Anderson localization of vector waves

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cnrs advancing the frontiers

Anderson localization in quantum mechanics

Quantum particle (electron, atom,...)

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

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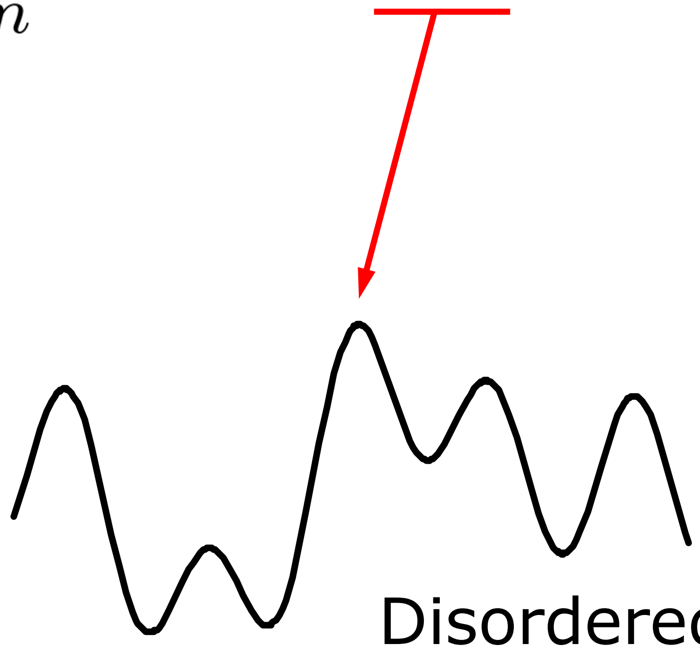

Disordered potential $V(\mathbf{r})$

Anderson, PR **109**, 1492 (1958)

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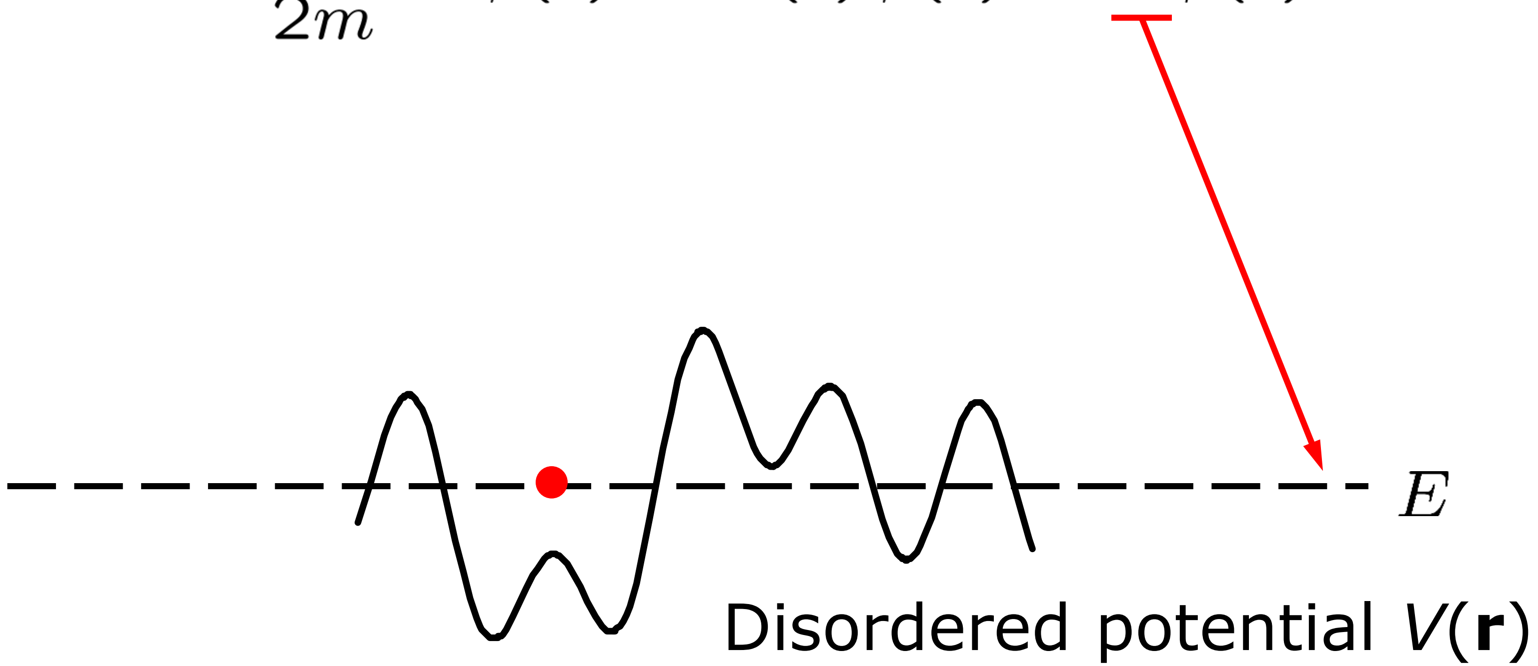

