Reflection in Non-Hermitian Optical Systems vs Symmetries of Transfer Matrix

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Contents:

1. J. Ramirez-Hernandes, F.M. Izrailev, N.M. Makarov, D.N. Christodoulides ***“$\mathcal{PT}$–symmetric transport in non- $\mathcal{PT}$–symmetric bi-layer optical arrays” Journal of Optics (Letter) 18, 09LT01 (2016), open access.***

2. J. Ramirez-Hernandes, F.M. Izrailev, N.M. Makarov ***“Double-sided unidirectional reflectivity” Phys Rev A 96, 013856 (2017).***

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1. Motivation

$\mathcal{PT}$–symmetric models ***are a particular class of non-Hermitian systems whose Hamiltonian is symmetric under the double action: space reflection and time reversal.***

The availability of optical attenuation and amplification makes possible to construct $\mathcal{PT}$–symmetric optical structures by artificial intermixing losses and gains. Therefore, such systems have been recently attracted much attention. As a result, there were predicted and observed their **surprising extraordinary properties**:

- **Real frequency spectrum in spite of the presence of loss and gain.**
- **Nonreciprocal transmission.**
- **Unidirectional reflection –** the most unusual property of the $\mathcal{PT}$–symmetry.

The study of transmission in discrete optical systems is properly based on of ***the transfer matrix formalism.*** This requires to pass ***from the symmetry of a Hamiltonian to that of the correspon-ding transfer matrix.***

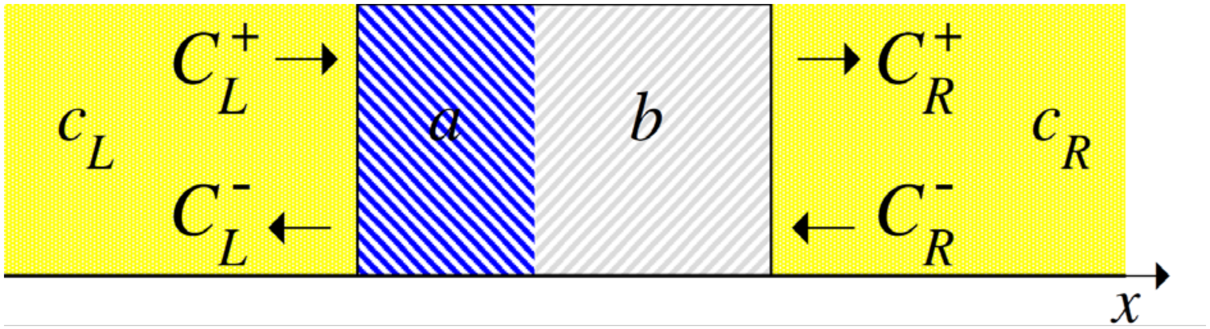
***In particular, the following questions naturally arise*:**

- **What symmetry should be inherent for the transfer matrix to observe $\mathcal{PT}$–symmetric optics?**
- **Can non $\mathcal{PT}$–symmetric system exhibits the $\mathcal{PT}$–symmetric behavior?**

The present talk is an attempt to shed light onto this problem.

2. Transfer Matrix Method

Consider propagation of an electromagnetic wave via a **unit bilayer setup**, specifically, through a 1D bilayer unit (a, b) cell embedded in a perfect c medium.



***The wave transfer via the system can be described by the matrix relation*:**

$$\begin{pmatrix} C_R^+ \\ C_R^- \end{pmatrix} = \widehat{M} \begin{pmatrix} C_L^+ \\ C_L^- \end{pmatrix} \quad \text{with} \quad \widehat{M} = \widehat{M}^{(ca)^{-1}} \widehat{M}^{(UC)} \widehat{M}^{(ca)} \quad \text{and} \quad \det \widehat{M} = 1.$$

This transfer relation transforms the amplitudes of incident, C_L^+, and reflected, C_L^-, waves at the left side of the $(c_L|a)$ interface into the amplitudes of incident, C_R^-, and reflected, C_R^+, waves at the right side of the $(b|c_R)$ interface.

Since the left, c_L, and the right, c_R, leads of an optical system are the same, the ***total transfer matrix is always unimodular.***

2.1. Transfer Matrix Structure

Interface transfer matrix $\widehat{M}^{(ca)}$ determines the wave transfer through the left interface $(c_L|a)$:

$$\widehat{M}^{(ca)} = \frac{1}{2}\begin{pmatrix} 1+Z_a/Z_c & 1-Z_a/Z_c \\ 1-Z_a/Z_c & 1+Z_a/Z_c \end{pmatrix} \quad \text{with} \quad \det \widehat{M}^{(ca)} = Z_a/Z_c \,.$$

Properties of the ITM: $\widehat{M}^{(ca)^{-1}} = \widehat{M}^{(ac)}$, $\widehat{M}^{(ac)}\widehat{M}^{(ba)} = \widehat{M}^{(bc)}$.

