Global robustness vs. local vulnerabilities in network-coupled dynamical systems

Philippe Jacquod

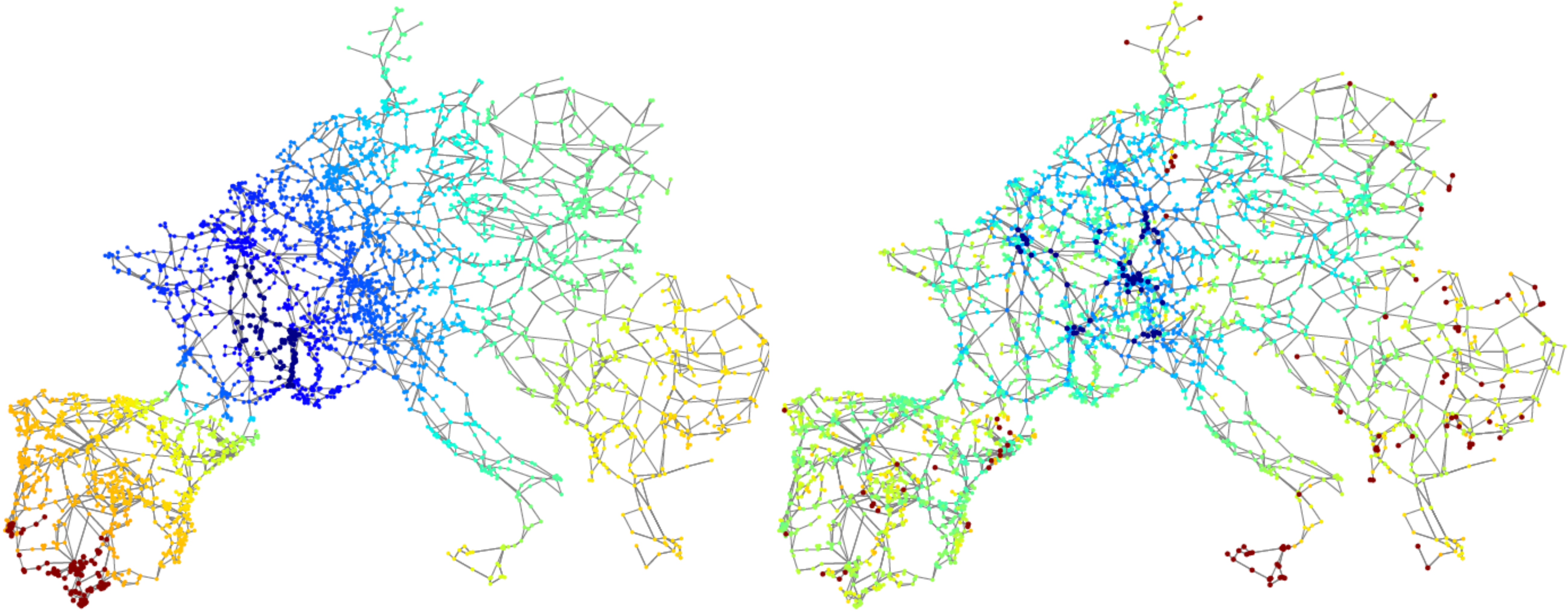
IBS-PCS/Daejon - 9.6.2018

FNSNF

Hes·so// VALAIS WALLIS
School of Engineering

The questions of interest

Given a set of dynamical systems with couplings between them defined on a certain graph :



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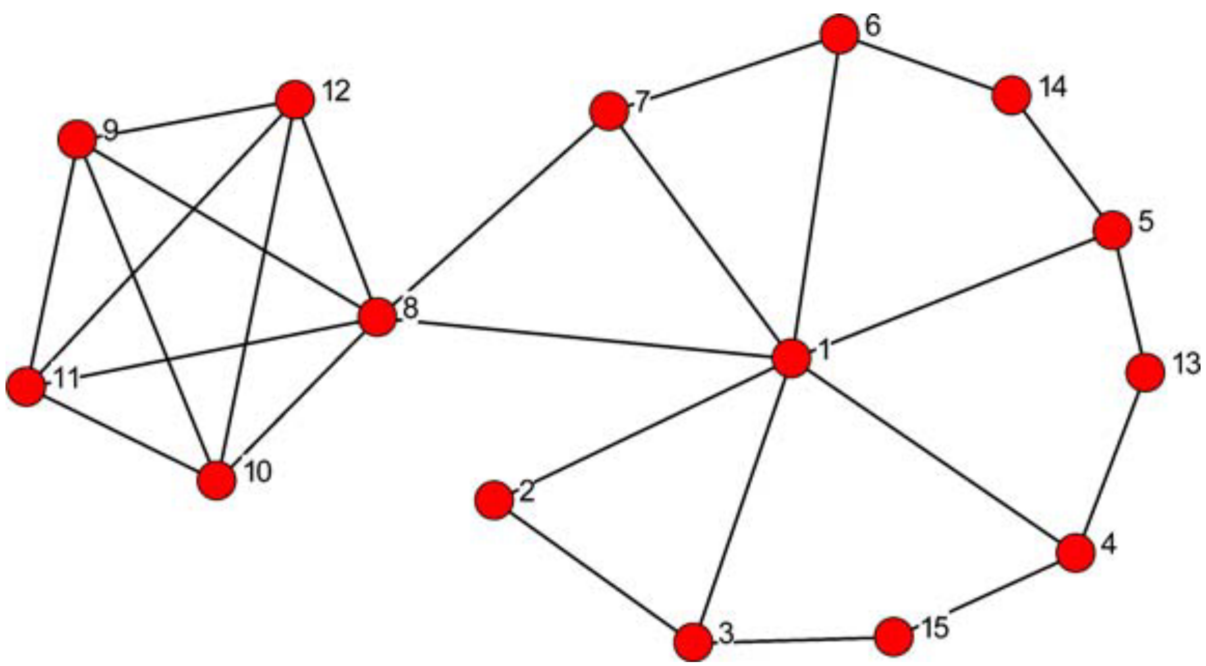
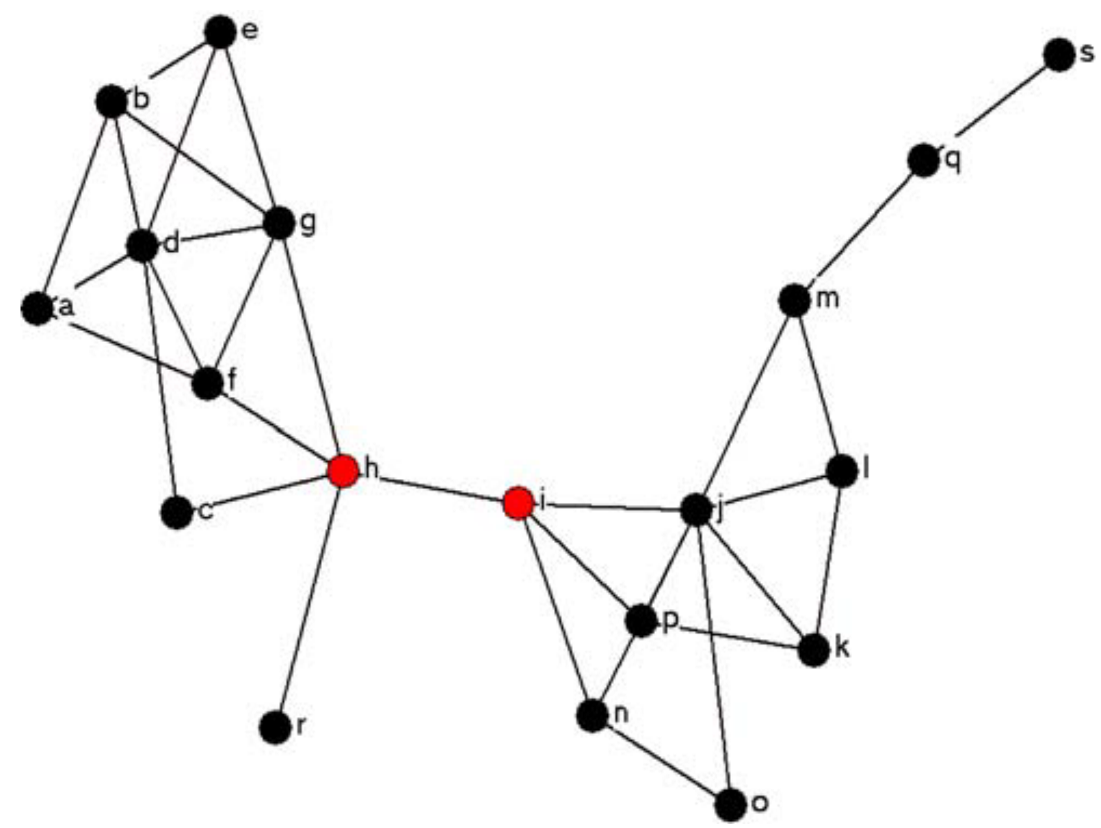
Global robustness vs. local vulnerabilities

The key player problem

1. Given a network, find the node which, if removed, would maximally disrupt communication among the remaining nodes.

2. Given a network, find the node that is maximally connected to all other nodes.

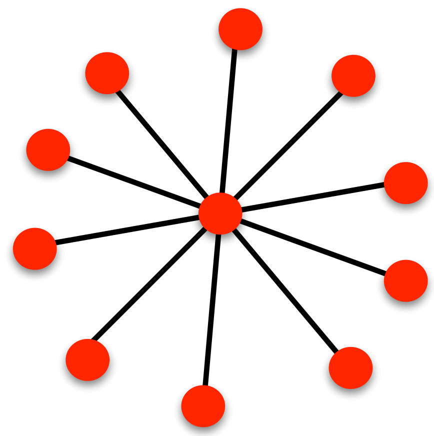




Borgatti :“The key player problem” (2003); Comput Math Organiz Theor (2006)

The key player problem : an answer

Star graph :



Which node should one remove to maximally disrupt communication among the remaining nodes ?

The key player problem : an answer

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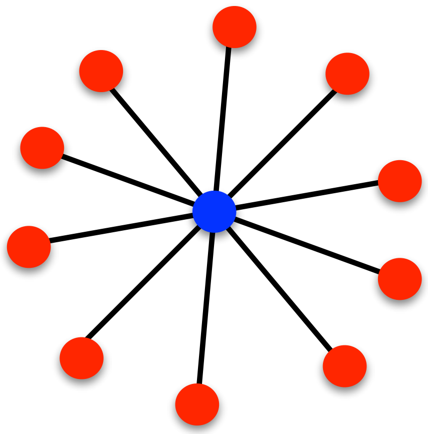

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A: the central one, obviously…

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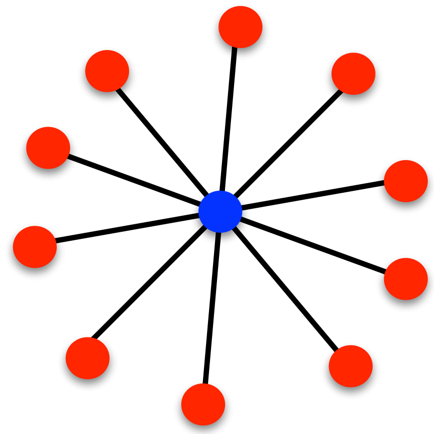

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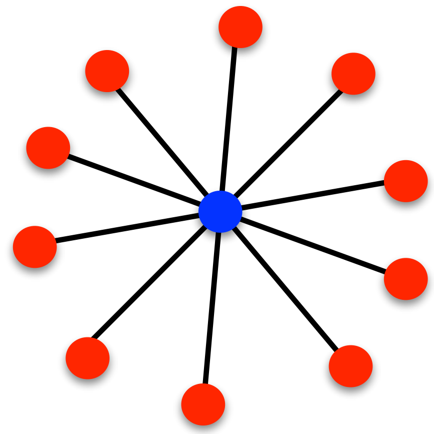

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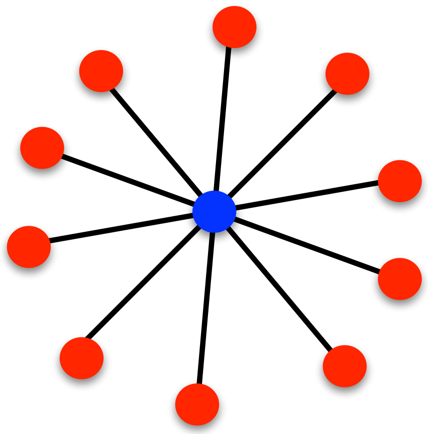

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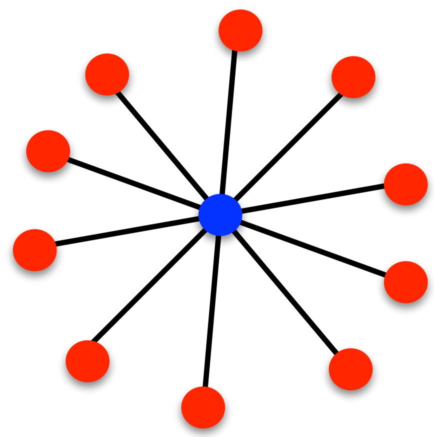

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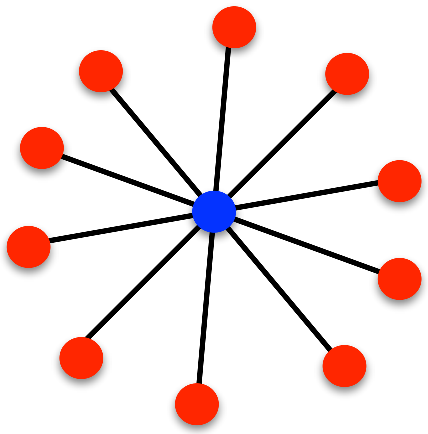

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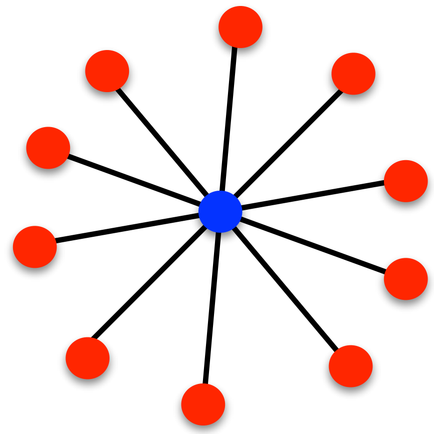

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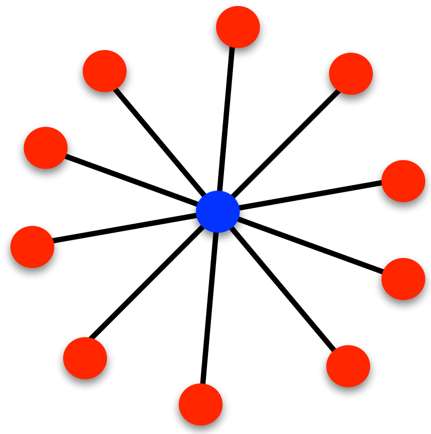

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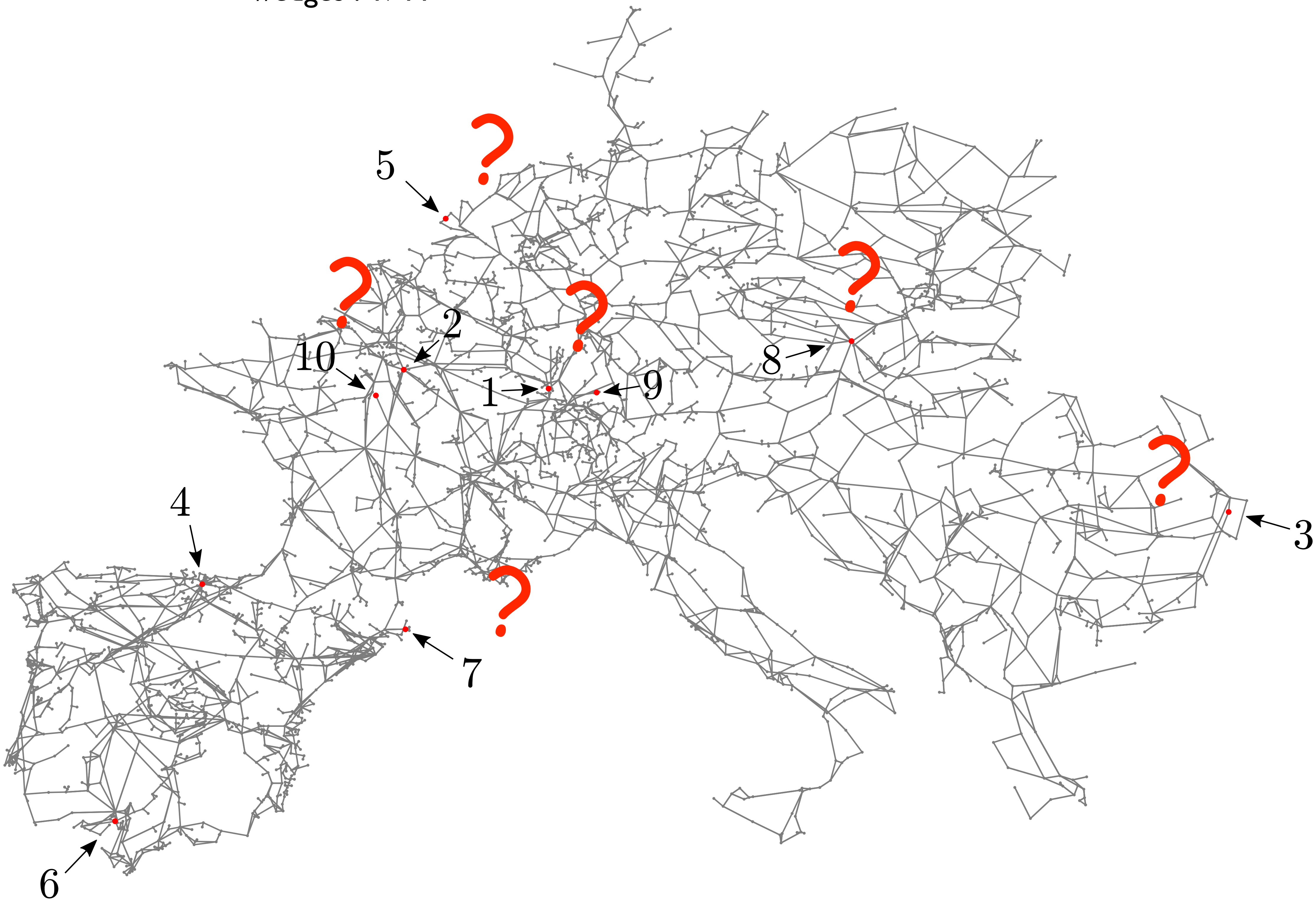
Which one of these property makes the central node the key player ?

#nodes : 3809
#edges : 4944



A bit of literature

Centralities have been introduced to solve the “key player” problem
*vs. graph/network matrix (geodesic, betweenness, Bonacich, Katz, PageRank…)

Comput Math Organiz Theor (2006) 12: 21–34
DOI 10.1007/s10588-006-7084-x

Identifying sets of key players in a social network

Stephen P. Borgatti

Econometrica, Vol. 74, No. 5 (September, 2006), 1403–1417

WHO’S WHO IN NETWORKS. WANTED: THE KEY PLAYER

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Axioms for Centrality

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Axioms for Centrality

Paolo Boldi and Sebastiano Vigna

…leading to purely graph-theoretic approaches (rather successful)

Collective dynamics of 'small-world' networks

Duncan J. Watts* **& Steven H. Strogatz**
Department of Theoretical and Applied Mechanics, Kim…
Cornell University, Ithaca, New York 14853, USA

THE ROYAL SOCIETY
doi 10.1098/rspb.2001.1767

Complexity and fragility in ecological networks

Curvature of co-links uncovers hidden thematic layers in the World Wide Web

Jean-Pierre Eckmann*† and Elisha Moses‡
*Département de Physique Théorique and Section de Mathématiques, Université de Genève, 32 Boulevard D'Yvoy, CH-1211 Genève 4, Switzerland; and
‡Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel

… REVIEW LETTERS
week ending
12 JUNE 2009

…rld Networks: Exact Results and New Insights

Cécile Caretta Cartozo and Paolo De Los Rios

BIRDS OF A FEATHER: Homophily in Social Networks

Miller McPherson[1], Lynn Smith-Lovin[1], and James M Cook[2]

Navigation in a small world

It is easier to find short chains between points in some networks than others.

