = \frac{\chi(r)}{\sqrt{r}}\Theta(\theta)$$

Angular part

$$\frac{d^2\Theta}{d\theta^2} + l^2\Theta = 0 \quad \Rightarrow \quad \Theta(\theta) = \frac{1}{\sqrt{2\pi}}e^{il\theta}, l \in \mathbf{Z}$$

Radial part

$$\frac{d^2\chi}{dr^2} + \left[E - V(r) - \frac{l^2 - 1/4}{r^2}\right]\chi = 0$$

HAMILTONIAN MAP APPROACH

Dynamical equation

$$\ddot{x}+\left[q^2-\frac{l^2-1/4}{t^2}-\sum_{n=1}^{\infty}(U+u_n)\delta(t-t_n)\right]x=0$$

Oscillator with time-varying frequency perturbed (weak disorder!) by a sequence of random kicks

Integrating over the time interval $[t_n^{(-)}, t_{n+1}^{(-)}]$ gives

$$\begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = \mathbf{T}_n \begin{pmatrix} x_n \\ p_n \end{pmatrix}$$

with

$$\mathbf{T}_n = \begin{pmatrix} (\mathbf{T}_n)_{11} & (\mathbf{T}_n)_{12} \\ (\mathbf{T}_n)_{21} & (\mathbf{T}_n)_{22} \end{pmatrix}$$

TRANSFER MATRIX

$$
\begin{aligned}
(\mathbf{T}_n)_{11} &= \frac{\pi}{2}\Big\{q\sqrt{t_{n+1}t_n}\left[J_l(qt_{n+1})Y_l'(qt_n) - Y_l(qt_{n+1})J_l'(qt_n)\right] \\
&+ \frac{1}{2}\sqrt{\frac{t_{n+1}}{t_n}}\left[J_l(qt_{n+1})Y_l(qt_n) - Y_l(qt_{n+1})J_l(qt_n)\right] \\
&+ (U+u_n)\sqrt{t_{n+1}t_n}\left[Y_l(qt_{n+1})J_l(qt_n) - J_l(qt_{n+1})Y_l(qt_n)\right]\Big\}
\end{aligned}
$$

$$
(\mathbf{T}_n)_{12} = \frac{\pi}{2}\sqrt{t_{n+1}t_n}\left[Y_l(qt_{n+1})J_l(qt_n) - J_l(qt_{n+1})Y_l(qt_n)\right]
$$

$$
\begin{aligned}
(\mathbf{T}_n)_{21} &= \frac{\pi}{2}\Big\{q^2\sqrt{t_{n+1}t_n}\left[J_l'(qt_{n+1})Y_l'(qt_n) - Y_l'(qt_{n+1})J_l'(qt_n)\right] \\
&+ \frac{1}{2}q\sqrt{\frac{t_{n+1}}{t_n}}\left[J_l'(qt_{n+1})Y_l(qt_n) - Y_l'(qt_{n+1})J_l(qt_n)\right] \\
&+ \frac{1}{2}q\sqrt{\frac{t_n}{t_{n+1}}}\left[J_l(qt_{n+1})Y_l'(qt_n) - Y_l(qt_{n+1})J_l'(qt_n)\right] \\
&+ \frac{1}{4\sqrt{t_{n+1}t_n}}\left[J_l(qt_{n+1})Y_l(qt_n) - Y_l(qt_{n+1})J_l(qt_n)\right] \\
&+ q(U+u_n)\sqrt{t_{n+1}t_n}\left[Y_l'(qt_{n+1})J_l(qt_n) - J_l'(qt_{n+1})Y_l(qt_n)\right] \\
&+ \frac{1}{2}q(U+u_n)\sqrt{\frac{t_n}{t_{n+1}}}\left[Y_l(qt_{n+1})J_l(qt_n) - J_l(qt_{n+1})Y_l(qt_n)\right]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{T}_n)_{22} &= \frac{\pi}{2}\Big\{q\sqrt{t_{n+1}t_n}\left[Y_l'(qt_{n+1})J_l(qt_n) - J_l'(qt_{n+1})Y_l(qt_n)\right] \\
&+ \frac{1}{2}\sqrt{\frac{t_n}{t_{n+1}}}\left[Y_l(qt_{n+1})J_l(qt_n) - J_l(qt_{n+1})Y_l(qt_n)\right]\Big\}
\end{aligned}
$$