If media to the right/left from an interface, get the ***same impedances,*** $Z_c = Z_a$, the **ITM** degenerates into the ***identity matrix***. The interface becomes transparent with no reflected waves.

The unit (a, b) **cell transfer matrix** $\widehat{M}^{(UC)}$ is a product of four transfer matrices**:**

$$\widehat{M}^{(UC)} = \widehat{M}^{(ba)}\widehat{M}^{(b)}\widehat{M}^{(ab)}\widehat{M}^{(a)} \quad \text{with} \quad \det \widehat{M}^{(UC)} = 1 \,.$$

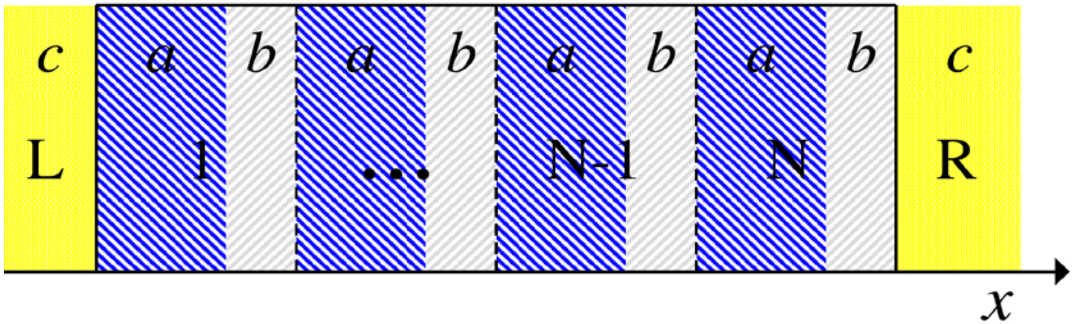
ITMs, $\widehat{M}^{(ab)}$and $\widehat{M}^{(ba)}$, describe the wave transfer through the interfaces $(a|b)$ and $(b|a)$. Their structure is identical to that of any ITM with corresponding impedances, Z_a and Z_b .

Free-flight transfer matrices $\widehat{M}^{(a)}$ and $\widehat{M}^{(b)}$ arise due to free wave traveling inside a or b slabs, e.g.

$$\widehat{M}^{(a)} = \begin{pmatrix} \exp(i\varphi_a) & 0 \\ 0 & \exp(-i\varphi_a) \end{pmatrix} \quad \text{with} \quad \det \widehat{M}^{(a)} = 1 \,.$$

2.2. 1D Superlattice

The system represents a ***1D regular array of*** N ***identical unit*** (a,b) ***cells*** attached to the left, c_L, and right, c_R, leads of the same homogeneous material c.



The basic transfer matrix $\widehat{M}$ of the ***unit bilayer setup*** is replaced with its Nth power: $\widehat{M} \to \widehat{M}^N$. In accordance with the ***Chebyshev theorem***

$$\widehat{M}^N = \begin{pmatrix} M_{11}J_N - J_{N-1} & M_{12}J_N \\ M_{21}J_N & M_{22}J_N - J_{N-1} \end{pmatrix}, \qquad J_N = \frac{\sin(N\mu_B)}{\sin\mu_B}, \quad \det\widehat{M}^N = 1.$$

Here J_N is the Chebyshev polynomial. The ***Bloch index*** μ_B associated with the eigenvalues $\exp(\pm i\mu_B)$ of the basic transfer matrix, is defined by the matrix trace,

$$2\cos\mu_B = M_{11} + M_{22}\ .$$

Remarkably: This ***dispersion relation*** is originated from the Bloch theorem for corresponding infinite array, defining the Bloch wave number $\kappa = \mu_B/d$. Thus, whereas $\widehat{M}^N$ describes a finite-size system, it contains the spectral parameter of the corresponding infinite counterpart.