A bit of literature

CHAOS **20**, 033122 (2010)

Do topological models provide good information about electricity infrastructure vulnerability?

Paul Hines,[1,a)] Eduardo Cotilla-Sanchez,[1,b)] and Seth Blumsack[2,c)]
[1]*School of Engineering, University of Vermont, Burlington, Vermont 05405, USA*
[2]*Department of Energy and Mineral Engineering, Pennsylvania State University, University Park, Pennsylvania 16802, USA*

(Received 7 April 2010; accepted 24 August 2010; published online 28 September 2010)

In order to identify the extent to which results from topological graph models are useful for modeling vulnerability in electricity infrastructure, we measure the susceptibility of power networks to random failures and directed attacks using three measures of vulnerability: characteristic path lengths, connectivity loss, and blackout sizes. The first two are purely topological metrics. The blackout size calculation results from a model of cascading failure in power networks. Testing the response of 40 areas within the Eastern U.S. power grid and a standard IEEE test case to a variety of attack/failure vectors indicates that directed attacks result in larger failures using all three vulnerability measures, but the attack-vectors that appear to cause the most damage depend on the measure chosen. While the topological metrics and the power grid model show some similar trends, the vulnerability metrics for individual simulations show only a mild correlation. We conclude that evaluating vulnerability in power networks using purely topological metrics can be misleading.
 [doi:10.1063/1.3489887]

The key player problem : deterministically coupled systems

node #	C_{geo}	Degree	PageRank	C_1	C_2	$\mathcal{P}_1^{num}$	$\mathcal{P}_2^{num}$ $[\gamma^2]$
1	7.84	4	3024	31.86	5.18	0.047	0.035
2	6.8	1	2716	22.45	5.68	0.021	0.118
3	5.56	10	896	22.45	2.33	0.32	0.116
4	4.79	3	1597	21.74	3.79	0.126	0.127
5	7.08	1	1462	21.74	5.34	0.026	0.125
6	4.38	6	2945	21.69	5.65	0.023	0.129
7	5.11	2	16	19.4	5.89	0.016	0.164
8	4.15	6	756	19.38	1.83	0.453	0.172
9	5.06	1	1715	10.2	5.2	0.047	0.449
10	2.72	4	167	7.49	2.17	0.335	0.64

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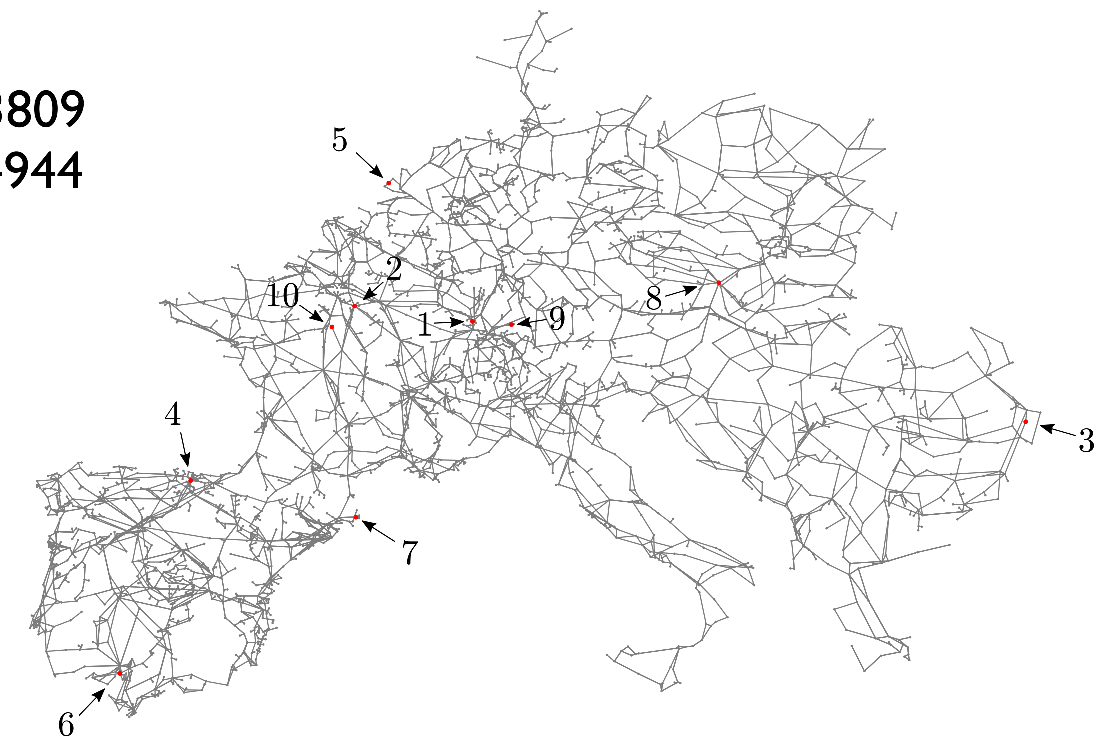
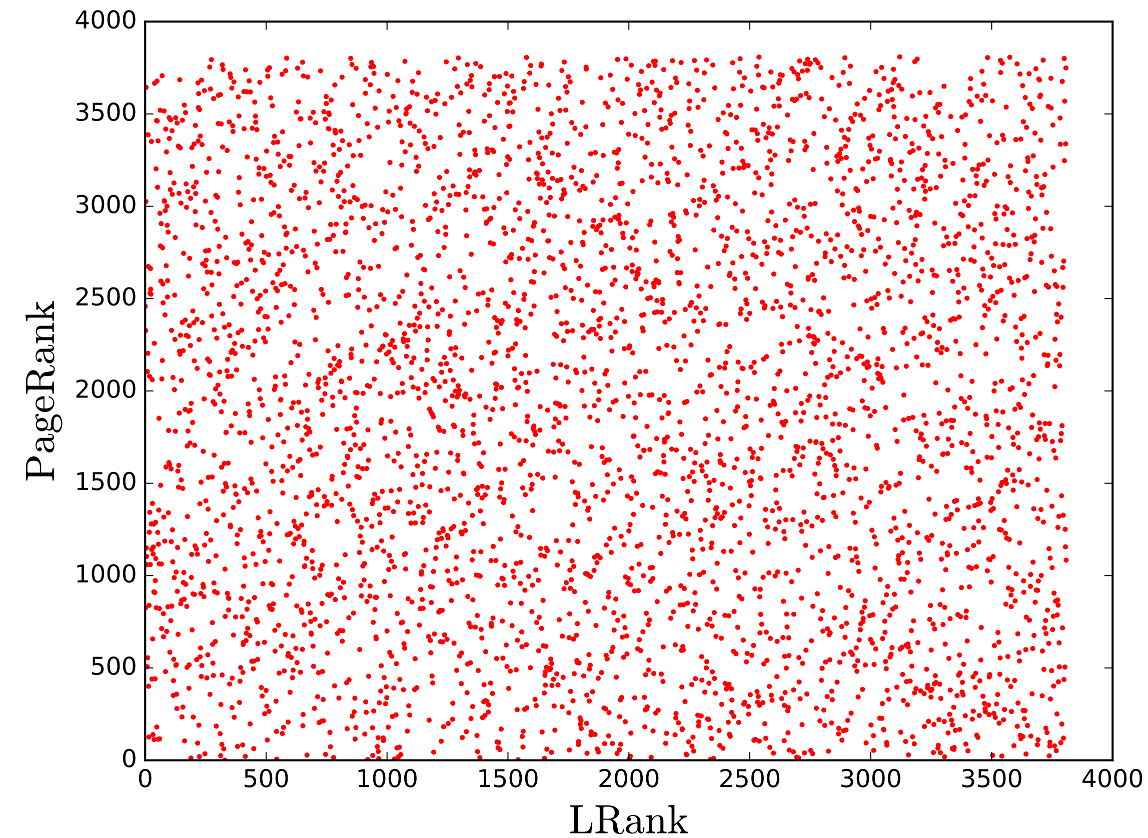


Resistance distance centrality a.k.a. LRank

Numerically computed performance measure



Tyloo, Pagnier and PJ : to be published (2018)

The program

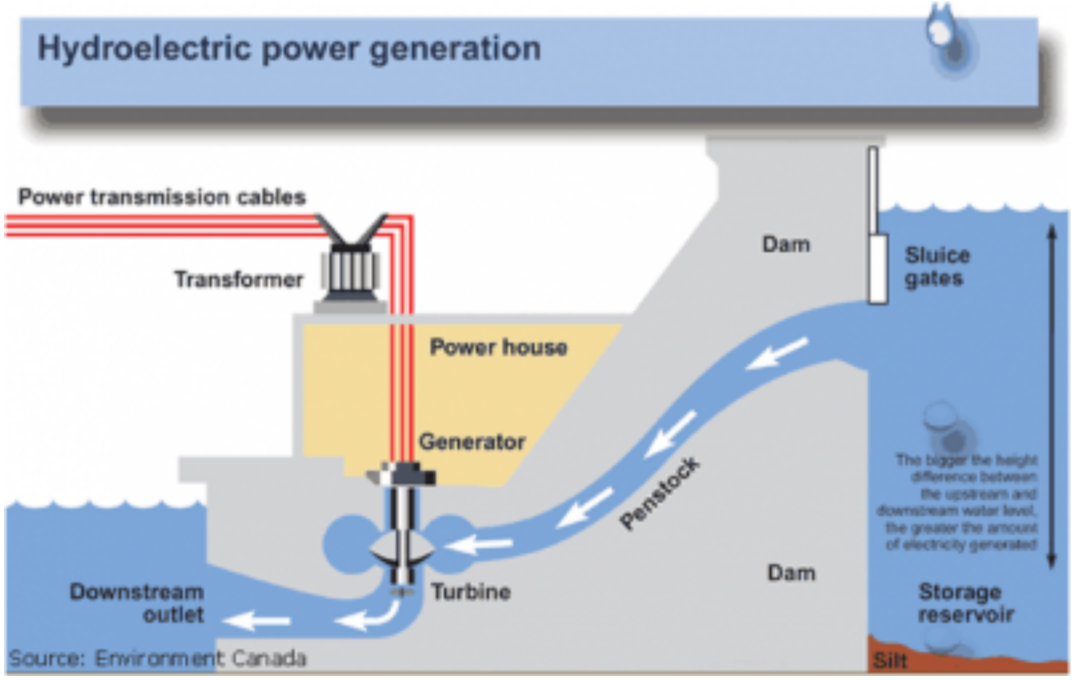
1) Dynamics of electric power grids (coupled oscillators)

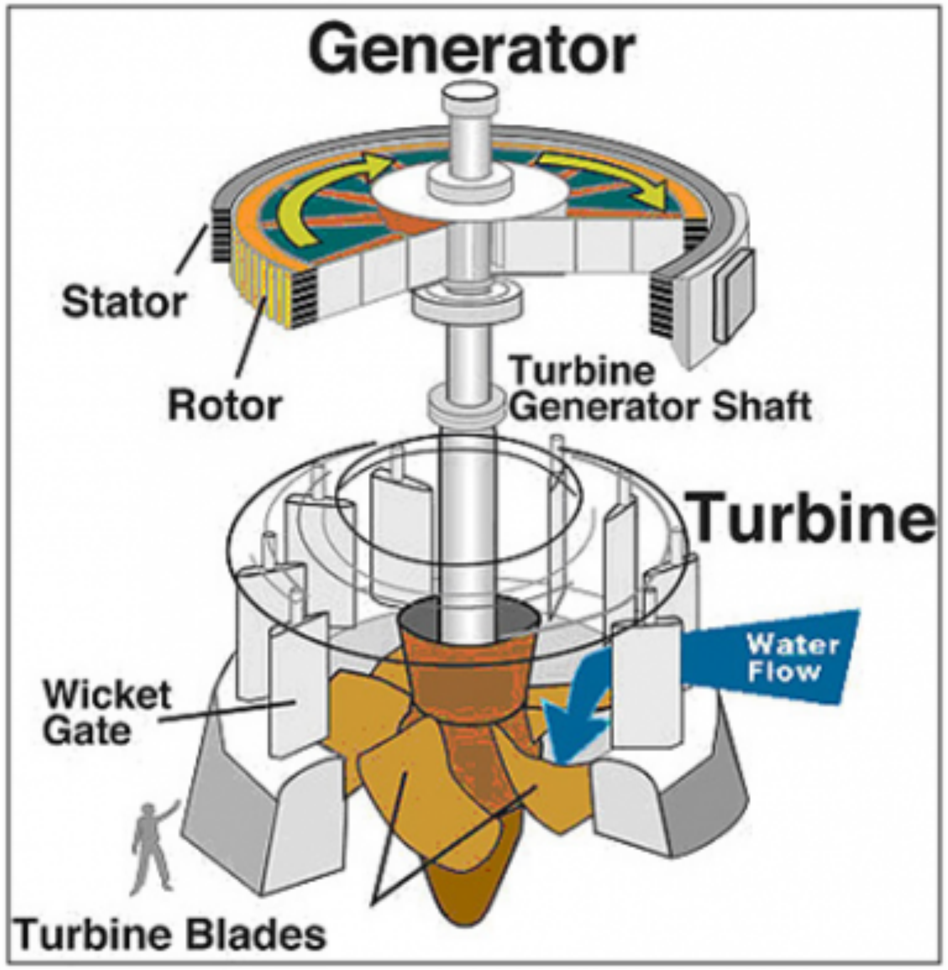
2) Synchronous operational setpoints

3) Transient dynamics under perturbations - local vs. averaged

A bit of electric power engineering

- Electricity production with rotating machines
- Potential, chemical, nuclear or thermal energy converted into mechanical energy (rotation)
- Mechanical energy converted into electric energy
- Time-dependence : think power instead of energy
- Balance between power in, power out and energy change in rotator : **SWING EQUATIONS**





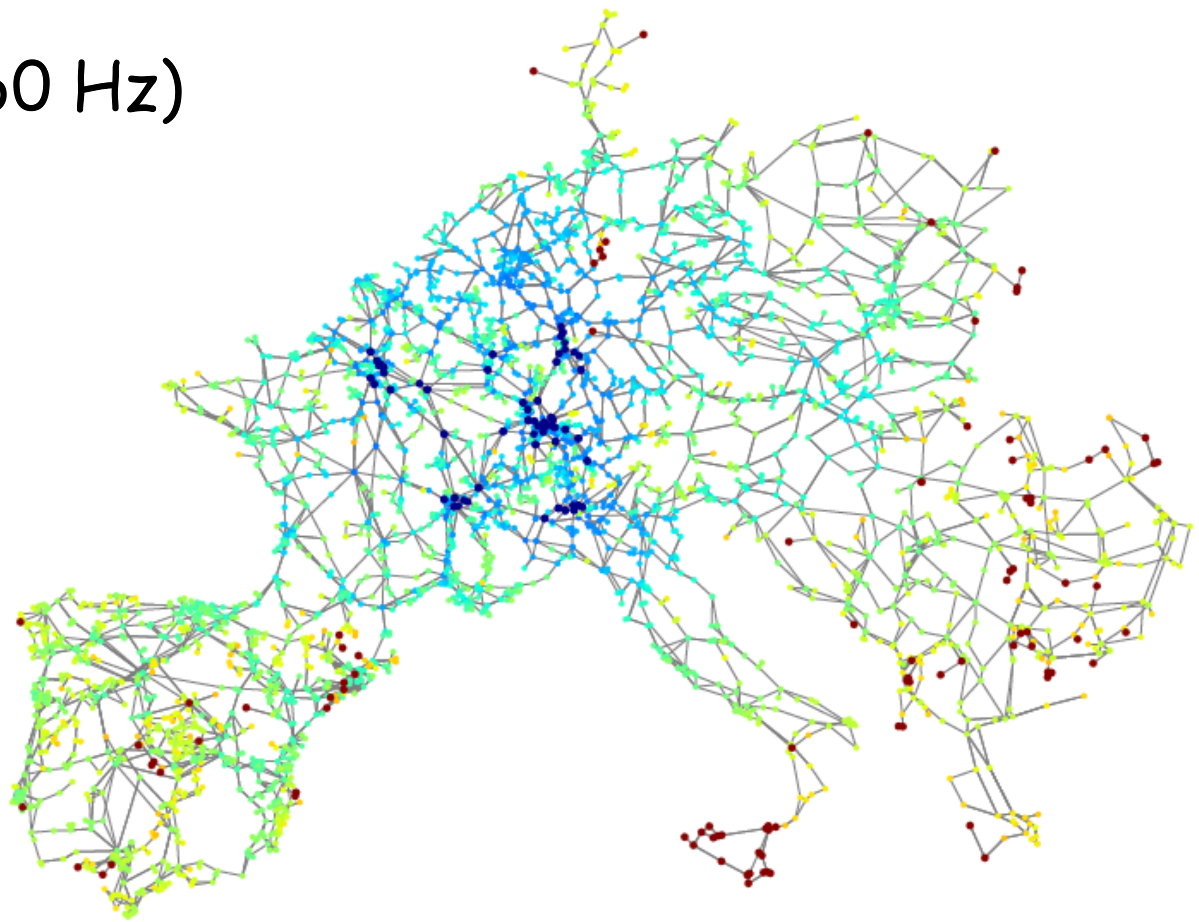
A bit of electric power engineering

Dynamics: swing Eqs. (neglect voltage variations from now on)

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum \left[B_{ij}\sin(\theta_i - \theta_j) + G_{ij}\cos(\theta_i - \theta_j)\right]$$

- i : node/bus index
- θ_i : voltage angle (rotating frame @ 50/60 Hz)
- P>0 : production
- P<0 : consumption
- I : inertia ~ rot. kinetic energy
- D : damping ~ control
- Admittance : y = g + i b;

  $G = g\,V_0$ $B = b\,V_0$



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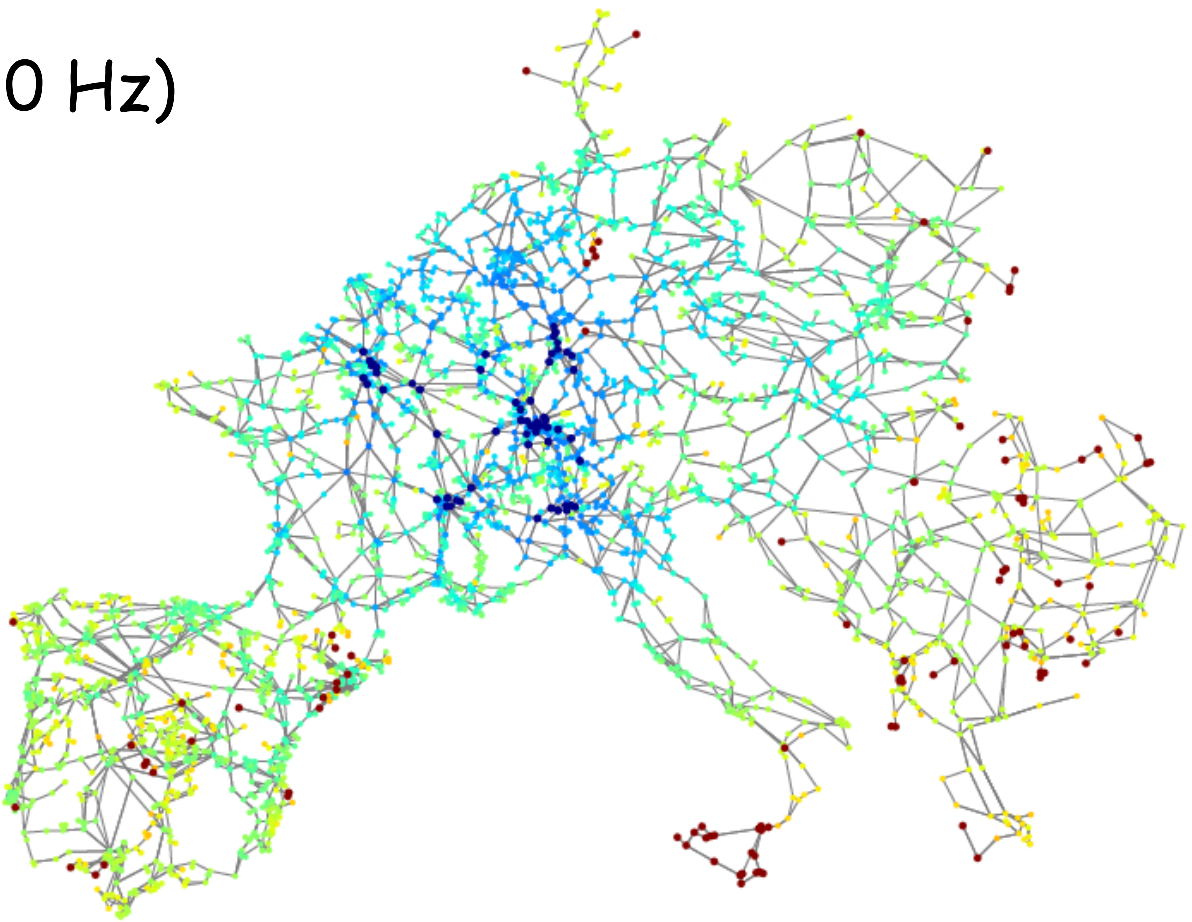
  $G = g\, V_0^2$ $\quad$ $B = b\, V_0^2$