ASYMPTOTIC BEHAVIOUR OF THE TRANSFER MATRIX

For large times, the evolution matrix can be expanded in power of $1/t$

$$\mathbf{T}_n = \mathbf{T}_n^{(0)} - \frac{l^2 - 1/4}{2qt_n^2}\mathbf{T}_n^{(2)} + \dots$$

Zero-th order term

$$\mathbf{T}_n^{(0)} = \begin{pmatrix} \cos\left[q\left(a+\Delta_n\right)\right] + (U+u_n)\,\frac{1}{q}\sin\left[q\left(a+\Delta_n\right)\right] & \frac{1}{q}\sin\left[q\left(a+\Delta_n\right)\right] \\ -q\sin\left[q\left(a+\Delta_n\right)\right] + (U+u_n)\cos\left[q\left(a+\Delta_n\right)\right] & \cos\left[q\left(a+\Delta_n\right)\right] \end{pmatrix}$$

Second-order term

$$\begin{aligned}
(\mathbf{T}_n^{(2)})_{11} &= \left[\tfrac{1}{q^2}(U+u_n) - a - \Delta_n\right]\sin[q(a+\Delta_n)] \\
&- \tfrac{1}{q}(U+u_n)(a+\Delta_n)\cos[q(a+\Delta_n)] \\
(\mathbf{T}_n^{(2)})_{12} &= -\tfrac{1}{q^2}\sin\left[q\left(a+\Delta_n\right)\right] + \tfrac{1}{q}(a+\Delta_n)\cos\left[q\left(a+\Delta_n\right)\right] \\
(\mathbf{T}_n^{(2)})_{21} &= \left[1+(U+u_n)(a+\Delta_n)\right]\sin\left[q\left(a+\Delta_n\right)\right] \\
&+ q(a+\Delta_n)\cos\left[q\left(a+\Delta_n\right)\right] \\
(\mathbf{T}_n^{(2)})_{21} &= -(a+\Delta_n)\sin\left[q\left(a+\Delta_n\right)\right]
\end{aligned}$$

where

$$\Delta_n = (\delta r)_{n+1} - (\delta r)_n$$

LYAPUNOV EXPONENT

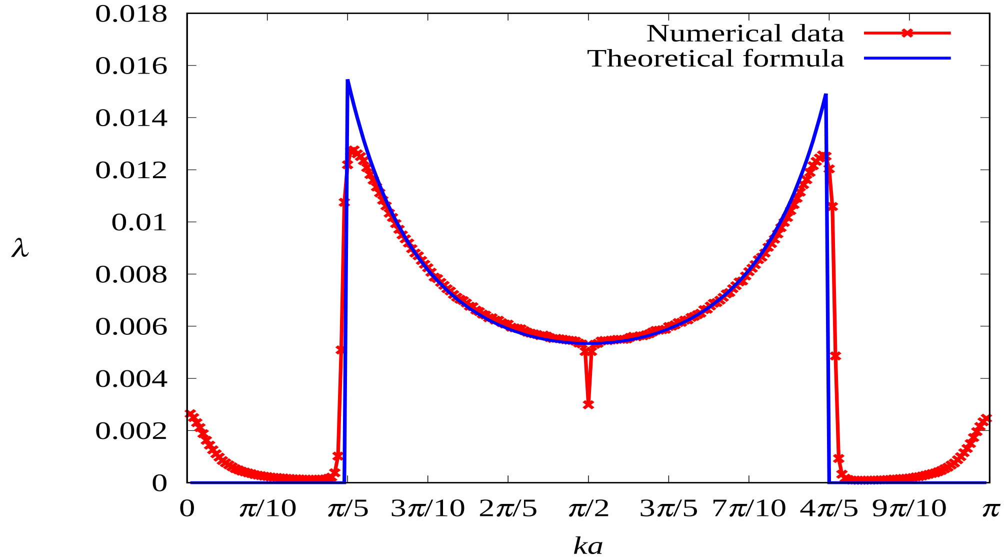
The $1/t_n^2$ correction is obscured by noise when

$$\frac{1}{t_n^2} \ll \sqrt{\langle \Delta_n^2 \rangle} \sim \sqrt{\langle u_n^2 \rangle}$$