3. Spectrum, Transmission and Reflection

In the case when the left c_L an right c_R connecting leads are of the same impedance Z_c , the total transfer matrix $\widehat{M}^{(T)}$ is always ***unimodular***,

$$\det \widehat{M}^{(T)} = 1.$$

Therefore, the ***transmittance*** of the system with such a ***symmetric connection***, is independent of the left/right direction of incident wave,

$$T = \left| M_{22}^{(T)} \right|^{-2} = \left[1 + M_{12}^{(T)} M_{21}^{(T)} + M_{22}^{(T)} \left({M_{22}^{(T)}}^{*} - M_{11}^{(T)} \right) \right]^{-1}$$

However, ***the left and the right reflectances***, in general, are different,

$$\frac{R^{(L)}}{T} = \left| M_{21}^{(T)} \right|^{2}, \qquad \frac{R^{(R)}}{T} = \left| M_{12}^{(T)} \right|^{2}$$

3.1. Hermitian System – Time-reversal Symmetry

The unimodular transfer matrix of the Hermitian system meets the condition,

$$M_{22} = M_{11}^* \,, \qquad M_{21} = M_{12}^* \,.$$

The transmittance gets the conventional form, and the left and right reflectances become equal

$$T_N = \left[1 + |M_{12}|^2 \frac{\sin^2(N\mu_B)}{\sin^2 \mu_B}\right]^{-1} \leq 1, \qquad \frac{R_N^{(L)}}{T_N} = \frac{R_N^{(R)}}{T_N} = |M_{12}|^2 \frac{\sin^2(N\mu_B)}{\sin^2 \mu_B}$$

The ***unimodularity + time-reversal symmetry*** provide the **flow conservation law**:

$$T_N + R_N = 1.$$

The ***dispersion relation*** for the Bloch index μ_B has ***real*** right-hand side,

$$\cos \mu_B = \mathrm{Re}\, M_{22} \,.$$

Therefore, it has ***real solution*** $\mu_B = \mu_B(\omega)$ when $|\mathrm{Re}\, M_{22}| \leq 1$ (***spectral pass bands***).

Otherwise, the spectrum is complex, $\mu_B = m\pi + i\alpha(\omega), \quad \alpha > 0, \; m = 0, \pm 1, \pm 2, \cdots$, giving rise to ***spectral gaps***, or the same, reflection bands.

3.2. Non-Hermitian System – Partial Time-reversal Symmetry

The TRS is conserved only for diagonal elements of transfer matrix $\widehat{M}$ of the ***unit bilayer setup***

$$M_{22} = M_{11}^{*}\,, \qquad\qquad M_{21} \neq M_{12}^{*}\,.$$

The transmittance can be of ***any positive value,***

$$T_N = \left[1 + M_{12}M_{21}\frac{\sin^2(N\mu_B)}{\sin^2\mu_B}\right]^{-1} \gtreqless 1.$$

The left and right reflectances are ***different* – nonreciprocal reflection,**

$$\frac{R_N^{(L)}}{T_N} = |M_{21}|^2\frac{\sin^2(N\mu_B)}{\sin^2\mu_B} \quad \neq \quad \frac{R_N^{(R)}}{T_N} = |M_{12}|^2\frac{\sin^2(N\mu_B)}{\sin^2\mu_B}\,.$$

The flow conservation low breaks down, changing into generalized relation,

$$|1 - T_N| = \sqrt{R_N^{(L)}R_N^{(R)}}\,.$$

The ***spectrum*** $\mu_B = \mu_B(\omega)$ of the Bloch index obeys, as before, the dispersion relation

$$\cos\mu_B = \mathrm{Re}\, M_{22}\,.$$

with purely real r.h.s., thus allowing the spectral band structure in the dependence $\mu_B = \mu_B(\omega)$.

4. General Consequences

- In discrete layered systems, composed of homogeneous slabs, multiple wave scattering, resulting in reflected waves, emerges only at ***mismatching interfaces*** where the impedances of two adjacent layers are different. When the impedances are the same, the interface is invisible.

- For **TRS** and **PTRS**, all optics are specified by off-diagonal elements of the basic transfer matrix. Diagonal elements define the spectrum $\mu_B = \mu_B(\omega)$ relevant to a superlattice only.

- For **PTRS**, there are ***two mechanisms*** of occurring the perfect transmission $T = 1$.

- The first one is the ***Fabry-Perot resonances*** associated with the total length N of superlattice,

$$J_N^2(\mu_B) = 0 \quad \Longrightarrow \quad \mu_B = m\pi/N, \qquad m = 1,2,\cdots, \qquad N \geq 2.$$

The transmittance $T_N = 1$, and both reflectances expectedly vanish, $R_N^{(L)} = R_N^{(R)} = 0$.