High to very high voltage approximation
G/B < 0.1 -> neglect G

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum B_{ij}\sin(\theta_i - \theta_j)$$

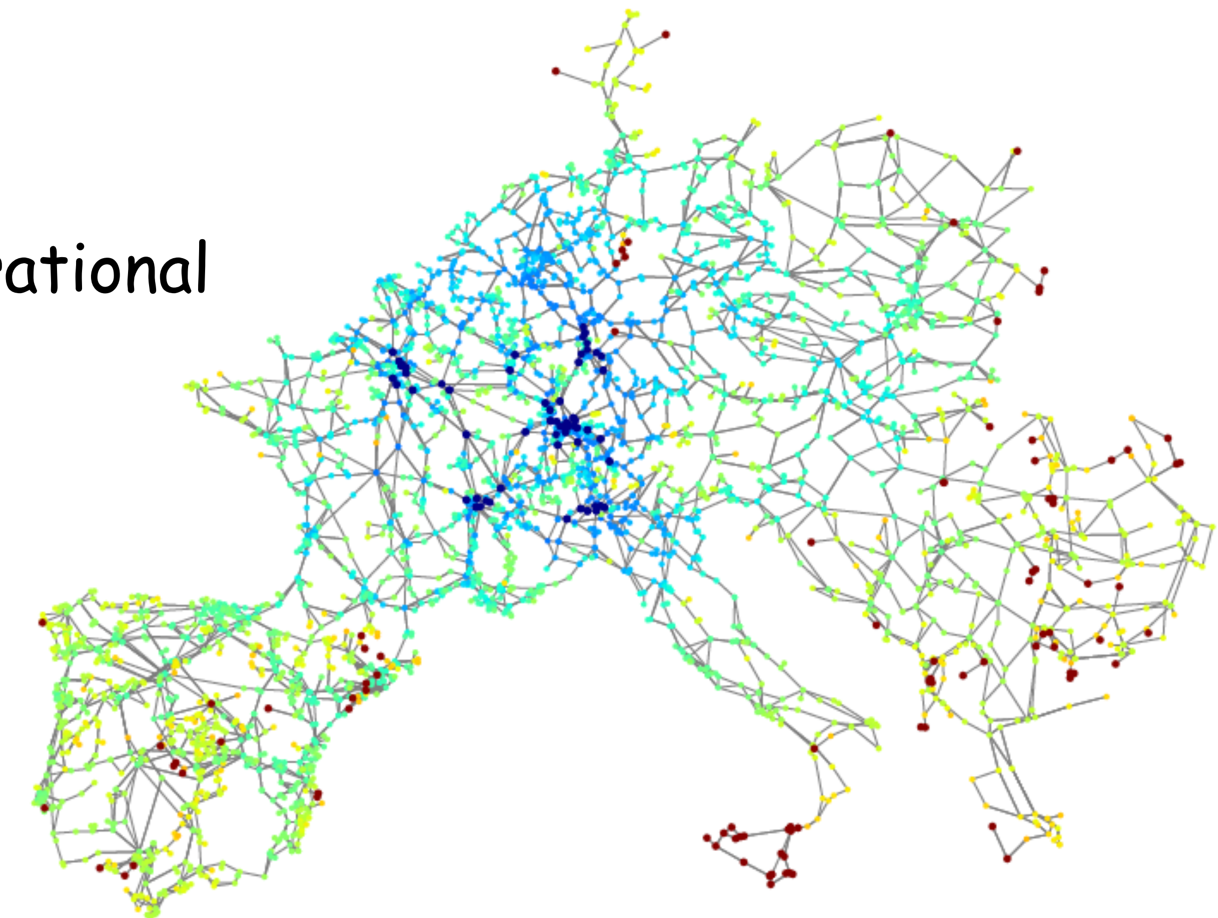
A bit of electric power engineering

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum B_{ij}\sin(\theta_i - \theta_j) \quad (*)$$

We are interested in

a) the synchronous fixed-points of (*) - operational states of the power grid

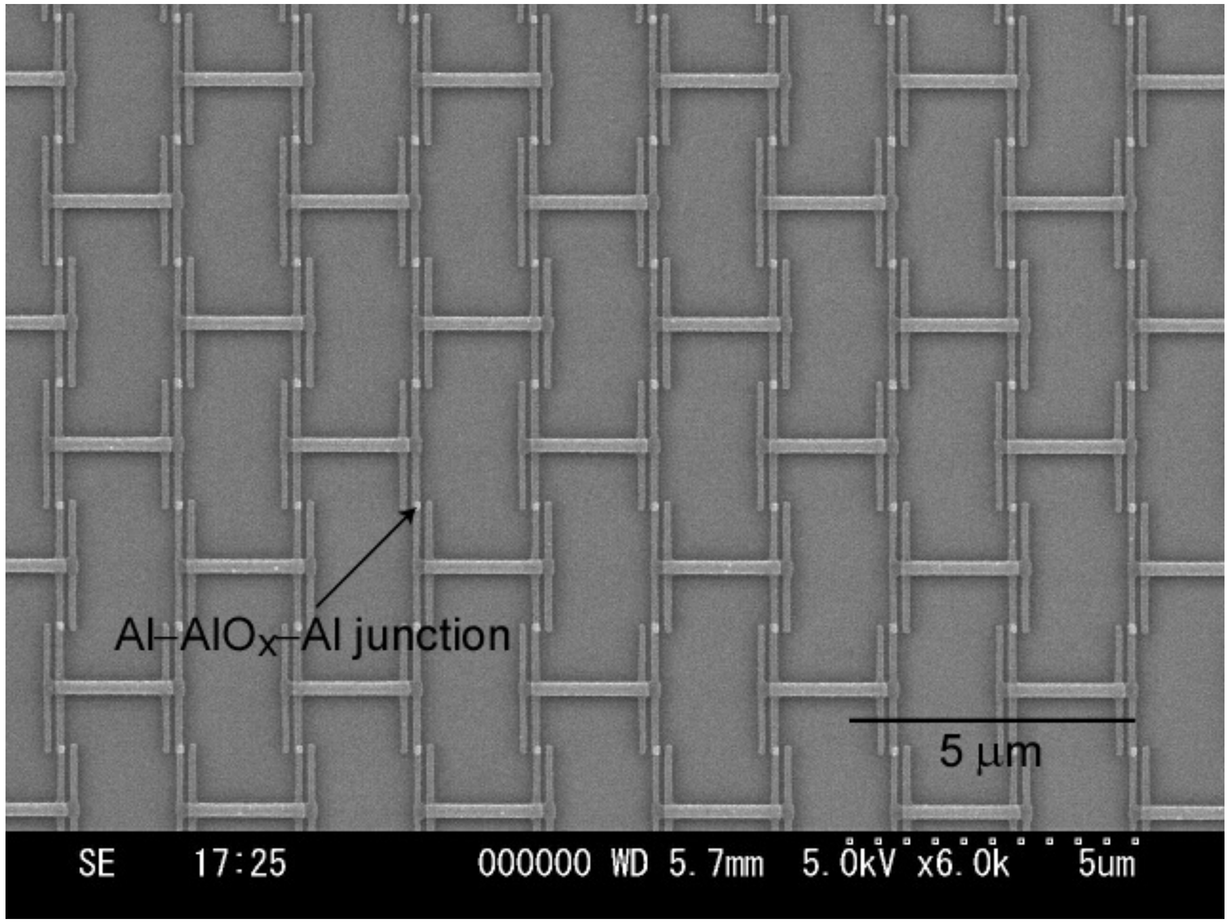
$$P_i = \sum_j B_{ij}\sin(\theta_i - \theta_j)$$



b) their stability - under specific or average disturbances (=local vulnerabilities vs. global robustness)

Sync : Kuramoto (1975); Strogatz "Sync: The Emerging Science of Spontaneous Order" (2004)

Synchronous fixed points vs. Josephson junctions

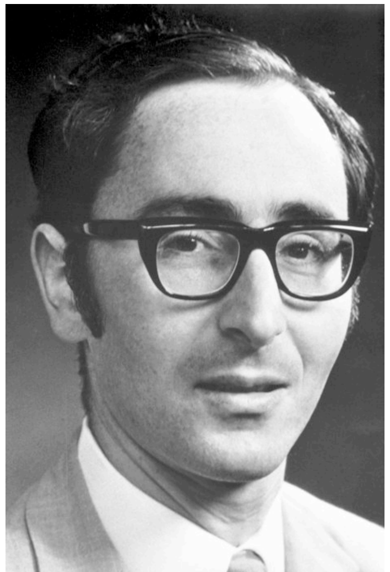


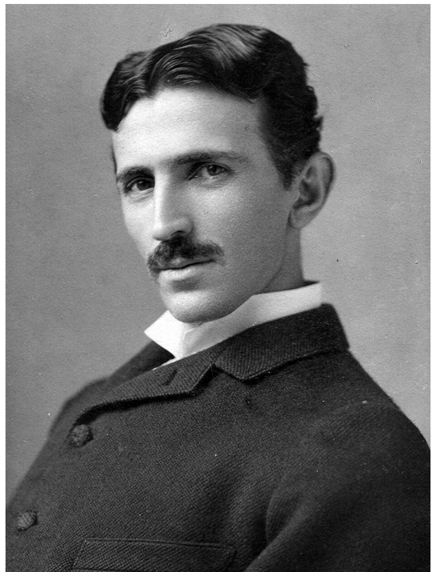
Josephson current

$$I_{ij} = I_c \sin(\theta_j - \theta_i)$$

AC transmitted power

$$P_{ij} = B_{ij} \sin(\theta_i - \theta_j)$$





Superconductivity vs. AC electric power grids !

	Superconductor	high voltage AC power grid
State	$\Psi(\mathbf{x}) = \lvert\Psi(\mathbf{x})\rvert e^{i\theta(\mathbf{x})}$	$V_i = \lvert V_i\rvert e^{i\theta_i}$
Current / power flow	$I_{ij} = I_c \sin(\theta_i - \theta_j)$ Josephson current	$P_{ij} \simeq B_{ij} \sin(\theta_i - \theta_j)$ Power flow; lossless line approx.
winding # $q=\Sigma_i \lvert\theta_{i+1}-\theta_i\rvert/2\pi$	Flux quantization Persistent currents	Circulating loop flows

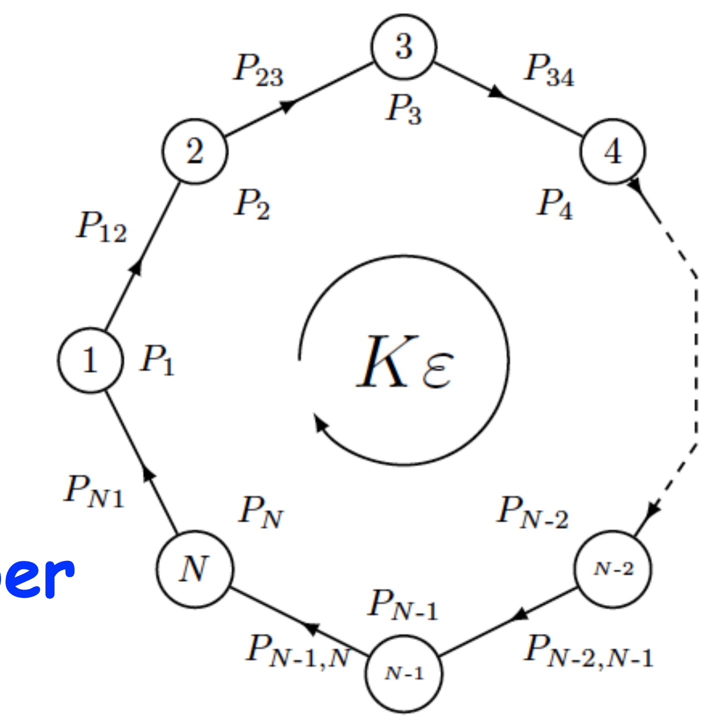
Circulating loop flows

*Thm: Different solutions to the following power-flow equation

$$P_i = \sum_j B_{ij} \sin(\theta_i - \theta_j)$$

may differ only by circulating loop current(s) **in any network**

Dörfler, Chertkov, Bullo, PNAS '13; Delabays, Coletta and PJ, JMP '16



*Voltage angle uniquely defined

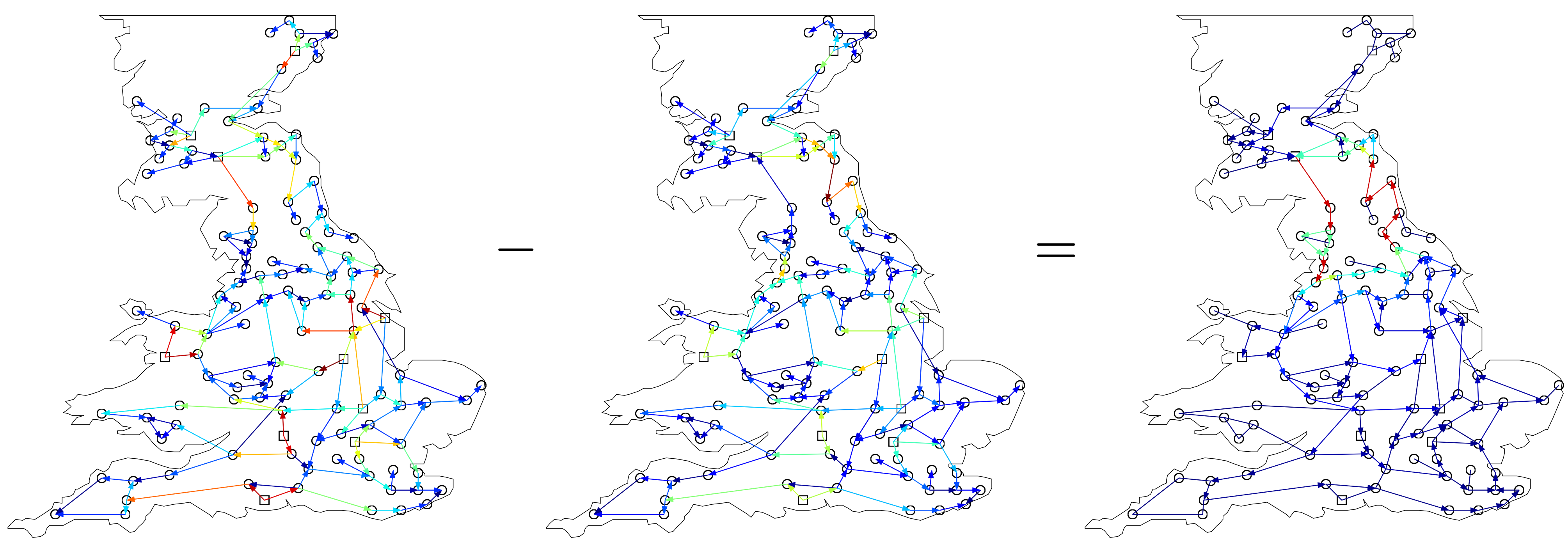
→ $q=\sum_i|\theta_{i+1}-\theta_i|_{2\pi}/2\pi \quad \in \mathbb{Z}$ **~topological winding number**

→ discretization of these loop currents **~vortex flows**

Janssens and Kamagate '03

→ number of stable solutions ~ number of possible vortex flows

Delabays, Coletta and PJ, JMP '16, JMP '17; Coletta, Delabays, Adagideli and PJ, NJP '16,;Delabays, Tyloo and PJ, Chaos '17



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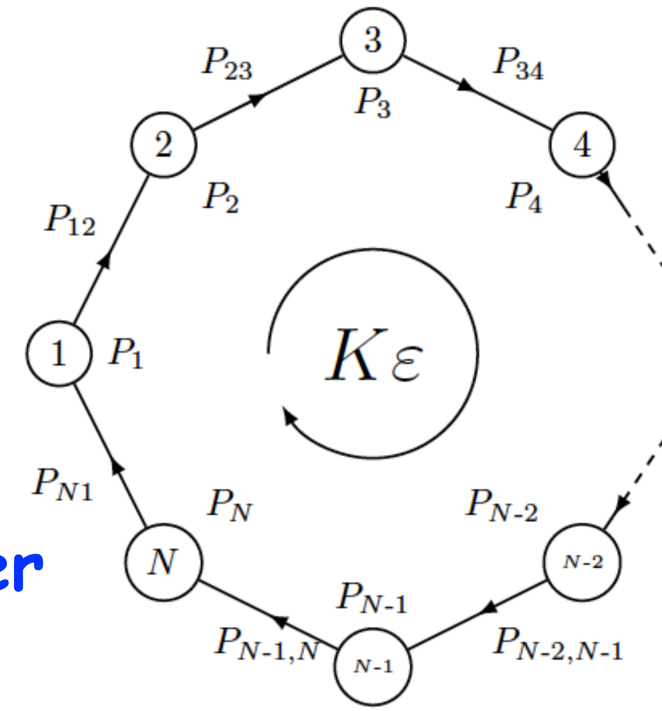

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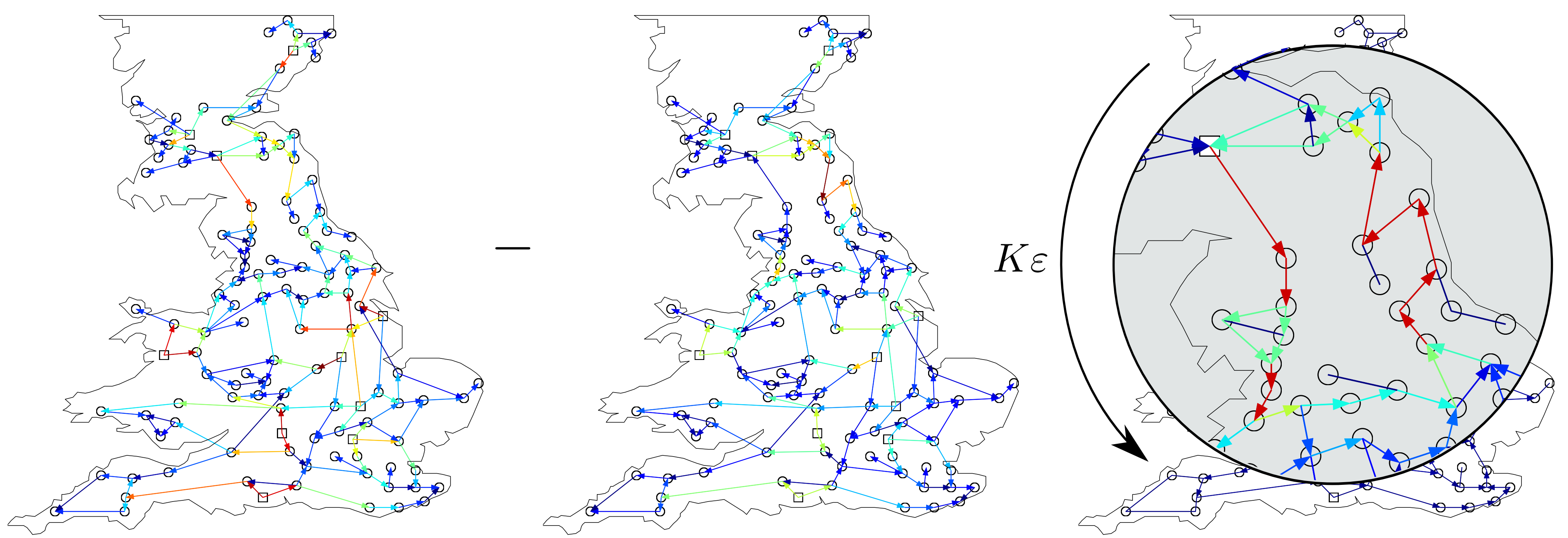