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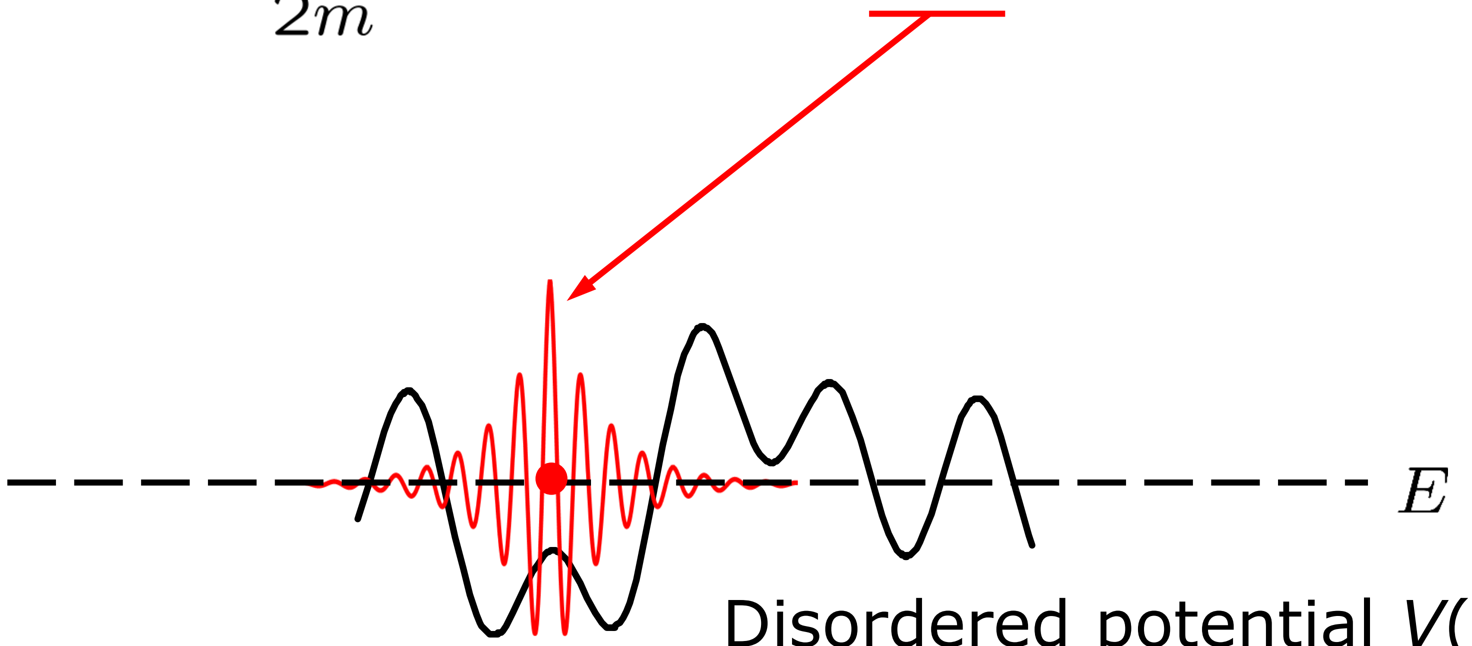

In **1D** & **2D** all eigenstates are **localized**
for arbitrarily weak disorder

Anderson, PR **109**, 1492 (1958)

Anderson localization in quantum mechanics

Quantum particle (electron, atom,...)