- The second mechanism manifests itself due to vanishing one of the off-diagonal elements of the basic ***unit bilayer setup matrix***, either M_{21} or M_{12},

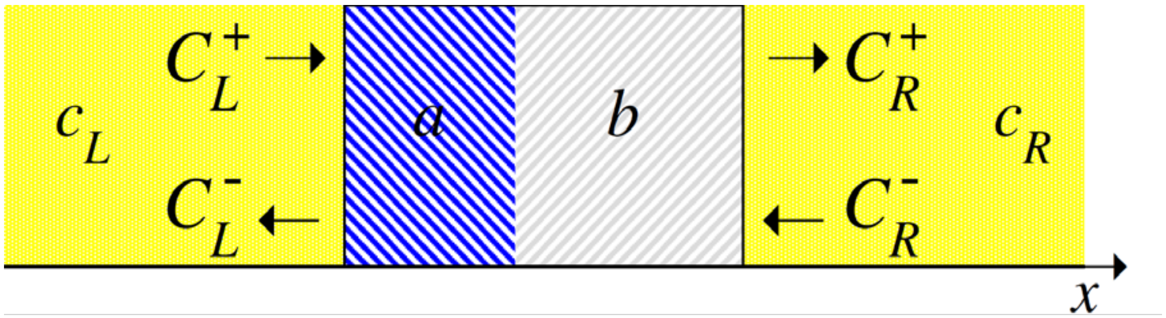
$$M_{21} = 0 \Longrightarrow \omega = \omega_U^+ \Longrightarrow T(\omega_U^+) = 1,\ R^{(L)} = 0,\ R^{(R)} \neq 0\,.$$

$$M_{12} = 0 \Longrightarrow \omega = \omega_U^- \Longrightarrow T(\omega_U^-) = 1,\ R^{(L)} \neq 0,\ R^{(R)} = 0\,.$$

Thus, the effect known as the ***unidirectional reflection***, arises where one reflectance vanishes, whereas the other remains finite. We term the $\omega_U^{\pm}$ points the unidirectional points or **U-points.**

5. The Model

Now, let us return to the ***unit bilayer setup*** ($N = 1$) introduced in the beginning of Section 2.



A dielectric 1D bilayer unit (a, b) cell is attached to dielectric left, c_L, and right, c_R, leads. The ***thicknesses*** of the slabs are d_a and d_b, the ***setup size*** is $d = d_a + d_b$. The optic parameters (***refractive index, impedance and wave phase shift***) of two constitutive a, b layers are chosen as

$$n_a = n_a^{(0)}(1 + i\gamma), \qquad Z_a = Z(1 + i\gamma)^{-1}, \qquad \varphi_a = \varphi(1 + i\gamma)/2$$

$$n_b = n_b^{(0)}(1 - i\gamma), \qquad Z_b = Z(1 - i\gamma)^{-1}, \qquad \varphi_b = \varphi(1 - i\gamma)/2$$

The a and b slabs are made of the materials ***absorbing or amplifying*** electromagnetic energy.

The dimensionless parameter $\gamma > 0$ measures the strength of loss/gain inside a/b layers.

Symmetric connection: The properties of leads are fully determined by their impedances that assumed to be the same Z_c for both leads.

6. No Loss/Gain ($\gamma = 0$)

All optic parameters are assumed to be real and positive constants.
The basic a and b layers turn out to be ***perfectly matched***, i.e. have the same impedance,

$$Z_a = Z_b = Z \quad \Longrightarrow \quad Z = \mu_a / n_a^{(0)} = \mu_b / n_b^{(0)}.$$

Also, the layers have ***equal optic paths***, i.e. the same phase shift $\varphi/2$,

$$n_a^{(0)} d_a = n_b^{(0)} d_b \quad \Longrightarrow \quad \varphi = 2\omega n_a^{(0)} d_a / c = 2\omega n_b^{(0)} d_b / c \ .$$

The ***mismatching*** emerges exclusively at the interfaces between the (a, b) bilayer and the leads,

$$Z_c = \chi Z, \qquad\qquad 0 < \chi \lesseqgtr 1.$$

The mismatching parameter χ measures the contrast between the impedances Z_c and Z, thus determining the impact of multiple wave scattering from the interfaces $(c_L|a)$ and $(b|c_R)$.