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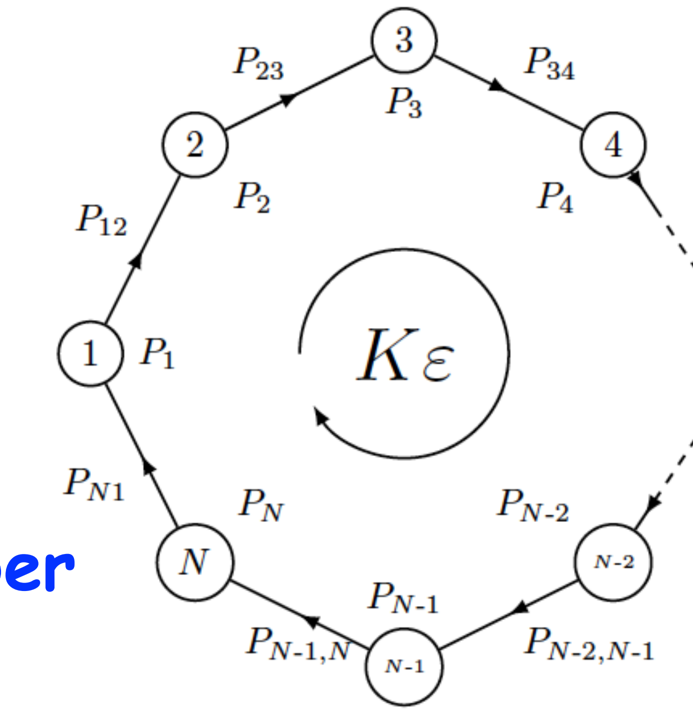

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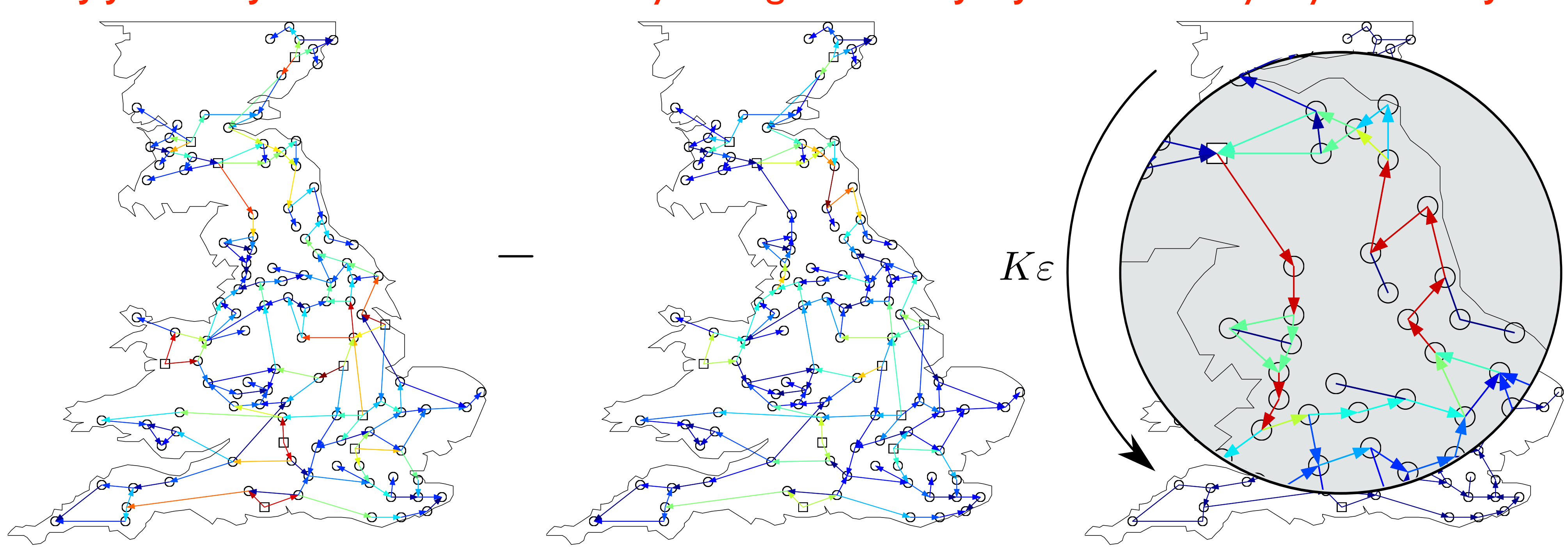
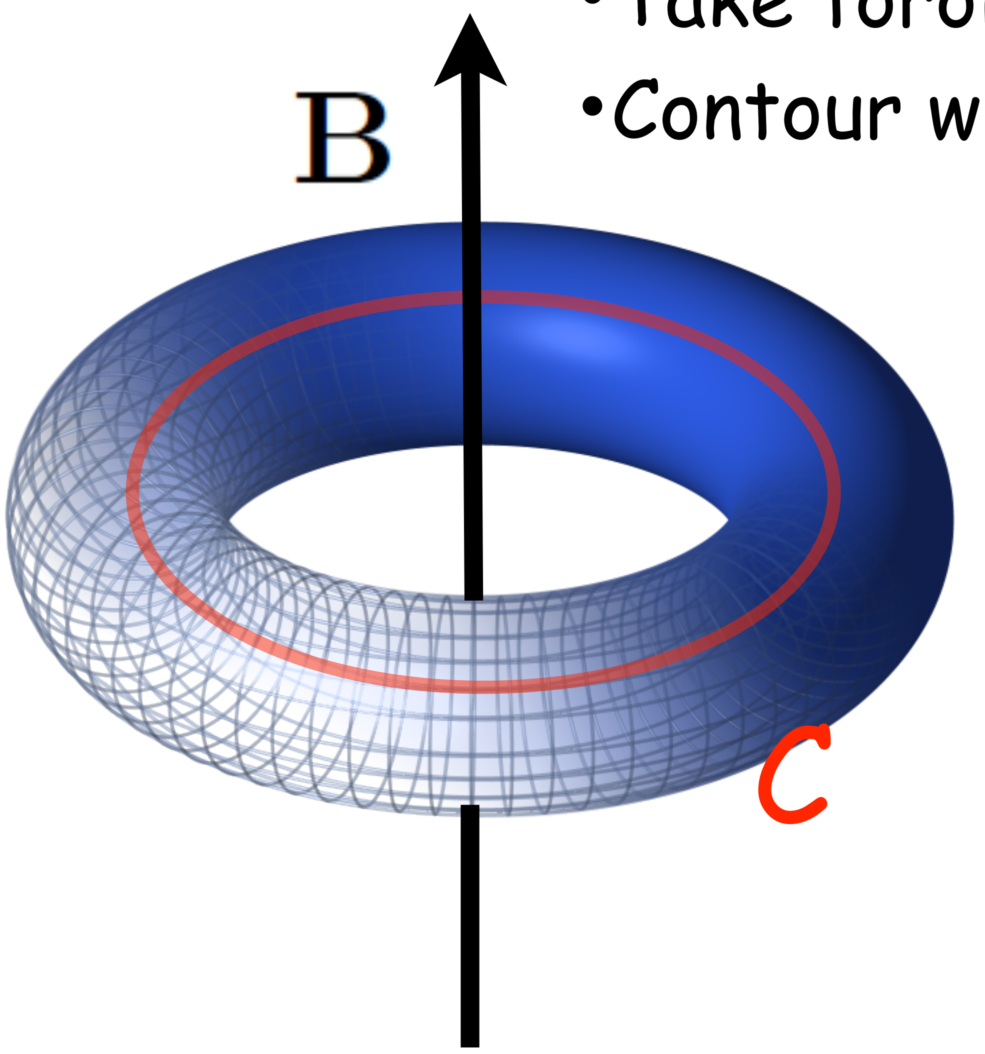


Topological quantum number : flux quantization with SC

- Landau theory of superconductivity - macroscopic wavefunction

$$\psi(\mathbf{x}) = |\psi(\mathbf{x})| e^{i\phi(\mathbf{x})}$$

- Gauge-invariant current $\mathbf{J}_s = \frac{e\hbar}{2m} n_s \left(\nabla\phi - \frac{2e}{\hbar}\mathbf{A} \right)$



- Take toroidal SC pierced by B-field
- Contour well inside SC : Meissner effect

$$\mathbf{B}\big|_C = \mathbf{J}_s\big|_C \equiv 0$$

-> $\oint_C \mathbf{A}d\mathbf{l} = \int\int_{\partial C} \mathbf{B}d\mathbf{f} = \varphi$

-> $\oint_C \mathbf{A}d\mathbf{l} = \frac{\hbar}{2e}(\phi_+ - \phi_-) = m\frac{h}{2e}$

-> flux quantization (winding number) $\varphi = m\varphi_0$

Exps.: Deaver and Fairbanks '61 (Sn cylinders); Gough et al. '87 (high Tc)

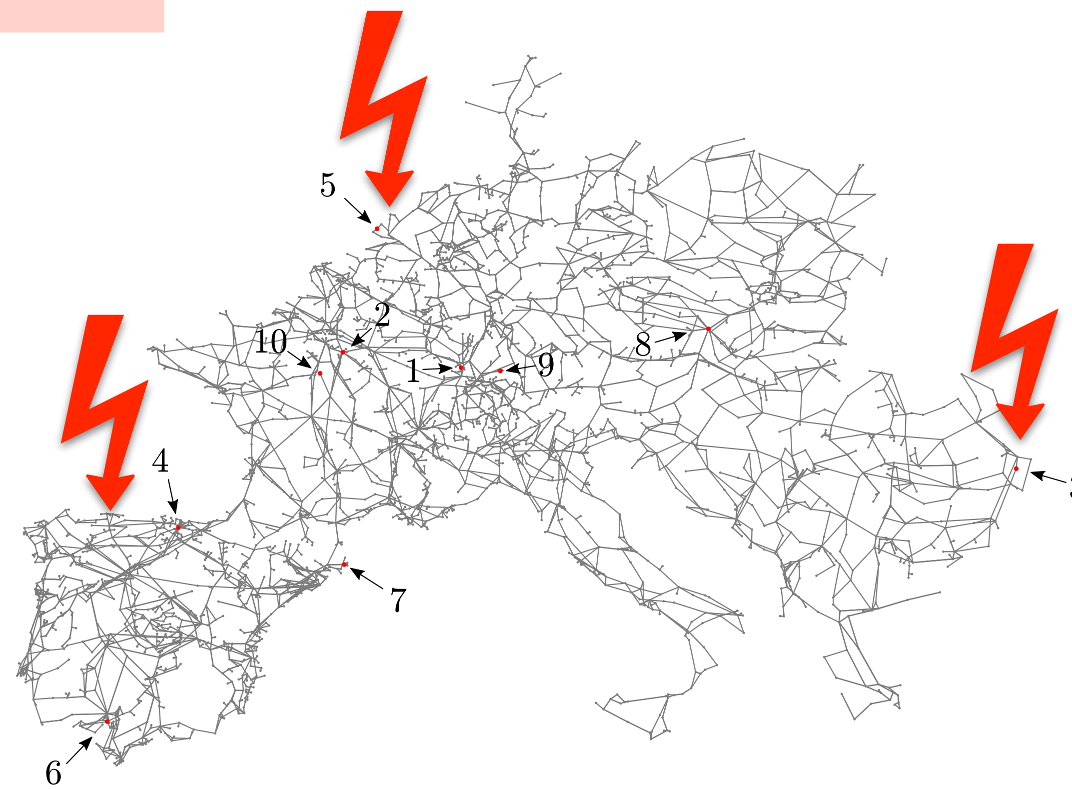
Nodal noise disturbance

$$I_i\ddot{\theta} + D_i\dot{\theta}_i = P_i - \sum [B_{ij}\sin(\theta_i - \theta_j)$$

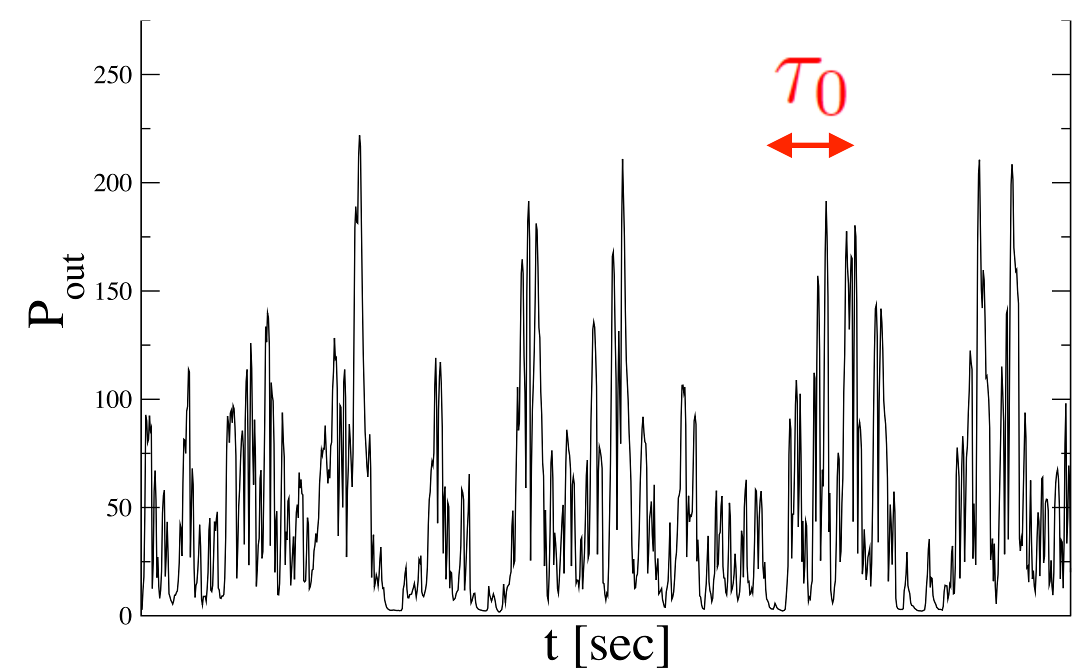
$$P_i(t) = P_i^{(0)} + \delta P_i(t)$$

$$\langle \delta P_i(t) \rangle = 0$$

$$\langle \delta P_i(t_1)\delta P_j(t_2) \rangle = \delta P_0^2 \delta_{ij} e^{-|t_1 - t_2|/\tau_0}$$



- No spatial correlation
- Characteristic time τ_0



Tyloo, Coletta and PJ, PRL '18
Tyloo, Pagnier and PJ, submitted

Nodal noise disturbance

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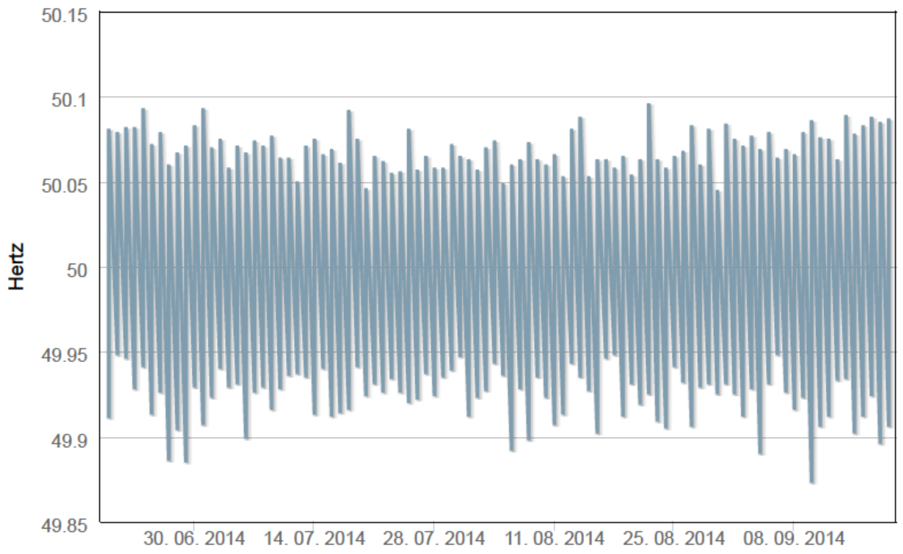

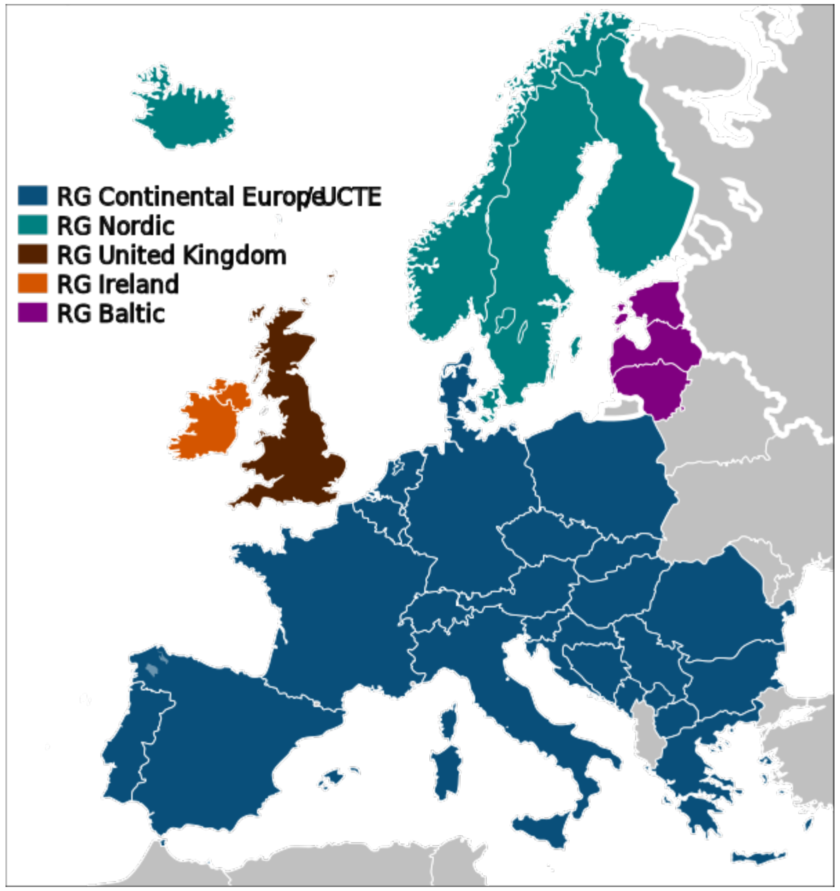

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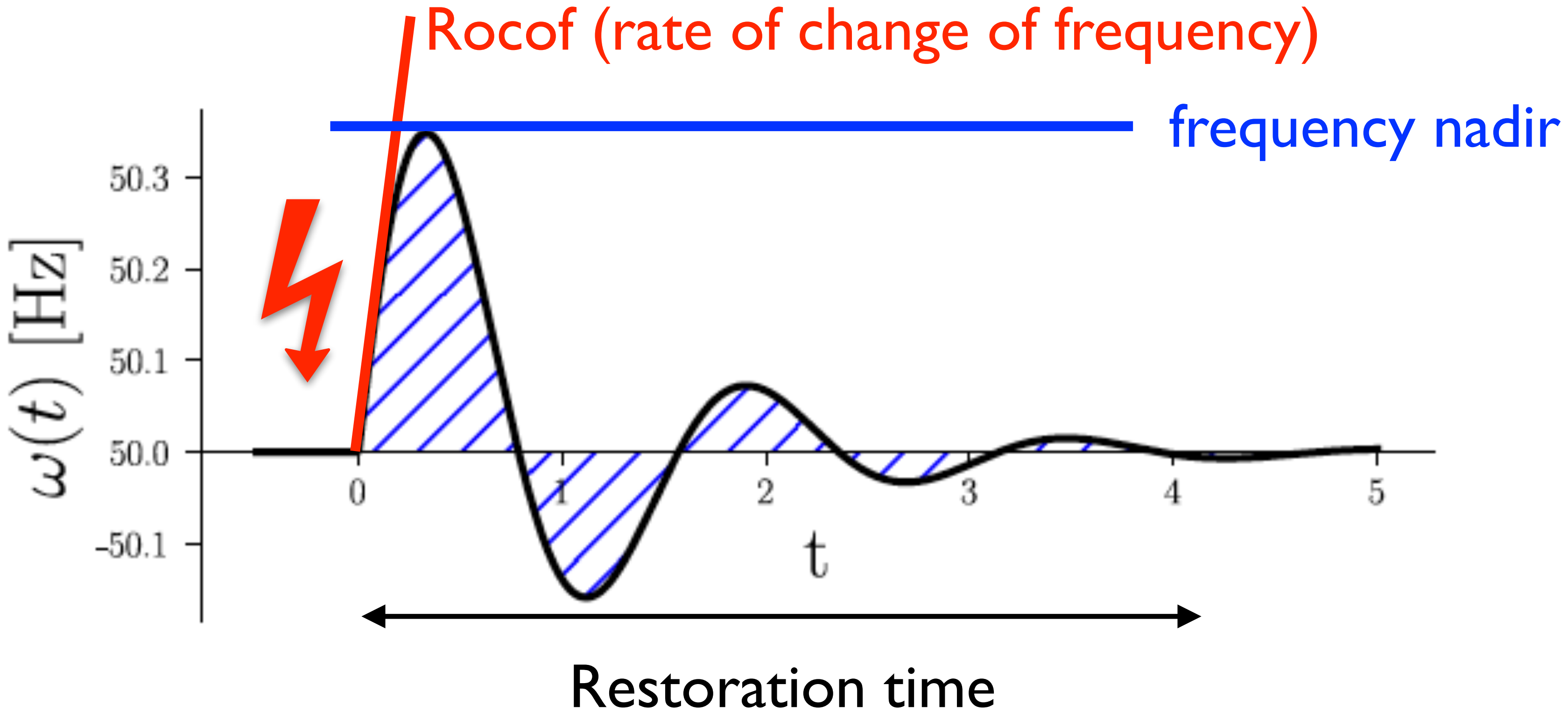