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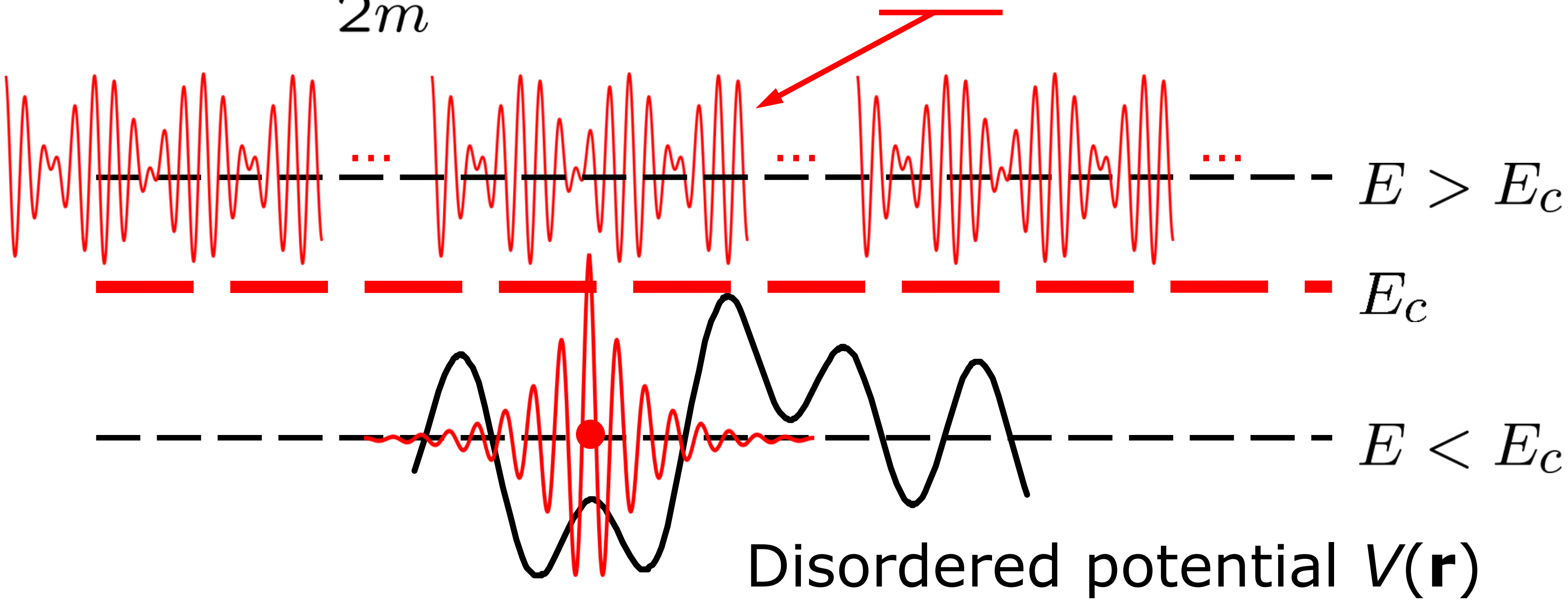