The transfer matrix $\widehat{M}$ obeys the **time-reversal symmetry**. Its off-diagonal elements, completely specifying the transmission and reflection, get the following explicit expressions:

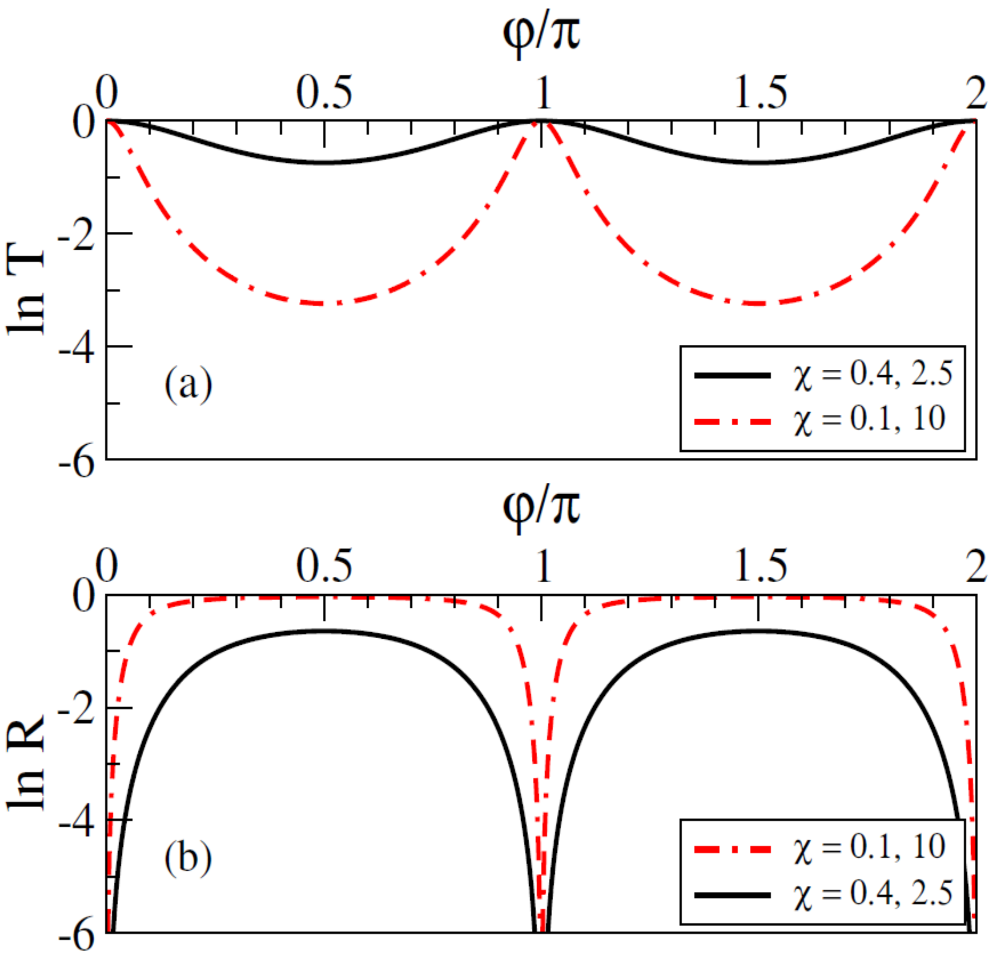
$$M_{12} = \frac{i}{2}(\chi - \chi^{-1}) \sin\varphi, \qquad M_{21} = -\frac{i}{2}(\chi - \chi^{-1}) \sin\varphi \ .$$

6.1. No Loss/Gain ($\gamma = 0$): Fabry-Perot Resonances

Thus, the transmittance and reflectance are described by

$$T = \left[1 + \frac{1}{4}(\chi - \chi^{-1})^2 \sin^2 \varphi\right]^{-1}, \qquad \frac{R}{T} = \frac{1}{4}(\chi - \chi^{-1})^2 \sin^2 \varphi \,.$$

As a consequence, the ***flow conservation law*** holds true.



The transmission displays the ***Fabry-Perot resonances*** associated with the reflections from the interfaces between bilayer and mismatching leads. ***At the resonance,***

$$\varphi = \varphi_{res} \equiv m\pi, \qquad m = 1, 2, \cdots .$$

the perfect transmission with $T = 1$ and $R = 0$ emerges. Otherwise,

$$T < 1 \text{ and } R = 1 - T < 1.$$

The ***amplitude of oscillations*** is specified by the contrast factor $(\chi - \chi^{-1})^2$: The stronger the mismatching between bilayer and leads, the larger the oscillations.