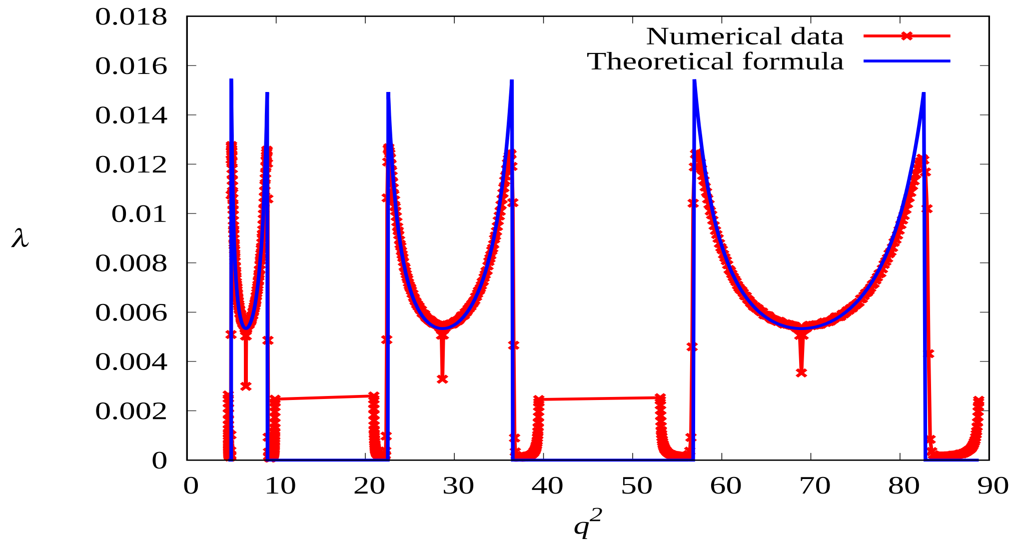
and *does not affect the Lyapunov exponent*

The Lyapunov exponent λ for the 2D K.-P. model with rotationally invariant disorder coincides with λ for the 1D K.-P. model. This conclusion is confirmed by numerical results.

NUMERICAL RESULTS



Lyapunov exponent λ vs. Bloch vector ka



Lyapunov exponent λ vs. angular energy q^2

Data obtained for $U = 8$, $\sqrt{\langle \Delta_n^2 \rangle} = 0.02$, and $\sqrt{\langle u_n^2 \rangle} = 0.02$. Self-correlations generate an effective mobility edge at $ka = \pi/5$; no cross-correlations.

TRANSMISSION THROUGH A FINITE NUMBER OF δ-BARRIERS

Statement of the problem: Central antenna irradiating circular waves from the origin which impinge on a sequence of N δ-barriers.

Schrödinger equation for the radial part $R(r)$

$$\frac{d^2 R_l}{dr^2}+\frac{1}{r}\frac{dR_l}{dr}+\left[q^2-\sum_{n=1}^{N}(U+u_n)\,\delta\,(r-r_n)-\frac{l^2}{r^2}\right]R_l=0$$

Solution in the central disk: Incoming wave + reflected wave

$$R_l(r)=A_0 H_l^{(1)}(qr)+B_0 H_l^{(2)}(qr)$$

Solution in the n-th well:

$$R_l(r)=A_n H_l^{(1)}(qr)+B_n H_l^{(2)}(qr)$$

Solution beyond the last barrier: Outgoing wave:

$$R_l(r)=A_N H_l^{(1)}(qr)$$

TRANSMISSION COEFFICIENT

Probability current

$$\mathbf{j} = \frac{1}{2m}\left[\psi^*(r,\theta)\mathbf{p}\psi(r,\theta) + c.c.\right]$$

Transmission coefficient: definition

$$T_N = \frac{\displaystyle\int_0^{2\pi} [\mathbf{j}_{\text{out}} \cdot \hat{\mathbf{r}}]_{r=r_N}\, r_N \mathrm{d}\theta}{\displaystyle\int_0^{2\pi} [\mathbf{j}_{\text{in}} \cdot \hat{\mathbf{r}}]_{r=r_1}\, r_1 \mathrm{d}\theta}$$