In **3D** a **mobility edge** E_c separates localized and extended states

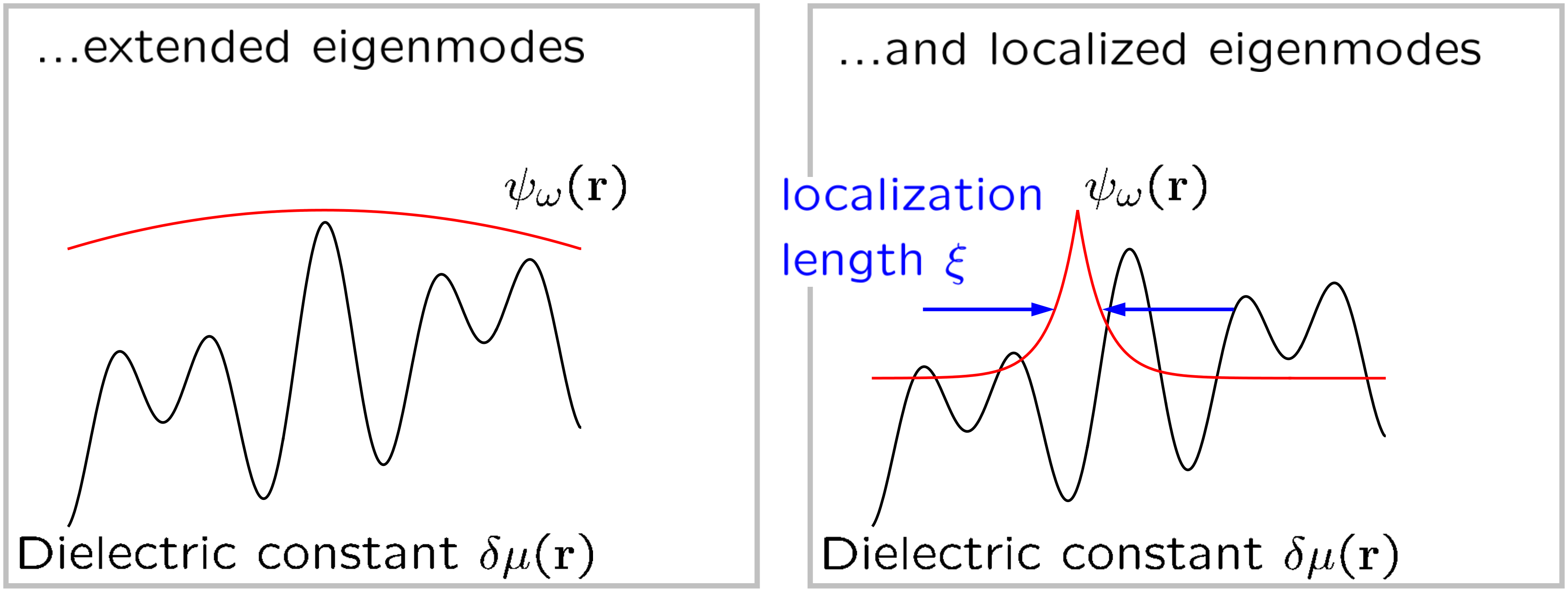
Anderson, PR **109**, 1492 (1958)

Localization of classical waves: light, sound, etc.

$$\nabla^2 \psi_\omega(\mathbf{r}) + k^2 \left[1 + \delta\mu(\mathbf{r})\right] \psi_\omega(\mathbf{r}) = 0$$

Fluctuating "dielectric constant"

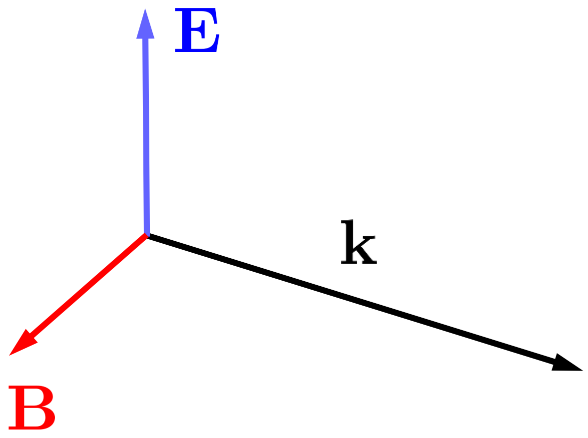
- In 3D mobility edges ω_c separate...