7. Balanced Loss/Gain ($\gamma \neq 0$)

A wave is attenuated or amplified by the same factor $\exp(\gamma\varphi/2)$ when traveling through the a or b slab. Due to the balance, ***the phase shift*** $\varphi_a + \varphi_b = \varphi$ of the wave passing the ***unit bilayer setup,*** turns out to be ***real and positive.***

The reflected waves emerge due to ***mismatching of all three interfaces*** provided by both ***mismatching*** χ and ***loss/gain*** γ parameters. For $\chi = 1$ and $\gamma = 0$ the system is transparent. In general, the interplay between these two parameters turns out to be highly nontrivial, giving rise to quite specific transport properties.

The transfer matrix $\widehat{M}$ obeys the **partial time-reversal symmetry**. Its off-diagonal elements, specifying the transmission and reflection, are described by the following expressions:

$$M_{12} = -\frac{iG(-\gamma,\chi,\varphi)}{2(1+\gamma^2)}, \qquad M_{21} = \frac{iG(\gamma,\chi,\varphi)}{2(1+\gamma^2)}$$

where we have introduced the characteristic ***real-valued*** function

$$G(\gamma,\chi,\varphi) = \gamma[\chi(1+\gamma^2)+\chi^{-1}]\sinh(\gamma\varphi) +$$

$$+2\gamma[\cos\varphi - \cosh(\gamma\varphi)] - [\chi(1+\gamma^2)-\chi^{-1}]\sin\varphi\,.$$

7.1. Balanced Loss/Gain ($\gamma \neq 0$): Transmission and Reflection

In accordance with definition, the analytical expressions for the transmittance T, left $R^{(L)}$ and right $R^{(R)}$ reflectances can be written in terms of the function $G(\gamma,\chi,\varphi)$ as follows:

$$T = \left[1 + \frac{G(\gamma,\chi,\varphi)\, G(-\gamma,\chi,\varphi)}{4(1+\gamma^2)^2}\right]^{-1},$$

$$\frac{R^{(L)}}{T} = \frac{G^2(\gamma,\chi,\varphi)}{4(1+\gamma^2)^2}, \qquad \frac{R^{(R)}}{T} = \frac{G^2(-\gamma,\chi,\varphi)}{4(1+\gamma^2)^2}.$$

As a consequence of the ***partial time-reversal symmetry***, the transmittance is an even function of the loss/gain parameter γ , while the reflectances transform into each other when changing the sign before γ,

$$T(-\gamma) = T(\gamma), \qquad R^{(R)}(-\gamma) = R^{(L)}(\gamma).$$

In addition, due to the partial time-reversal symmetry, the transmittance and reflectances satisfy the generalized “flow conservation low”.
It is important to stress that for $\chi \neq 1$ both functions, $G(\gamma,\chi,\varphi)$ and $G(-\gamma,\chi,\varphi)$, can be either positive or negative as functions of the phase shift φ.

8. Unidirectional Points

When the balanced loss/gain are turned on, the perfect transmission with $T = 1$ emerges at two different kinds of U-points, $\varphi = \varphi_U^{\pm}$, at which either the left or the right reflectance vanishes.

$$G(+\gamma,\chi,\varphi) = 0 \implies \varphi = \varphi_U^{+}(\gamma,\chi) \implies T(\varphi_U^{+}) = 1,\ R^{(L)} = 0,\ R^{(R)} \neq 0\,.$$