Transmission coefficient: value

$$T_N = \frac{|A_N|^2}{|A_0|^2}$$

TRANSFER MATRIX APPROACH

Transfer matrix in wave representation

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \mathbf{M}_n \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

with

$$\begin{aligned}
(\mathbf{M}_n)_{11} &= 1 - i\tfrac{\pi}{4} r_{n+1}(U + u_{n+1}) H_l^{(1)}(q r_{n+1}) H_l^{(2)}(q r_{n+1}) \\
(\mathbf{M}_n)_{12} &= -i\tfrac{\pi}{4} r_{n+1}(U + u_{n+1}) \left[H_l^{(2)}(q r_{n+1}) \right]^2 \\
(\mathbf{M}_n)_{21} &= i\tfrac{\pi}{4} r_{n+1}(U + u_{n+1}) \left[H_l^{(1)}(q r_{n+1}) \right]^2 \\
(\mathbf{M}_n)_{22} &= 1 + i\tfrac{\pi}{4} r_{n+1}(U + u_{n+1}) H_l^{(1)}(q r_{n+1}) H_l^{(2)}(q r_{n+1})
\end{aligned}$$

Total transfer matrix

$$\mathcal{M}_N = \mathbf{M}_N \cdot \mathbf{M}_{N-1} \dots \mathbf{M}_1$$

Transmission coefficient

$$T_N = \frac{1}{|(\mathcal{M}_N)_{22}|^2}$$

RECURSIVE RELATION

After eliminating the B_n coefficient from the map

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \mathbf{M}_n \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

one obtains the recursive relation for the ratio $R_n = A_n/A_{n-1}$:

$$\begin{aligned} R_{n+1} &= -\frac{r_{n+1}(U+u_{n+1})\left[H_l^{(2)}(qr_{n+1})\right]^2}{r_n(U+u_n)\left[H_l^{(2)}(qr_n)\right]^2}\frac{1}{R_n}+1 \\ &- i\frac{\pi}{4}r_{n+1}(U+u_{n+1})H_l^{(1)}(qr_{n+1})H_l^{(2)}(qr_{n+1}) \\ &+ \frac{r_{n+1}(U+u_{n+1})\left[H_l^{(2)}(qr_{n+1})\right]^2}{r_n(U+u_n)\left[H_l^{(2)}(qr_n)\right]^2} \\ &\times \left[1+i\frac{\pi}{4}r_{n+1}(U+u_{n+1})H_l^{(1)}(qr_{n+1})H_l^{(2)}(qr_{n+1})\right] \end{aligned}$$