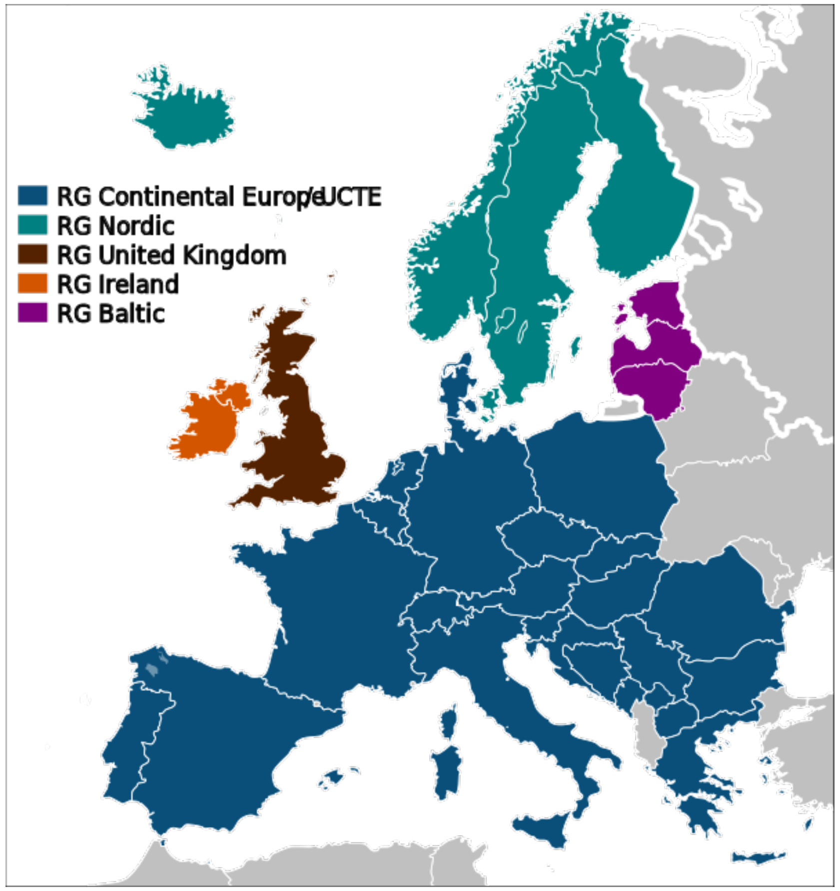

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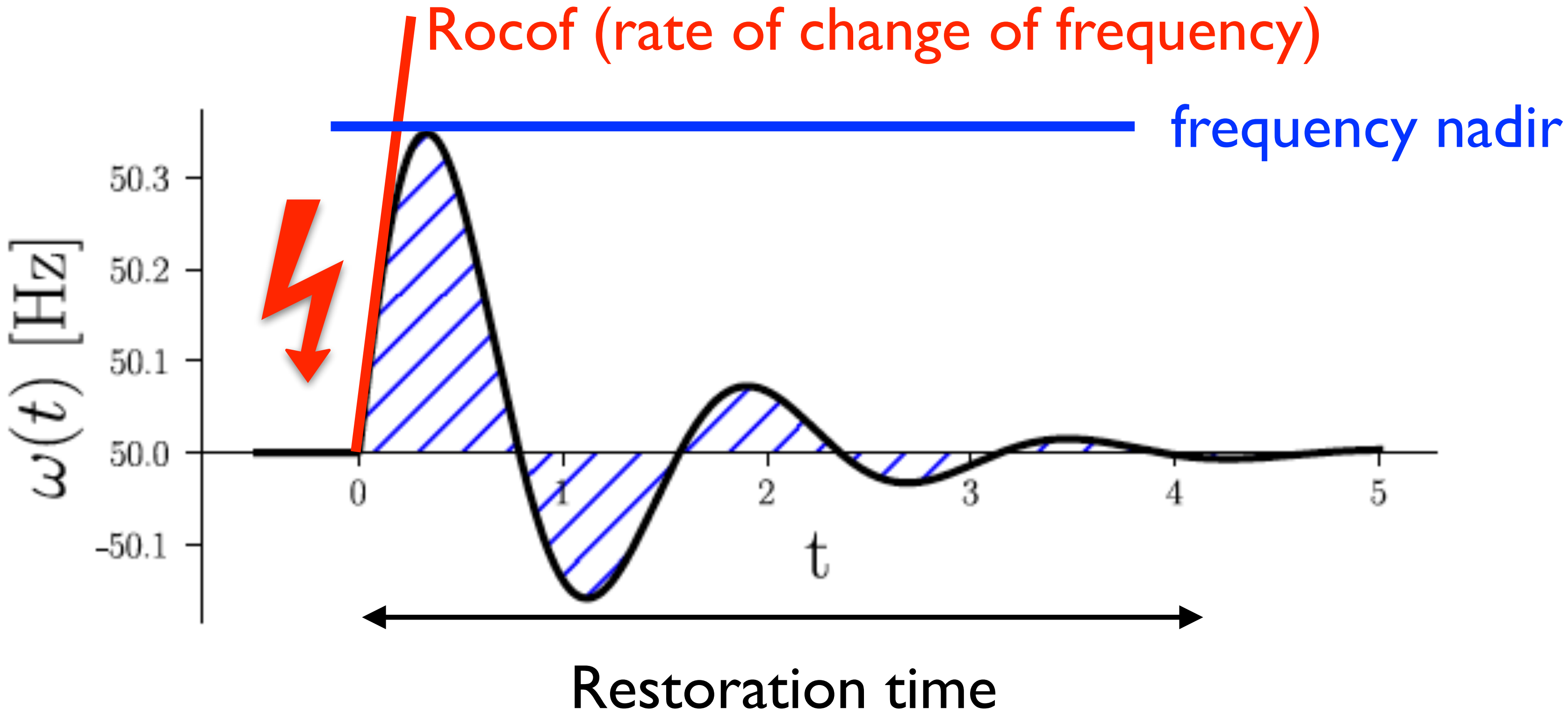
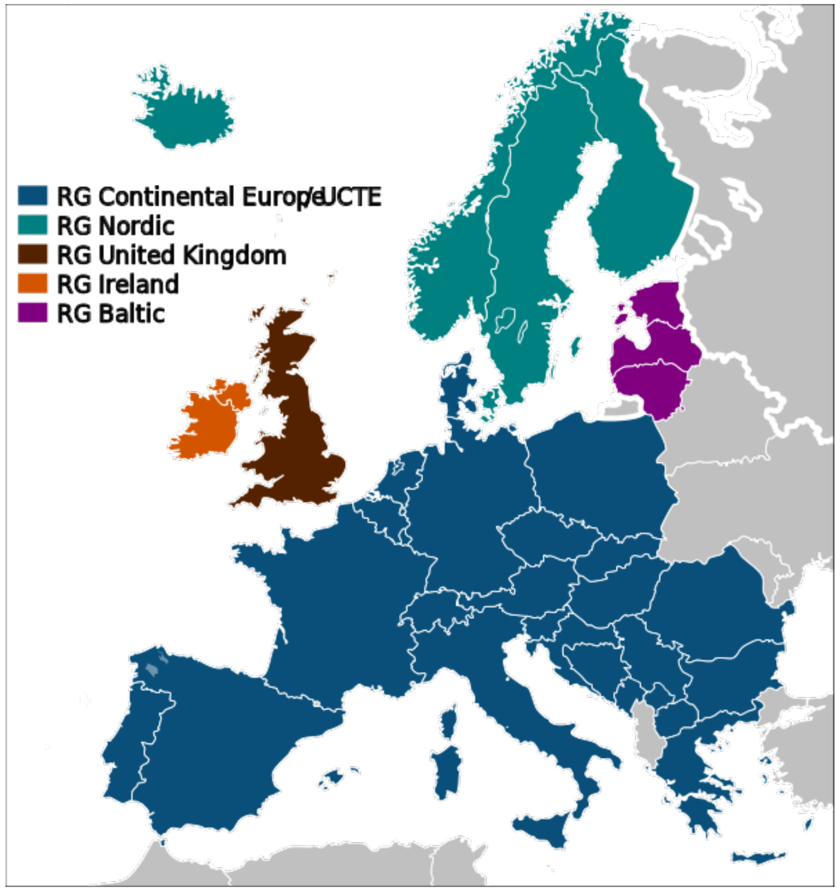




Our performance measures

$$\mathcal{P}_1(T) = \int_0^T \mathrm{d}t\, \delta\boldsymbol{\theta}^2(t)$$

$$\mathcal{P}_2(T) = \int_0^T \mathrm{d}t\, \delta\dot{\boldsymbol{\theta}}^2(t)$$

Take limit $T \to \infty$ when possible
Divide by T otherwise

Power grid with fluctuating feed-in

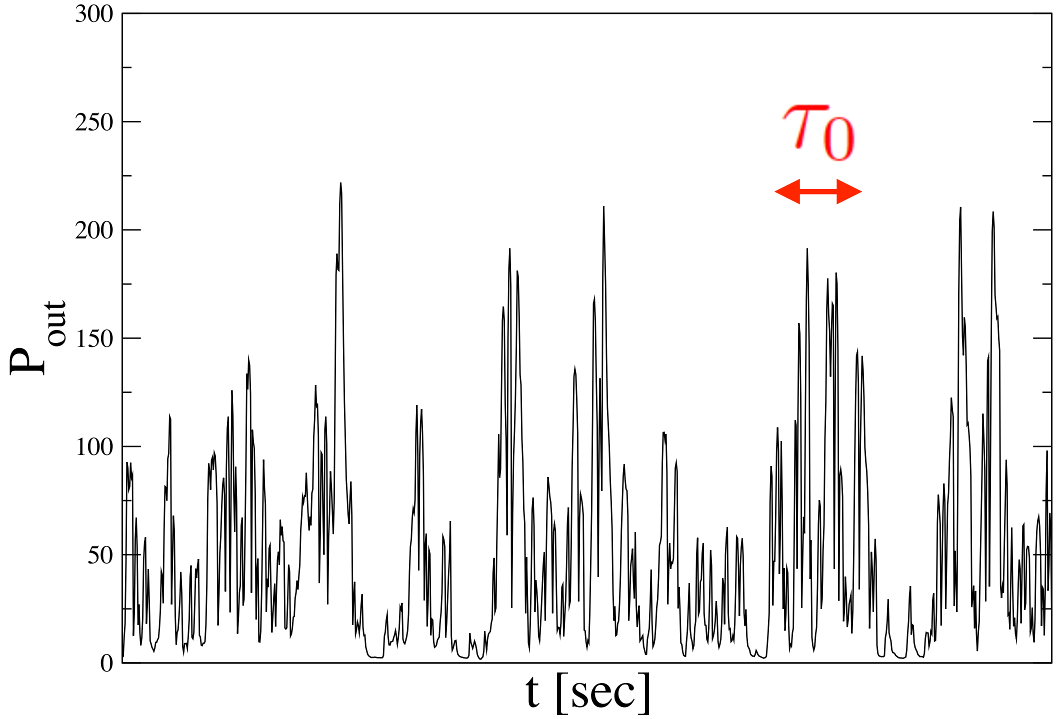
Angle dynamics

$$I_i \ddot{\theta} + D_i \dot{\theta}_i = P_i - \sum [B_{ij} \sin(\theta_i - \theta_j)$$

$$P_i(t) = P_i^{(0)} + \delta P_i(t)$$

$$\theta_i(t) = \theta_i^{(0)} + \delta\theta_i(t)$$

$$P_i^{(0)} = \sum B_{ij} \sin(\theta_i^{(0)} - \theta_j^{(0)})$$



Can one characterize $\delta\theta_i(t)$ given $\delta P_i(t)$?

A: (i) linearize the dynamics about a fixed-point solution

$$\delta\dot{\vec{\theta}} = \delta\vec{P} + \mathbb{L}(\vec{\theta}^{(0)})\, \delta\vec{\theta}$$

(ii) spectral decomposition, i.e. expand angles over eigenmodes of stability matrix

→ *get equation for coefficients of expansion !*

$$\delta\vec{\theta}(t) = \sum_\alpha c_\alpha(t) \vec{\phi}_\alpha$$

$$\mathbb{L}\vec{\phi}_\alpha = \lambda_\alpha \vec{\phi}_\alpha$$

Stability / weighted Laplacian matrix

Linearized dynamics about fixed point

$$\delta\dot{\vec{\theta}} = \delta\vec{P} + \mathbb{L}(\vec{\theta}^{(0)})\,\delta\vec{\theta} \qquad\qquad \mathbb{L}\vec{\phi}_\alpha = \lambda_\alpha\vec{\phi}_\alpha$$

Stability matrix

$$[\mathbb{L}(\vec{\theta}^{(0)})]_{ij} = -B_{ij}\cos(\theta_i^{(0)} - \theta_j^{(0)}) \qquad i \neq j$$

$$[\mathbb{L}(\vec{\theta}^{(0)})]_{ii} = \sum_k B_{ik}\cos(\theta_i^{(0)} - \theta_k^{(0)})$$

General property + special case

- It's a Laplacian matrix

$$\lambda_1 = 0 \qquad\qquad \vec{\phi}_1 = (N^{-1/2}, N^{-1/2}, \ldots, N^{-1/2})$$

- Limit of no flow -> graph Laplacian

$$\theta_i^{(0)} \equiv 0 \qquad\qquad \mathbb{L}(\vec{\theta}^{(0)}) \to \mathbb{L}_0$$

$$\mathcal{P}_1(T) = \lim_{T\to\infty} T^{-1} \int_0^T \mathrm{d}t\, \delta\boldsymbol{\theta}^2(t) = \delta P_0^2 \sum_{\alpha>2} \frac{\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2 (I/D + \tau_0)}{\lambda_\alpha(\lambda_\alpha\tau_0 + D + I\tau_0^{-1})}$$

$$\mathcal{P}_2(T) = \lim_{T\to\infty} T^{-1} \int_0^T \mathrm{d}t\, \delta\dot{\boldsymbol{\theta}}^2(t) = \delta P_0^2 \sum_{\alpha\geq 2} \frac{\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2}{D(\lambda_\alpha\tau_0 + D + I\tau_0^{-1})}$$

Local vulnerabilities

$$\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2 \to |\phi_{k,\alpha}|^2$$

$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1 \cong \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

$$\mathcal{P}_2 \cong \frac{\delta P_0^2 \tau_0}{DI} \frac{(n-1)}{n}$$

$\lambda_\alpha\tau_0 \gg D$

$$\mathcal{P}_1 \cong \delta P_0^2 \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha^2}$$

$$\mathcal{P}_2 \cong \frac{\delta P_0^2}{D\tau_0} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

Global robustness

$$\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2 \to 1$$

$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1(T) = \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{1}{\lambda_\alpha}$$

$$\mathcal{P}_2(T) = \frac{\delta P_0^2 \tau_0}{D(D + I\tau_0^{-1})} \frac{n-1}{n}$$

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Local vulnerabilities

$$\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2 \to |\phi_{k,\alpha}|^2$$

$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1 \cong \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

They all depend on

$$\sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha^p}$$!!!

$\lambda_\alpha\tau_0$ >

$$\mathcal{P}_2 \cong \frac{\delta P_0^2}{D\tau_0} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

Global robustness

$$\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2 \to 1$$

$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1(T) = \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{1}{\lambda_\alpha}$$

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$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1 \cong \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

$\lambda_\alpha\tau_0 >$ … $1)$ … $\phi_{\alpha,k}$ … λ_α^2 … $\alpha\geq 2$

$$\mathcal{P}_2 \cong \frac{\delta P_0^2}{D\tau_0} \sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha}$$

They all depend on

$$\sum_{\alpha\geq 2} \frac{\phi_{\alpha,k}^2}{\lambda_\alpha^p} \quad !!!$$

Global robustness

$$\sum_{\mathrm{noisy}\,i} |\phi_{i,\alpha}|^2 \to 1$$

$\lambda_\alpha\tau_0 \ll D$

$$\mathcal{P}_1(T) = \frac{\delta P_0^2 \tau_0}{D} \sum_{\alpha\geq 2} \frac{1}{\lambda_\alpha}$$

$\lambda_\alpha\tau$ … $\frac{n-1}{n}$ … $\frac{1}{\lambda_\alpha^2}$ … ≥ 2

$$\mathcal{P}_2(T) = \frac{\delta P_0^2}{D\tau_0} \sum_{\alpha\geq 2} \frac{1}{\lambda_\alpha}$$

They all depend on

$$\sum_{\alpha\geq 2} \frac{1}{\lambda_\alpha^p} \quad !!!$$

Results : (i) global robustness vs. Kirchhoff indices

PHYSICAL REVIEW LETTERS 120, 084101 (2018)

Robustness of Synchrony in Complex Networks and Generalized Kirchhoff Indices

M. Tyloo,[1,2] T. Coletta,[1] and Ph. Jacquod[1]
[1]*School of Engineering, University of Applied Sciences of Western Switzerland HES-SO, CH-1951 Sion, Switzerland*
[2]*Institute of Physics, EPF Lausanne, CH-1015 Lausanne, Switzerland*

Introduce "generalized Kirchhoff indices" $Kf_p = n \sum_{\alpha \geq 2} \lambda_\alpha^{-p}$

τ_0 is shortest time scale

$$\overline{\mathcal{P}}_1 \cong \frac{\delta P_0^2 \tau_0}{D n^2} Kf_1 \qquad \mathcal{P}_2 \cong \frac{\delta P_0^2 \tau_0}{DI} \frac{(n-1)}{n}$$

τ_0 is longest time scale

$$\overline{\mathcal{P}}_1 \cong \frac{\delta P_0^2}{n^2} Kf_2 \qquad \overline{\mathcal{P}}_2 \cong \frac{\delta P_0^2}{n^2 \tau_0} Kf_1$$

Kirchhof index Kf1 : Klein and Randic, JMC '93.