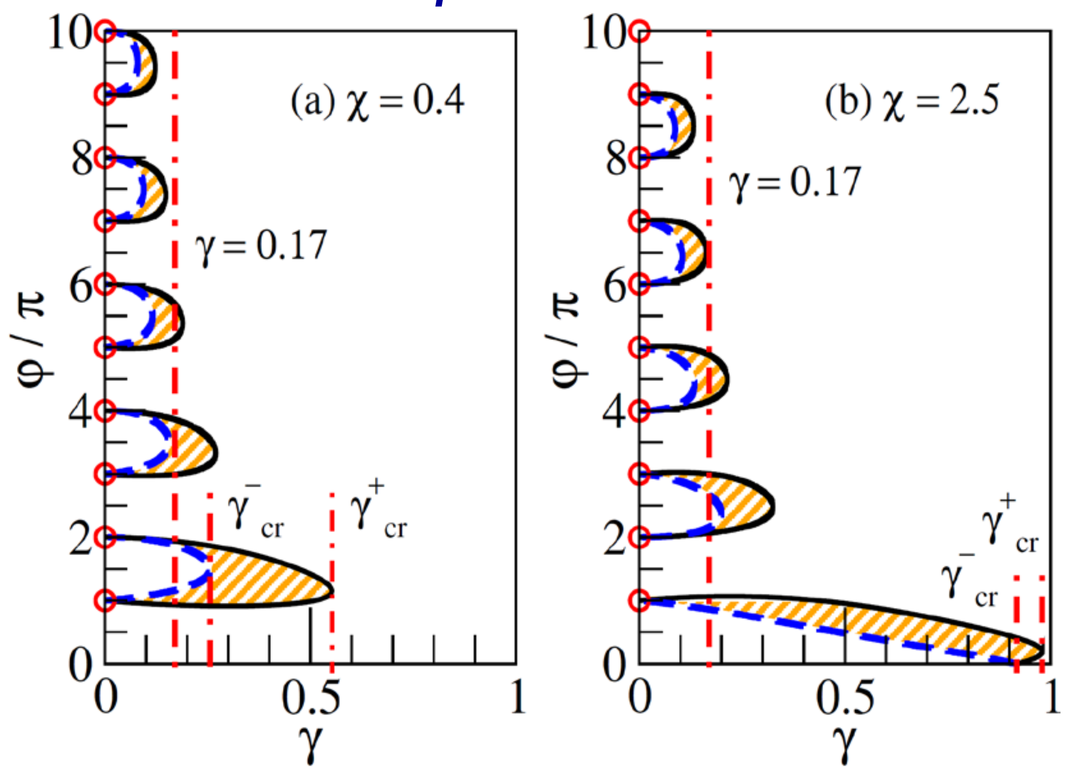
$$G(-\gamma,\chi,\varphi) = 0 \implies \varphi = \varphi_U^{-}(\gamma,\chi) \implies T(\varphi_U^{-}) = 1,\ R^{(L)} \neq 0,\ R^{(R)} = 0\,.$$

Thus, when $\varphi = \varphi_U^{\pm}$, two different effects of *unidirectional reflection* arise.

The figure shows the 2D phase space $(\gamma, \varphi/\pi)$ with $\varphi_U^{+}(\gamma)$ as solid curve and $\varphi_U^{-}(\gamma)$ being dashed curve for two mutually inverse values of χ .

Inside dashed regions $T(\varphi) > 1$.
Outside these regions $T(\varphi) < 1$.

The dependence of $\varphi = \varphi_U^{\pm}(\gamma)$ is quite sophisticated and sensitive to whether the value of mismatching χ is greater or smaller than unit.



8.1. Unidirectional Points: Threshold

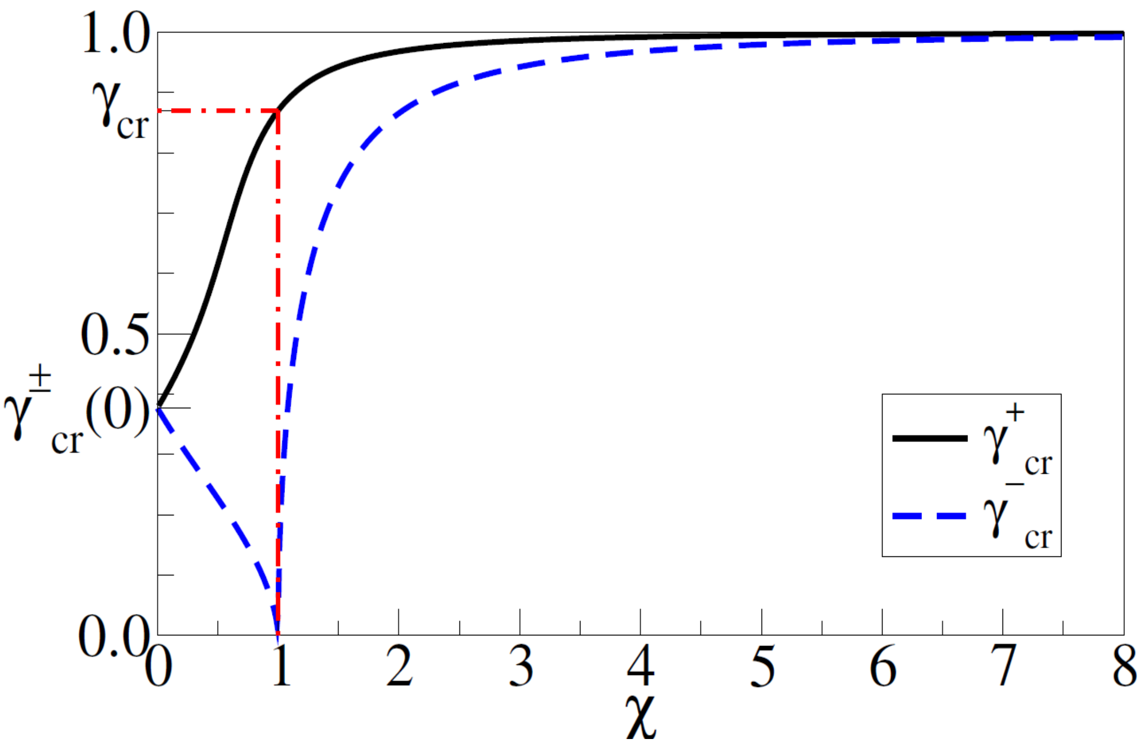
At $\gamma = 0$, and $\chi \neq 1$, both U-points start out with the Fabry-Perot resonances discussed above,