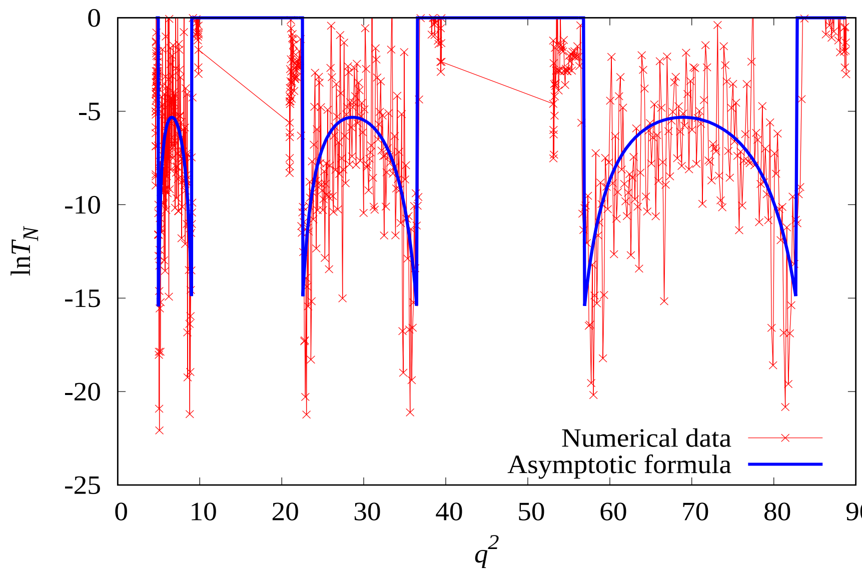
Transmission coefficient

$$\ln T_n = 2\sum_{n=1}^{N} \ln|R_n|$$

TRANSMISSION COEFFICIENT: THEORETICAL EXPECTATIONS AND NUMERICAL RESULTS

SPS hypothesis $\iff T_N = f\left(\frac{r_N - r_1}{l_{\text{loc}}}\right)$

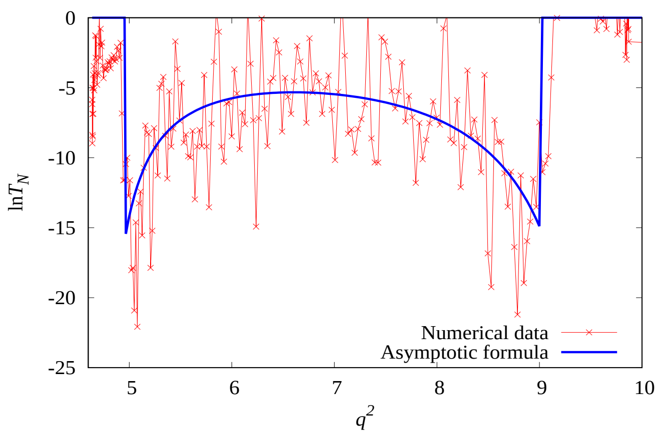
We expect $T_N \simeq 1$ whenever $\lambda \simeq 0$. The prediction agrees with the numerical results.



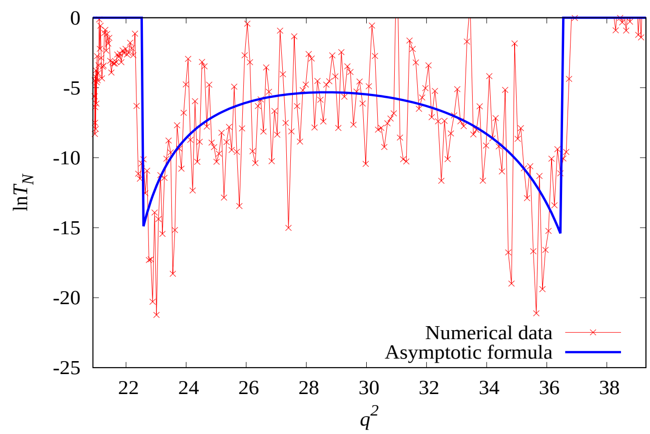
ln T_N vs q^2. The blue line represents the asymtptotic formula ln $T_N \simeq -2\lambda N$ valid in the localised regime. Data obtained for $N = 200$ barriers with $r_1 = 200$, $U = 8$, $\sqrt{\langle \Delta_n^2 \rangle} = 0.02$, and $\sqrt{\langle u_n^2 \rangle} = 0.02$. Self-correlations generate an effective mobility edge at $ka = \pi/5$; no cross-correlations.

TRANSMISSION COEFFICIENT: NUMERICAL RESULTS

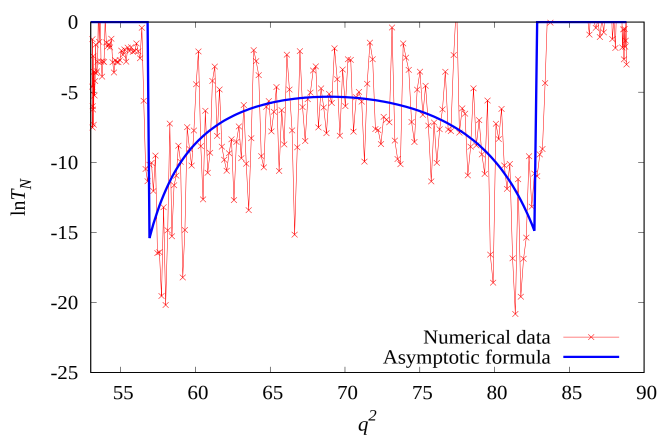
$\ln T_N$ vs q^2



First energy band



Second energy band



Third energy band

CONCLUSIONS

- Two 2D disordered models with separable Hamiltonians: one with angular disorder, the other invariant under rotations

- Mathematical problems reduced to manageable 1D cases although the models retain 2D features

- Correlated angular disorder can be used to focus beams in specific directions; correlated radial disorder can be used to select the energy of the transmitted beams.