Results : (i) global robustness vs. Kirchhoff indices

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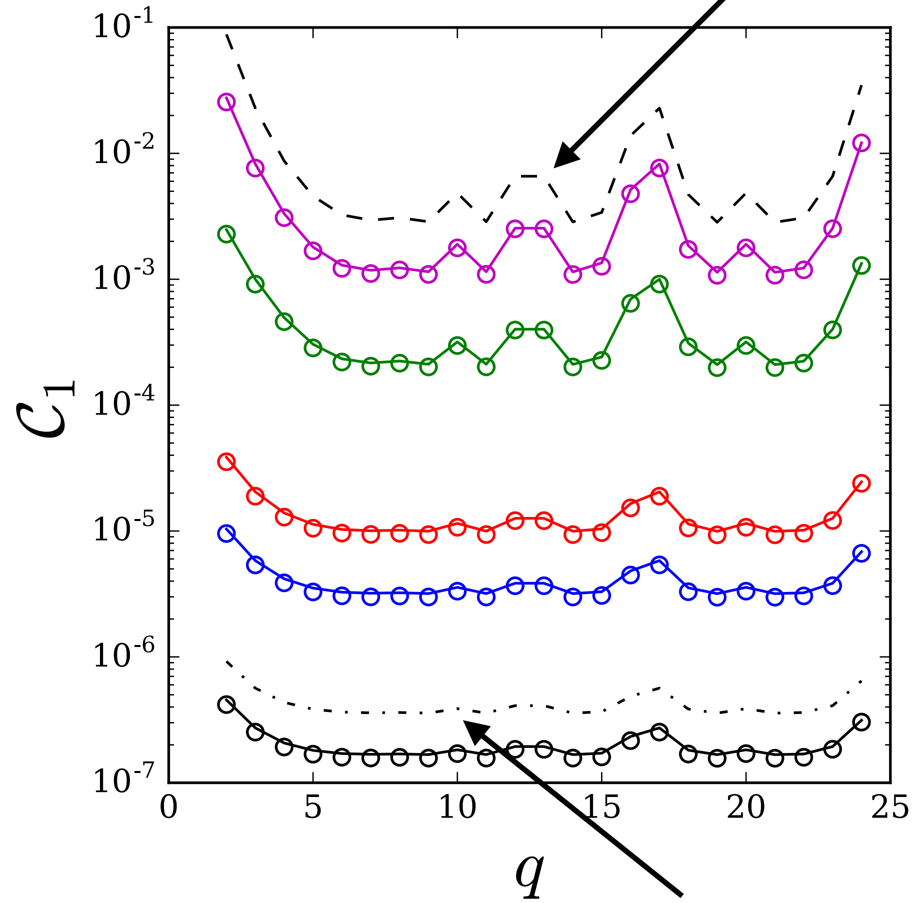
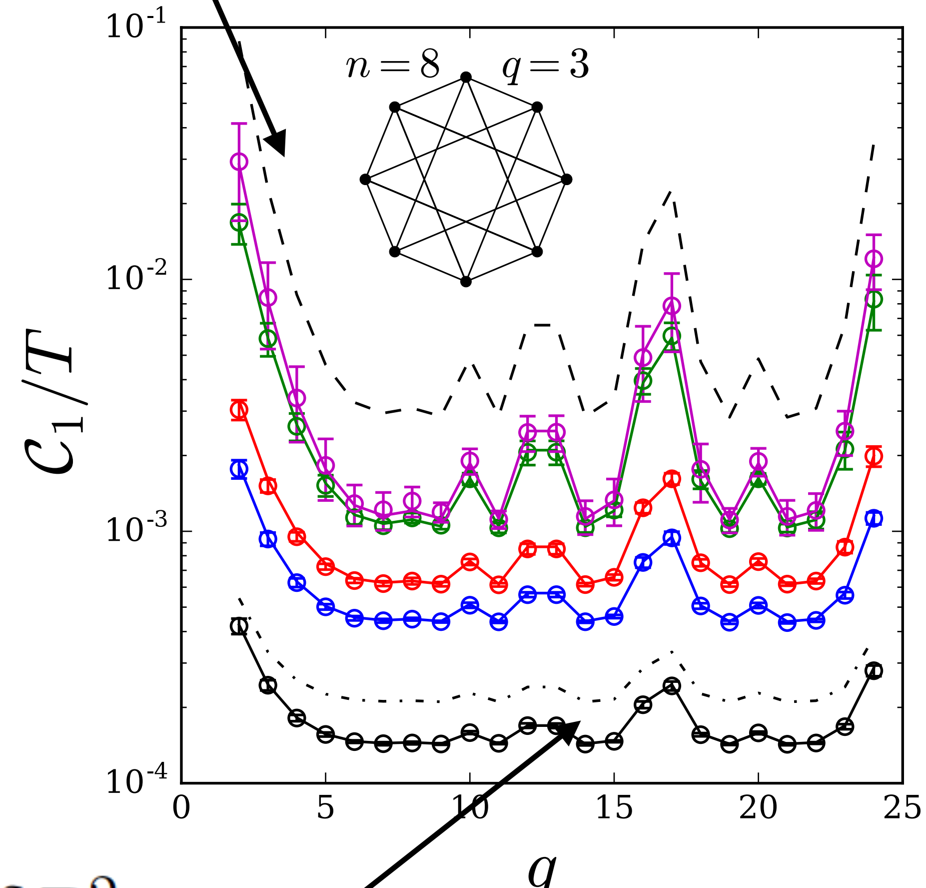
$$\overline{\mathcal{P}}_1 \cong \frac{\delta P_0^2}{n^2} Kf_2$$

Box disturbance

Noise disturbance





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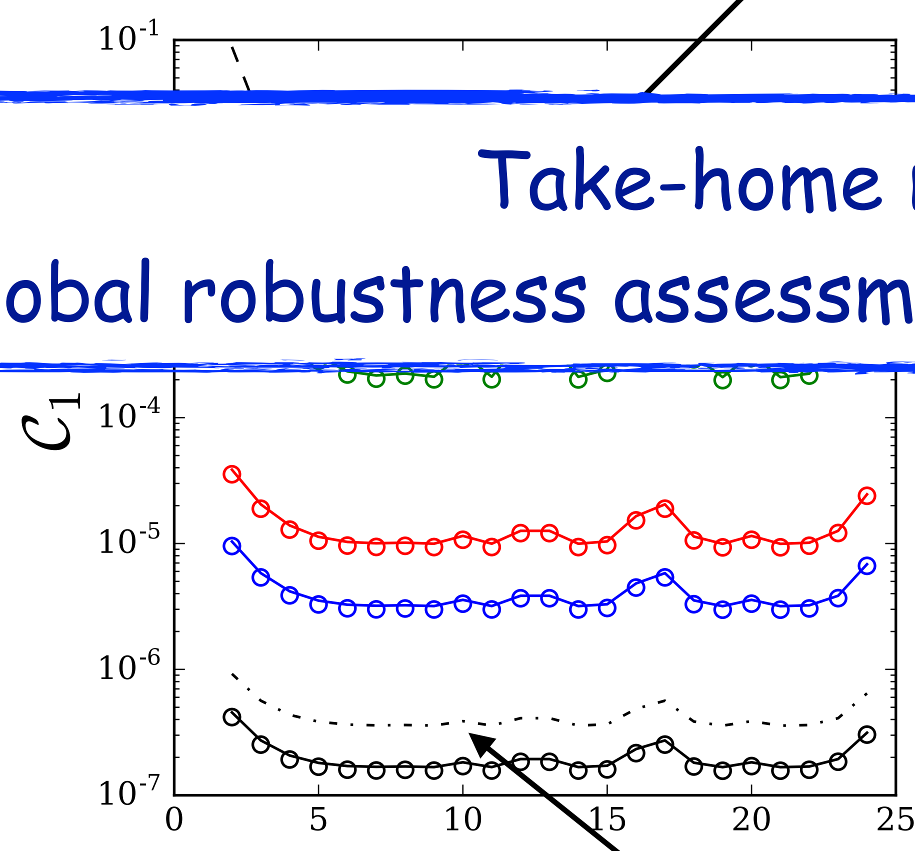

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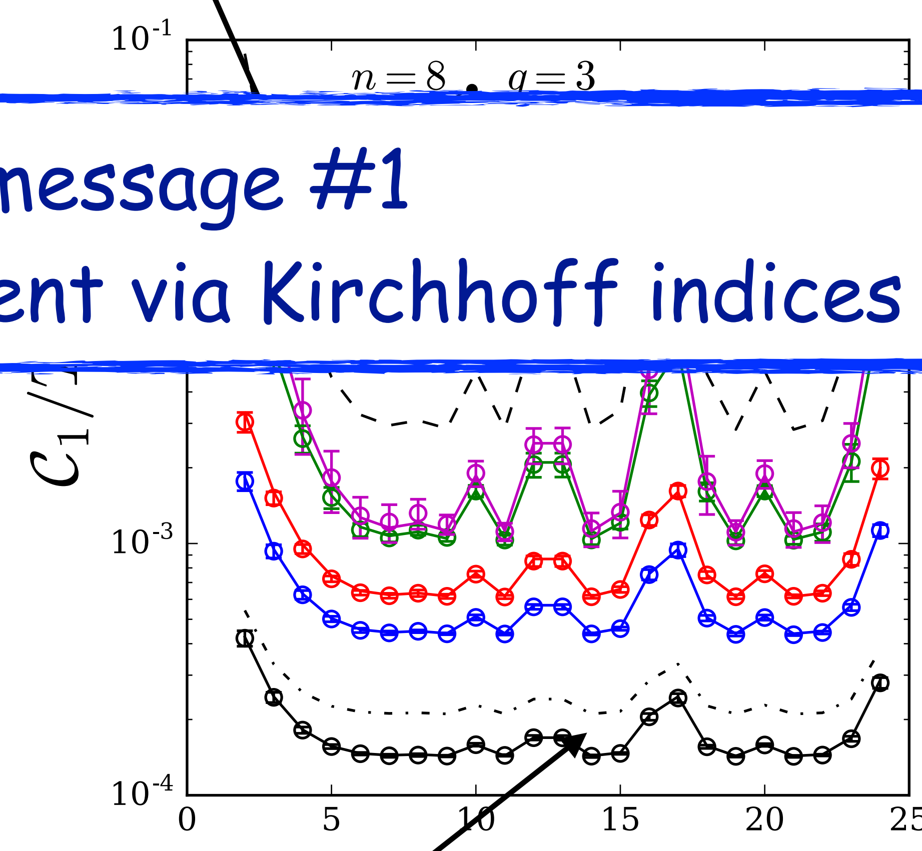

Take-home message #1

- Global robustness assessment via Kirchhoff indices

$$\overline{\mathcal{P}}_1 \cong \frac{\delta P_0^2 \tau_0}{D n^2} K f_1$$

Results : (ii) specific / local vulnerabilities

Resistance distance vs. Laplacian matrix (its pseudoinverse)

$$\Omega_{ik} = \mathbb{L}^{\dagger}_{ii} + \mathbb{L}^{\dagger}_{kk} - 2\mathbb{L}^{\dagger}_{ik} = \sum_{\alpha \geq 2} \frac{(\phi_{\alpha,i} - \phi_{\alpha,k})^2}{\lambda_\alpha}$$

~effective resistance between i and k, for equivalent network of resistors

$$\sum_{\alpha \geq 2} \frac{\phi^2_{\alpha,k}}{\lambda_\alpha} = \sum_i \Omega_{ik} - \frac{Kf_1}{n^2}$$

Tyloo, Pagnier and PJ, in preparation

Results : (ii) specific / local vulnerabilities

Resistance distance vs. Laplacian matrix (its pseudoinverse)

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~resistive centrality

$$C_i^{(1)} = \Big[n^{-1} \sum_j \Omega_{ij}\Big]^{-1} = \Big[\sum_{\alpha\geq 2} \frac{\phi^2_{\alpha,i}}{\lambda_\alpha} + n^{-2} Kf_1\Big]^{-1}$$

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~resistive centrality
for squared Laplacian

$$C_i^{(2)} = \left[\sum_{\alpha \geq 2} \frac{\phi^2_{\alpha,i}}{\lambda^2_\alpha} + n^{-2} Kf_2 \right]^{-1}$$

Tyloo, Pagnier and PJ, in preparation

Results : (ii) specific / local vulnerabilities

Resistance distance vs. Laplacian matrix (its pseudoinverse)

$$\Omega_{ik} = \mathbb{L}^{\dagger}_{ii} + \mathbb{L}^{\dagger}_{kk} - 2\mathbb{L}^{\dagger}_{ik} = \sum_{\alpha \geq 2} \frac{(\phi_{\alpha,i} - \phi_{\alpha,k})^2}{\lambda_\alpha}$$

~effective resistance between i and k, for equivalent network of resistors

$$\sum_{\alpha \geq 2} \frac{\phi^2_{\alpha,k}}{\lambda_\alpha} = \sum_i \Omega_{ik} - \frac{Kf}{n^2}$$

τ_0 is shortest time scale

$$\mathcal{P}_1 = \frac{\delta P_0^2 \tau_0}{D} \left(C_k^{(1)-1} - n^{-2} K f_1 \right) \qquad \mathcal{P}_2 \cong \frac{\delta P_0^2 \tau_0}{DI} \frac{(n-1)}{n}$$

τ_0 is longest time scale

$$\mathcal{P}_1 = \delta P_0^2 \left({C_k^{(2)}}^{-1} - n^{-2} K f_2 \right) \qquad \mathcal{P}_2 \cong \frac{\delta P_0^2}{D\tau_0} \left(C_k^{(1)-1} - n^{-2} K f_1 \right)$$

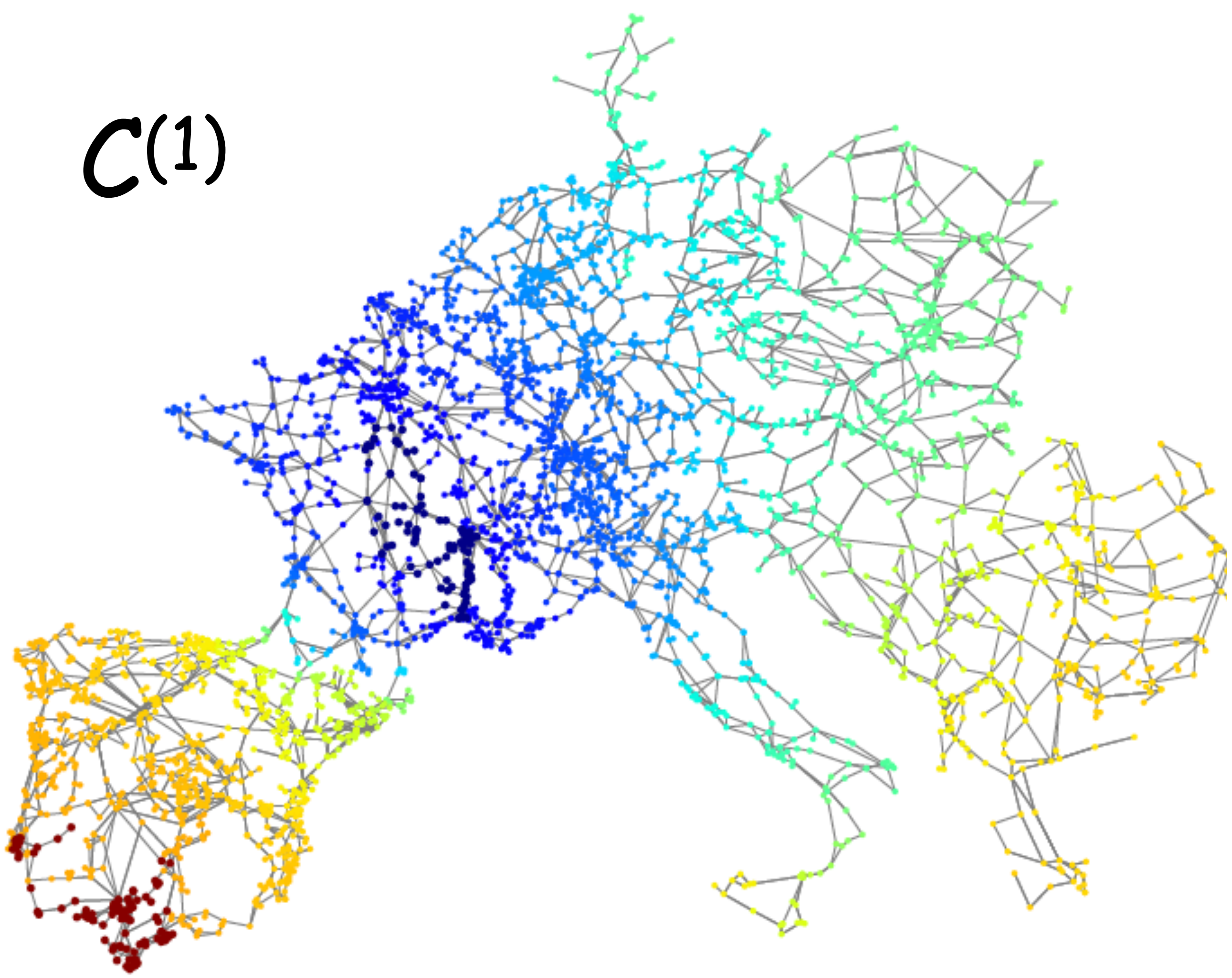
Tyloo, Pagnier and PJ, submitted

Results : (ii) specific / local vulnerabilities

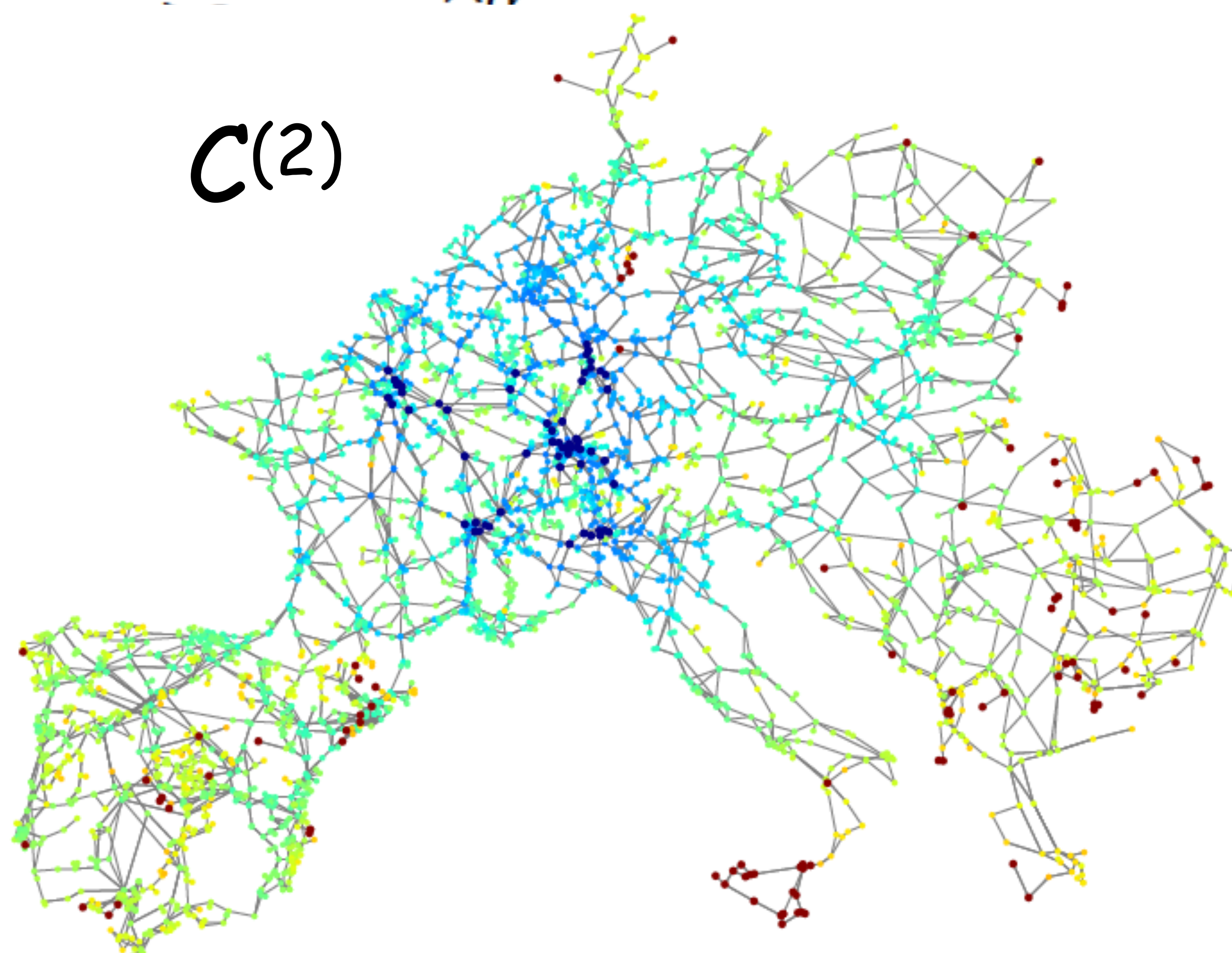
Resistance distance vs. Laplacian matrix (its pseudoinverse)

$$\Omega_{ik} = \mathbb{L}^{\dagger}_{ii} + \mathbb{L}^{\dagger}_{kk} - 2\mathbb{L}^{\dagger}_{ik} = \sum \frac{(\phi_{\alpha,i} - \phi_{\alpha,k})^2}{\lambda_\alpha}$$