$$\varphi_U^{\pm}(\gamma = 0, \chi) = \varphi_{res} \,.$$

The ***balanced loss/gain splits the Fabry-Perot resonances into the U-points, giving rise to range of the anomalous transmission with*** $T > 1$ ***and to the unidirectional reflections.***

The solutions $\varphi = \varphi_U^{\pm}$ occur only when γ does not exceed its threshold (critical) values,

$$\varphi = \varphi_U^{+}(\gamma, \chi) \quad \text{exists if} \quad \gamma \leq \gamma_{cr}^{+}(\chi) \,, \qquad \varphi = \varphi_U^{-}(\gamma, \chi) \quad \text{exists if} \quad \gamma \leq \gamma_{cr}^{-}(\chi) \,.$$



These thresholds meet the conditions

$$0 \leq \gamma_{cr}^{-}(\chi) < \gamma_{cr}^{+}(\chi) < 1.$$

Both thresholds tend to the same, but different, limits as $\chi \to 0$ or $\chi \to \infty$,

$$\gamma_{cr}^{\pm}(0) \approx 0.4 \,, \quad \gamma_{cr}^{\pm}(\infty) = 1 \,.$$

9. Mismatching vs Balanced Loss/Gain

Perfect matching $(Z_c = Z)$**:** Remarkably, for $\chi = 1$, the lower threshold vanishes,

$$\gamma_{cr}^{+}(1) \approx 0.87, \quad \gamma_{cr}^{-}(1) = 0.$$

The U-point $\varphi_U^-(\gamma, \chi = 1)$ ***does not exist! The only kind of the right unidirectional reflection remains on hand.*** **Quite unexpected result!!!**

Strong contrast between the unperturbed Z of the bilayer and Z_c of the external leads:

$$|\chi - \chi^{-1}| \gg 1 \quad \Leftrightarrow \quad \chi \ll 1 \;\text{ or }\; \chi \gg 1.$$

Here the limit values of the U-points coincide

$$\varphi_U^+(\gamma, 0) = \varphi_U^-(\gamma, 0), \qquad \varphi_U^+(\gamma, \infty) = \varphi_U^-(\gamma, \infty).$$

Then, both the left and the right reflectances are asymptotically equal

$$R^{(L)}(\gamma, \chi = 0) = R^{(R)}(\gamma, \chi = 0), \qquad R^{(L)}(\gamma, \chi = \infty) = R^{(R)}(\gamma, \chi = \infty).$$

This means that the flow conservation law is restored

$$T + R = 1.$$

The strong mismatching overcomes the loss/gain restoring the time-reversal symmetry, despite the fact that transmittance and reflectance still depend on loss/gain.

Extremely nontrivial result!!!

10. Summary

❖ **Starting with general concepts of the transfer-matrix theory, we have defined the symmetry condition that is the origin of universal properties peculiar for the $\mathcal{PT}$–symmetric systems.**

❖ **With the widely used model of a quarter-stack structure, we have demonstrated that the $\mathcal{PT}$–symmetric effects can emerge even when the system itself is not $\mathcal{PT}$–symmetric.**

❖ **We have unexpectedly found a new effect of alternating unidirectional reflection which occurs both for the left and for the right incident wave for the same setup, depending on the value of the wave frequency.**

❖ **In the case of the perfect matching between the structure and the external leads, only the right unidirectional reflection remains on hand that is typical for the $\mathcal{PT}$–symmetric systems.**

❖ **The strong mismatching restores the time-reversal symmetry in contrast to the fact that all transport characteristics still depend on the value of balanced lose/gain.**

Thank you!