$C^{(1)}$



$C^{(2)}$

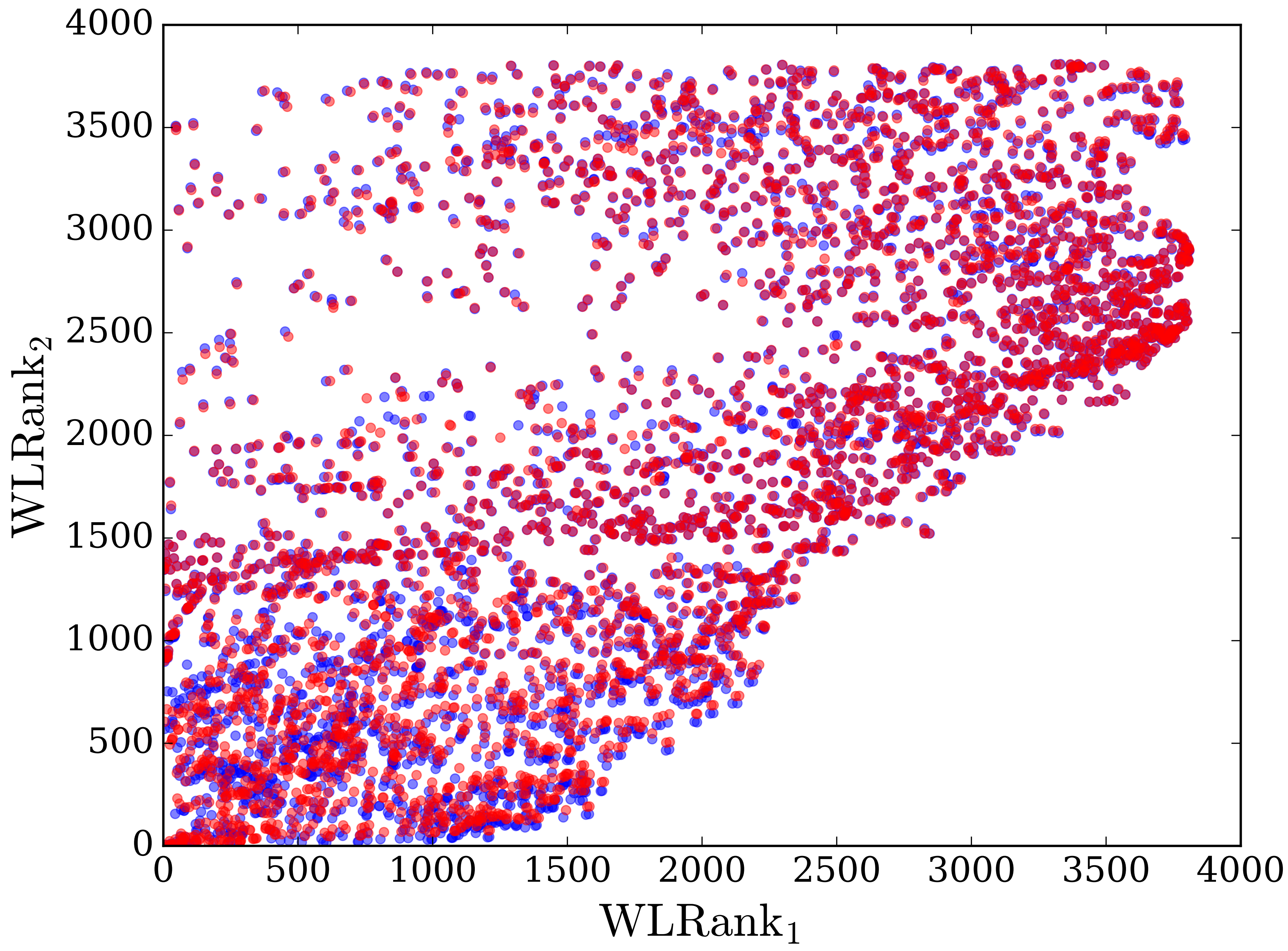


$$\mathcal{P}_1 = \delta P_0^2 \left({C_k^{(2)}}^{-1} - n^{-2} K f_2 \right) \qquad \mathcal{P}_2 \cong \frac{\delta P_0^2}{D \tau_0} \left(C_k^{(1)-1} - n^{-2} K f_1 \right)$$

Tyloo, Pagnier and PJ, submitted

Results : (ii) specific / local vulnerabilities

!!! Resulting ranking depends on performance measure of interest !!!



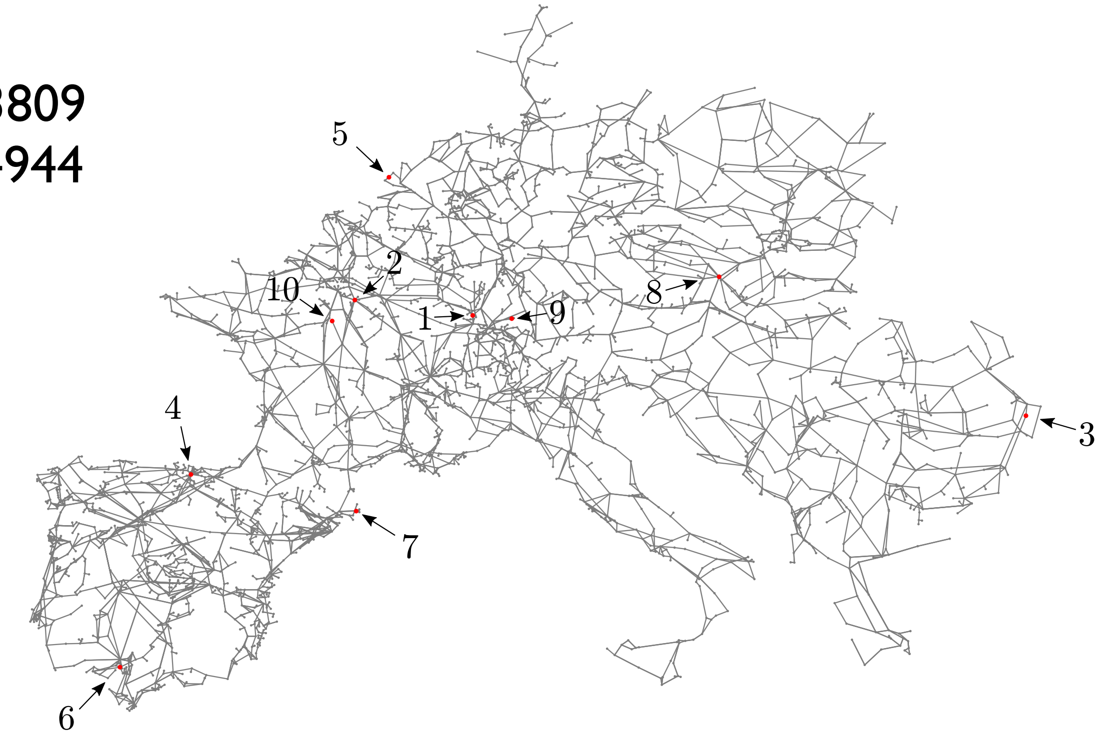
$$\mathcal{P}_2(T) = \int_0^T \mathrm{d}t\, \delta\dot{\boldsymbol{\theta}}^2(t)$$

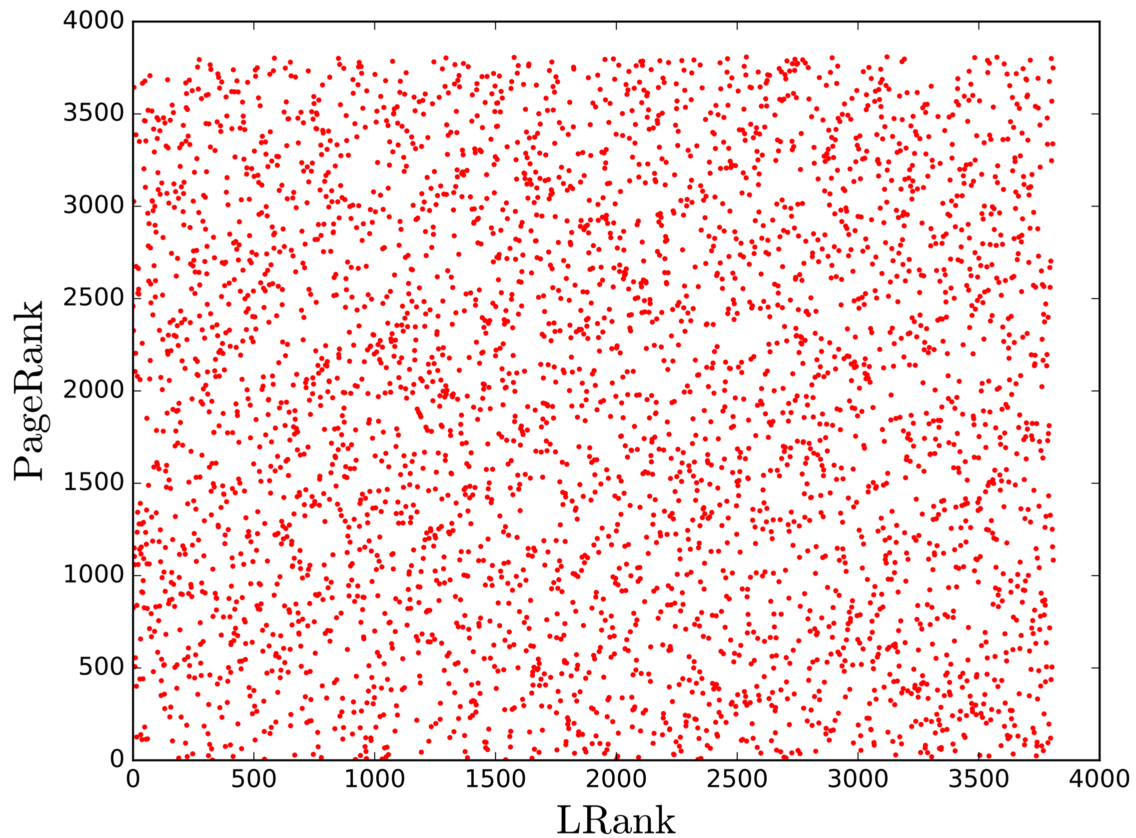
$$\mathcal{P}_1(T) = \int_0^T \mathrm{d}t\, \delta\boldsymbol{\theta}^2(t)$$

The key player problem : deterministically coupled systems

node #	C_{geo}	Degree	PageRank	C_1	C_2	$\mathcal{P}_1^{num}$	$\mathcal{P}_2^{num}$ $[\gamma^2]$
1	7.84	4	3024	31.86	5.18	0.047	0.035
2	6.8	1	2716	22.45	5.68	0.021	0.118
3	5.56	10	896	22.45	2.33	0.32	0.116
4	4.79	3	1597	21.74	3.79	0.126	0.127
5	7.08	1	1462	21.74	5.34	0.026	0.125
6	4.38	6	2945	21.69	5.65	0.023	0.129
7	5.11	2	16	19.4	5.89	0.016	0.164
8	4.15	6	756	19.38	1.83	0.453	0.172
9	5.06	1	1715	10.2	5.2	0.047	0.449
10	2.72	4	167	7.49	2.17	0.335	0.64

#nodes : 3809
#edges : 4944





Resistance distance centrality a.k.a. LRank

Numerically computed performance measure

Tyloo, Pagnier and PJ : to be published (2018)

The key player problem : deterministically coupled systems

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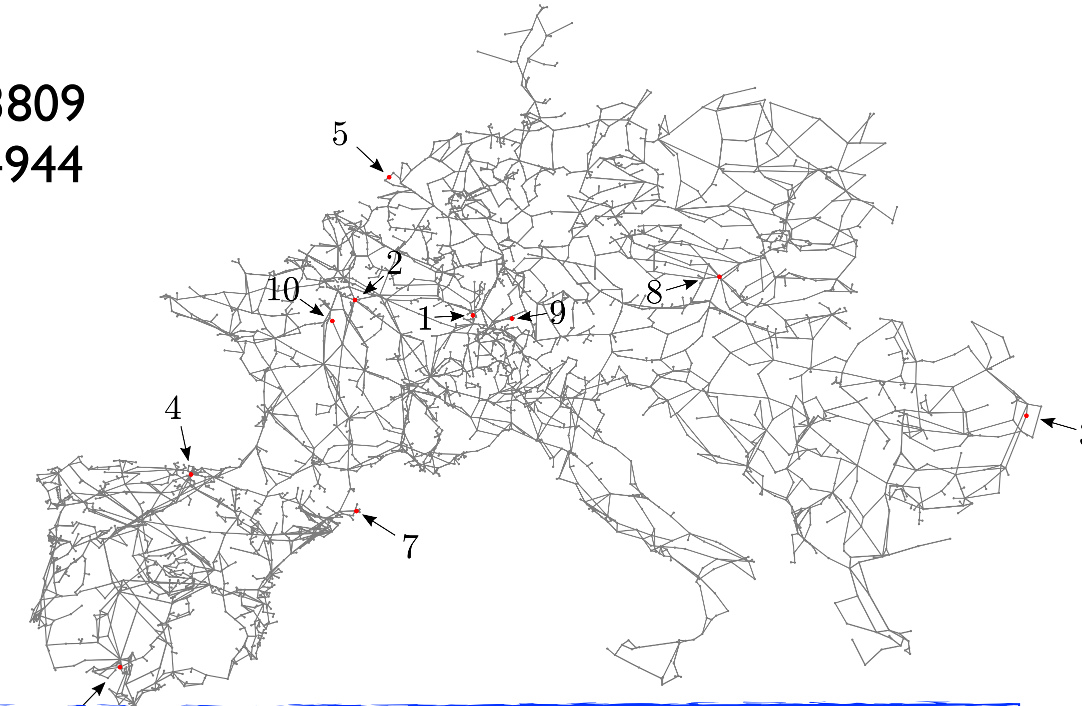
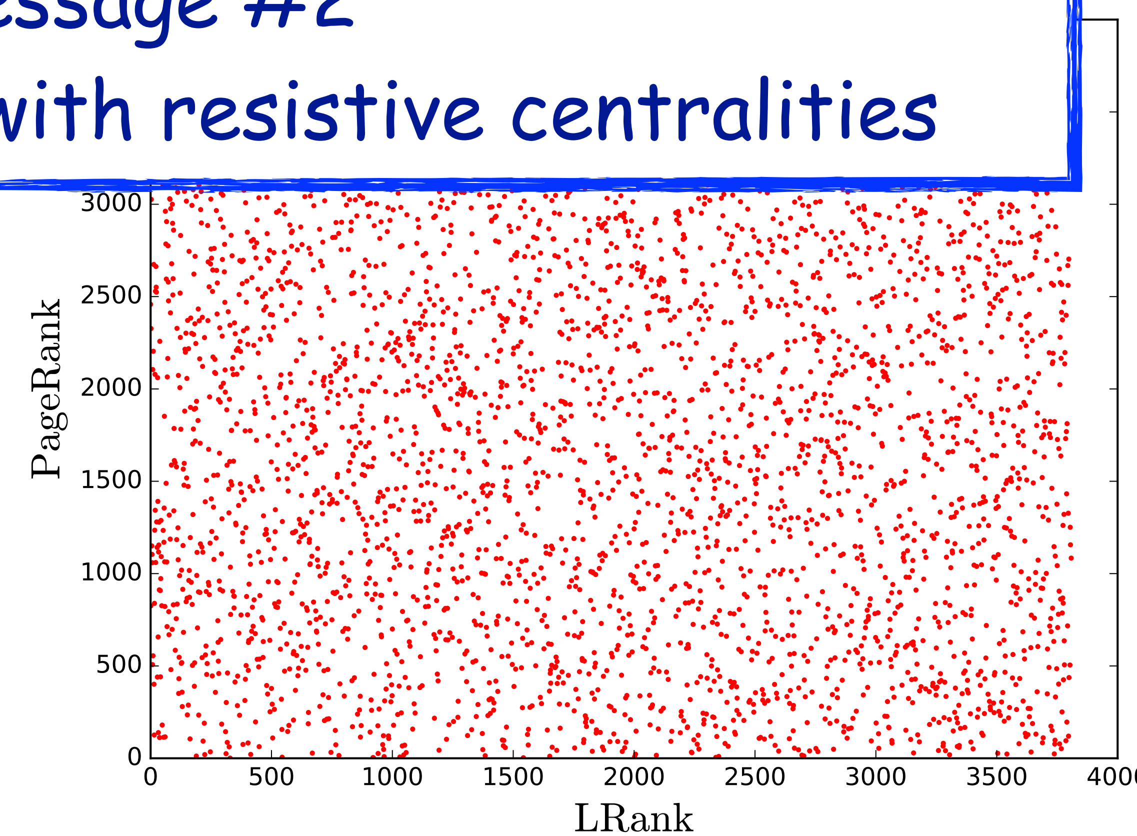


Take-home message #2

- Local vulnerabilities ranked with resistive centralities

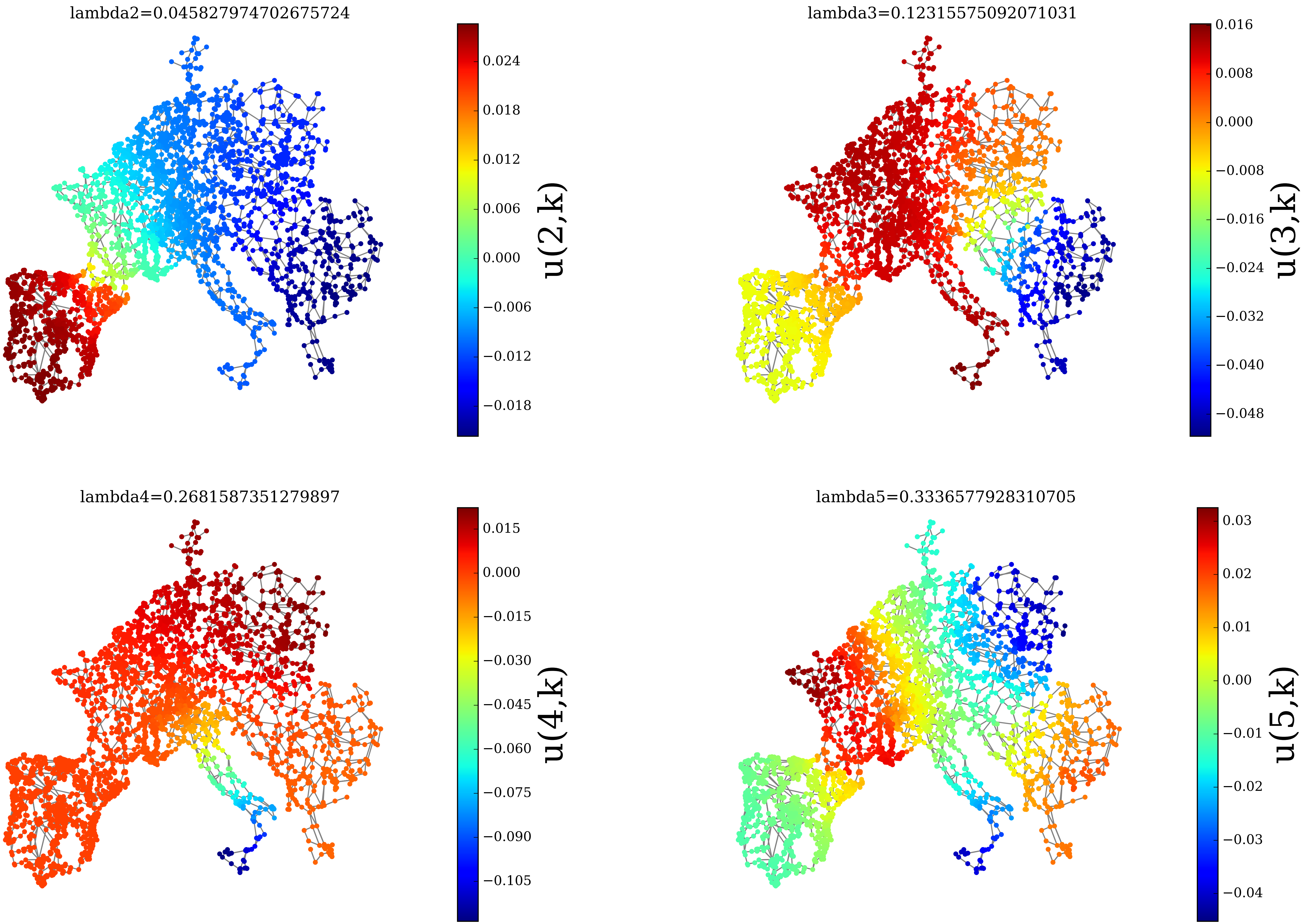
Resistance distance centrality a.k.a. LRank

Numerically computed performance measure

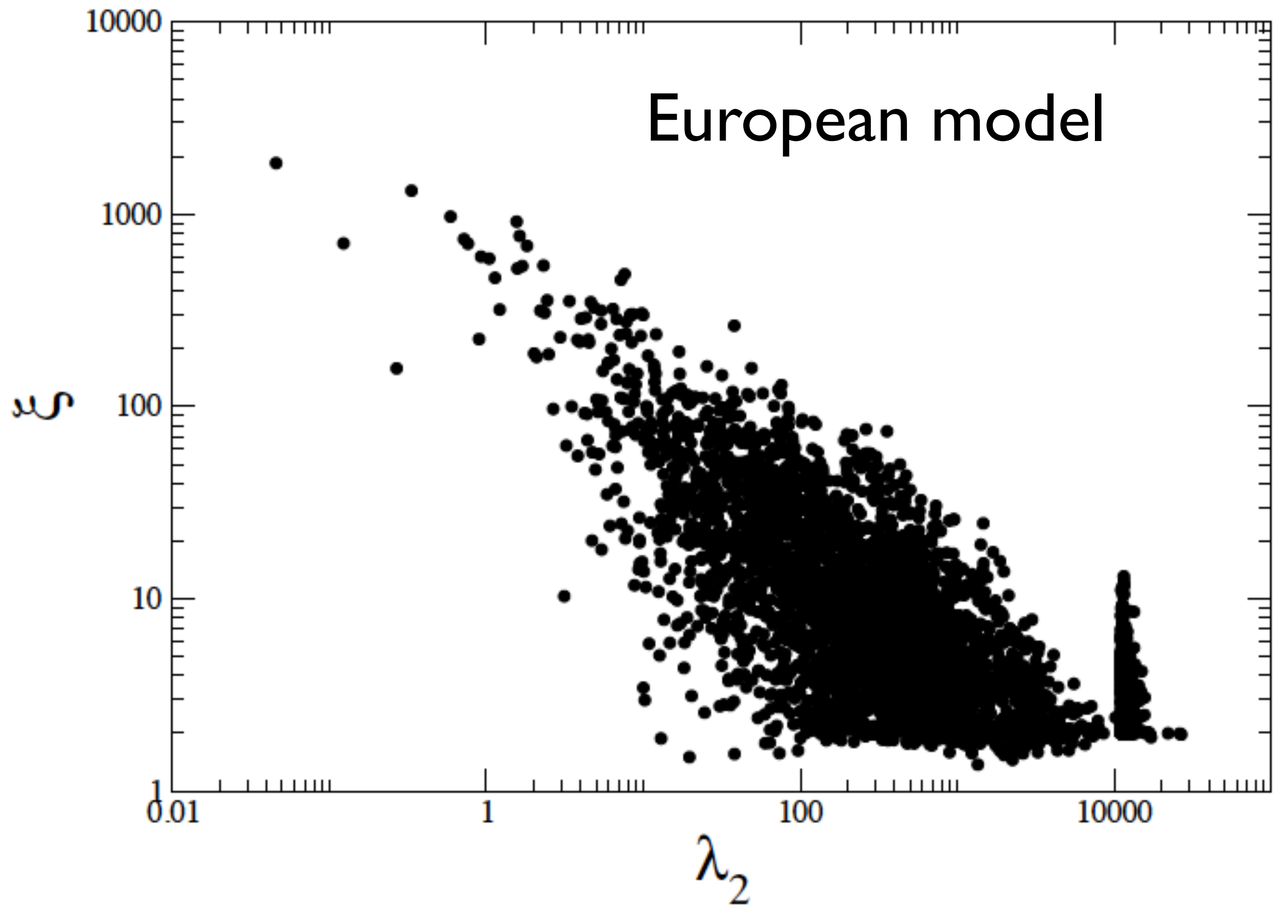


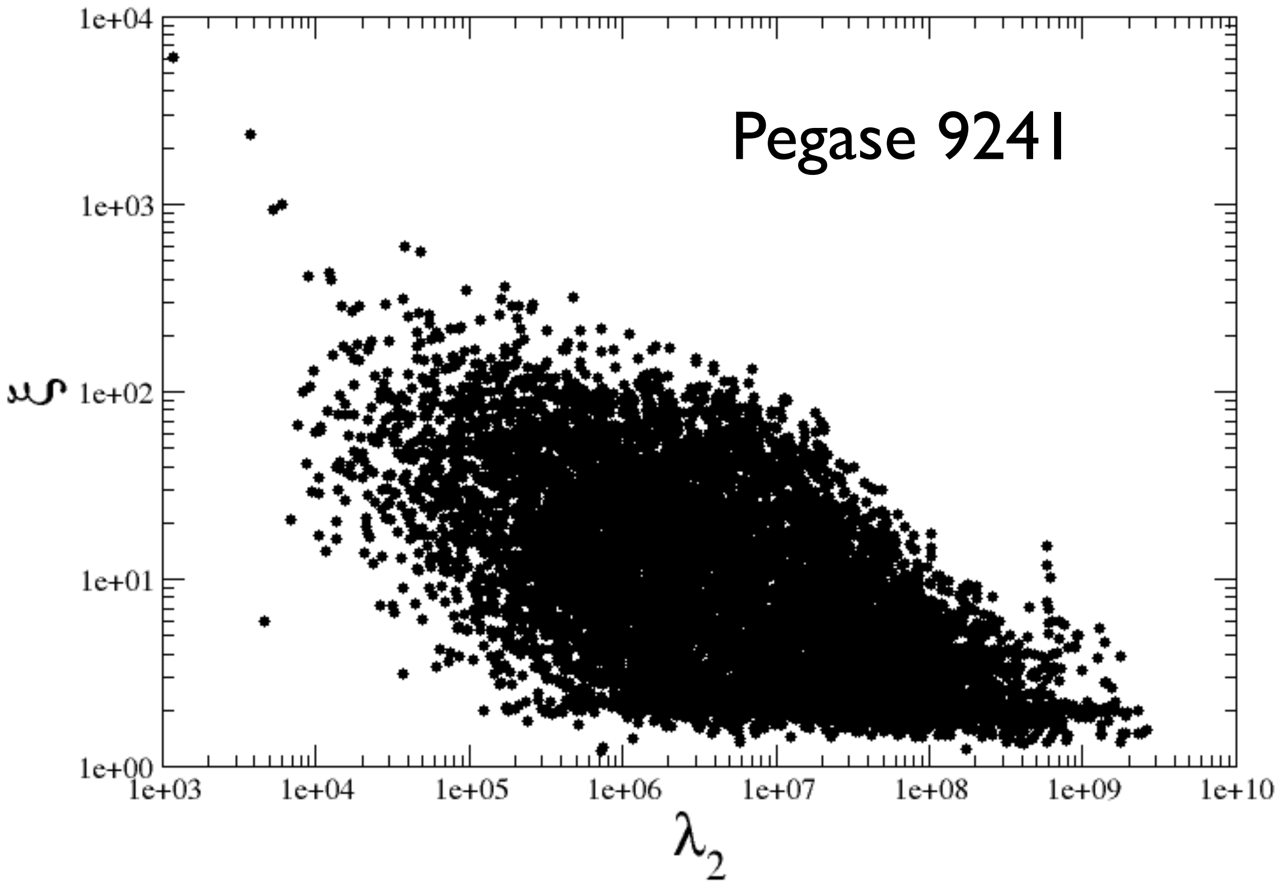
Tyloo, Pagnier and PJ : to be published (2018)

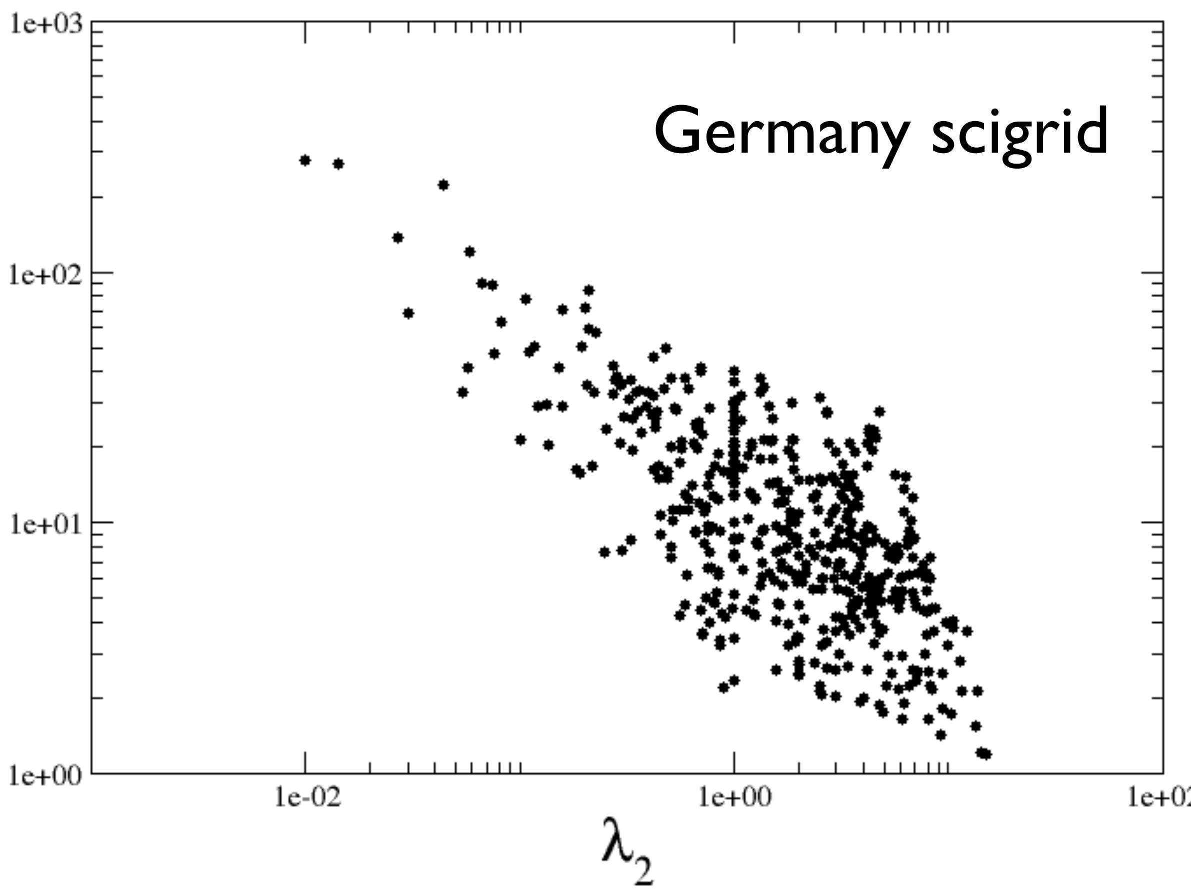
Low-lying modes of the Laplacian matrix



Connection between e-values and extension of e-vectors







Conclusion

Robustness assessment and local vulnerability ranking / key player problem
in deterministic, network-coupled dynamical systems

$$\dot{\mathbf{x}} = \mathbf{P} - \mathbb{M}\mathbf{x} \qquad\qquad \mathbb{I}\ddot{\mathbf{x}} + \mathbb{D}\dot{\mathbf{x}} = \mathbf{P} - \mathbb{M}\mathbf{x}$$

Look at distances, centralities, indices related to the matrix M !

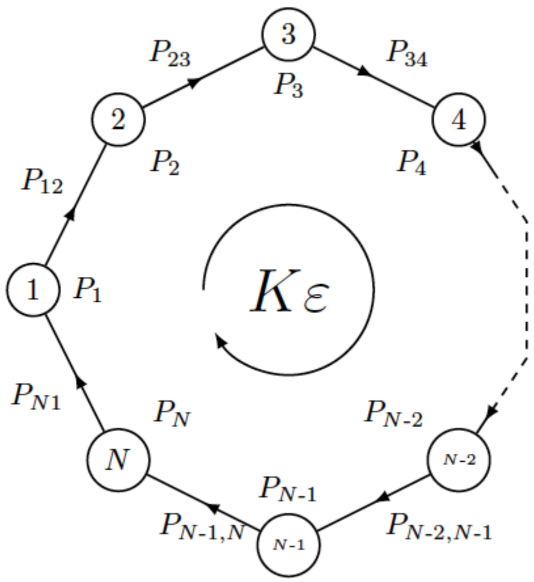
Impact :
planning of electric power grids
real-time assessment of grid stability

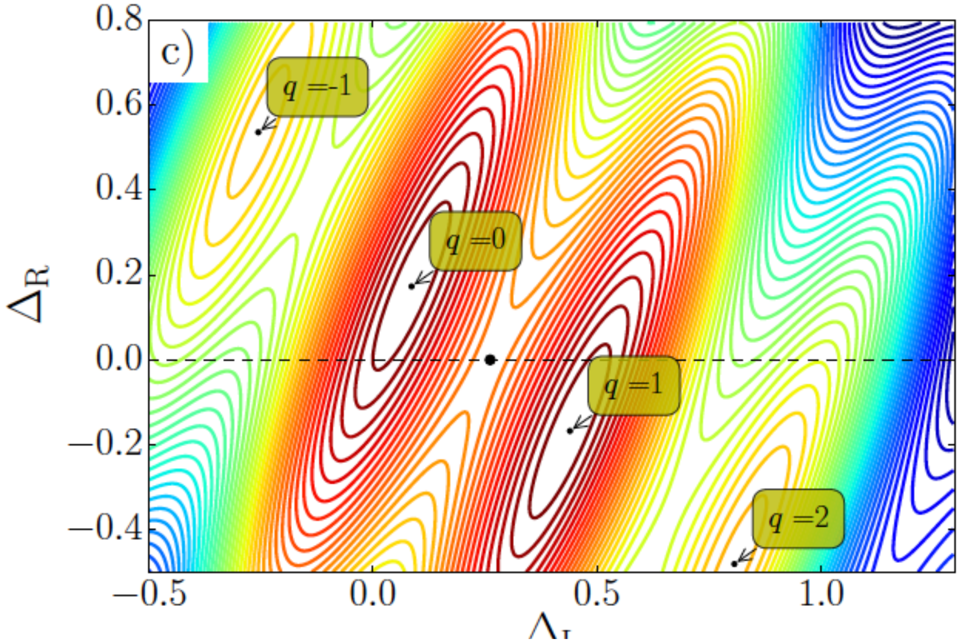
Note : -method based on gradient and Lyapunov equation also applicable
Coletta, Bamieh and PJ arXiv:1807.09048
-*even for line faults*
Coletta and PJ arXiv:1711.10348

One open question (in progress)

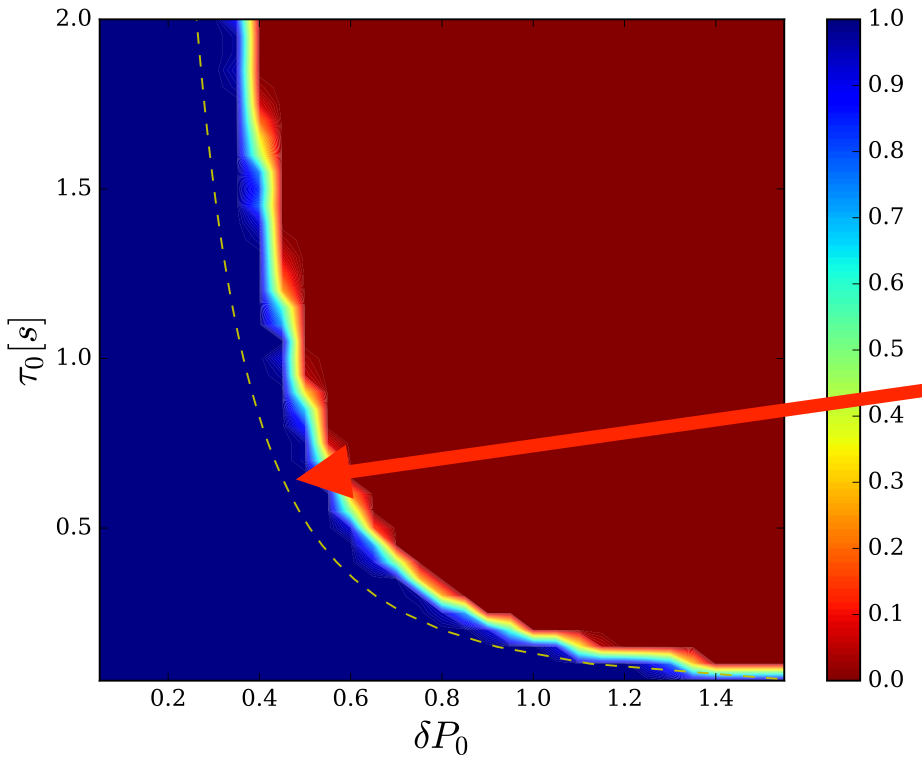
topological winding number : $q=\sum_i |\theta_{i+1}-\theta_i|_{2\pi}/2\pi$

different solutions ~ different vortex flows





Noise-induced transition from one sync state to another ~configuration space picture

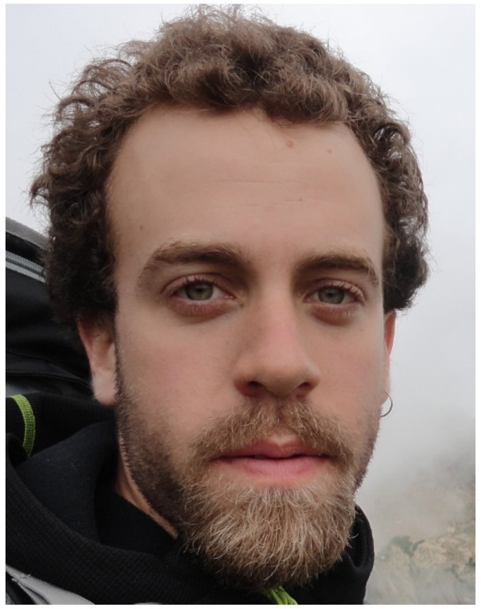


Theory based on $\delta\theta^2(t) > \Delta$ with
Δ = distance from stable sync point to first saddle node

The team

Tommaso Coletta, postdoc
(now with Cargil)

Robin Delabays, PhD student
(now postdoc)



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