- In 1D and 2D all modes are localized whatever ω

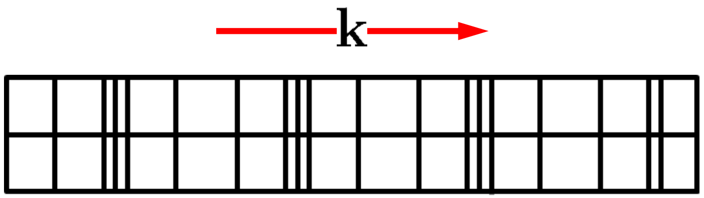
Classical waves are often vector waves

Light

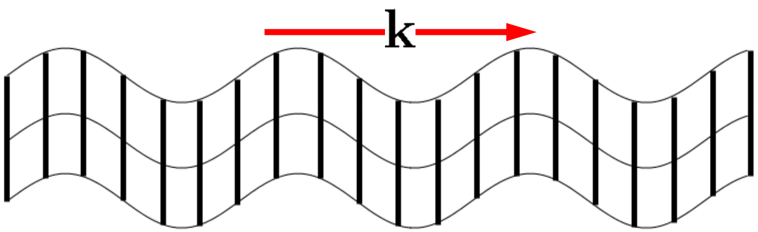


Elastic wave

compressional waves:



shear waves:



Does the vector character of excitations have any importance?

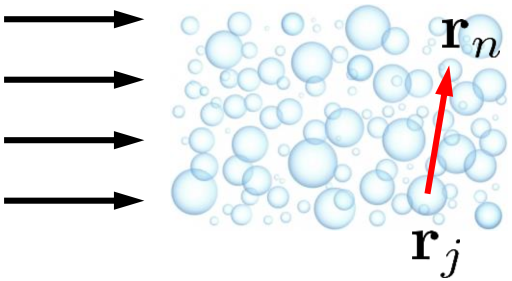
Foldy-Lax equations for multiple scattering

Wave field
on the scatterers

Incident
monochromatic
wave $\boldsymbol{\psi}_0$

Scatterers

$$\psi(\mathbf{r}_n) = \psi_0(\mathbf{r}_n) + \alpha(\omega) \sum_{j \neq n} G_\omega(\mathbf{r}_n, \mathbf{r}_j)\psi(\mathbf{r}_j)$$



Foldy, Phys. Rev. **67**, 107 (1945)
Lax, Rev. Mod. Phys. **23**, 287 (1951)

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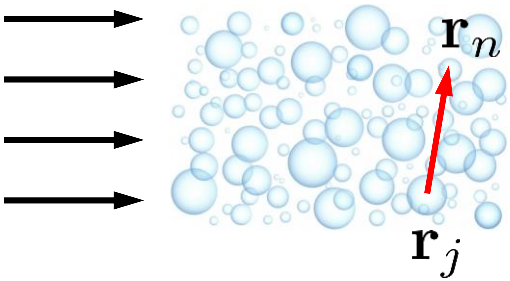

Incident
field

Scatterer
properties

Green's function
of Helmholtz equation

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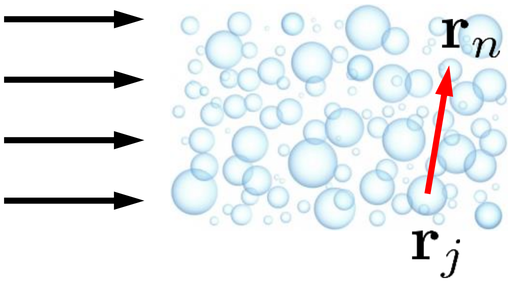

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Incident field

Scatterer properties

Green's function of Helmholtz equation

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 + \alpha(\omega)\left[\hat{G}(\omega) - i\mathbb{1}\right]\boldsymbol{\psi} \qquad \boldsymbol{\psi} = \{\psi(\mathbf{r}_1), \dots, \psi(\mathbf{r}_N)\}^T$$

$$G_{jn}(\omega) = i\delta_{jn} + (1 - \delta_{jn})\frac{\exp(ik|\mathbf{r}_n - \mathbf{r}_j|)}{k|\mathbf{r}_n - \mathbf{r}_j|}$$

Foldy, Phys. Rev. **67**, 107 (1945)
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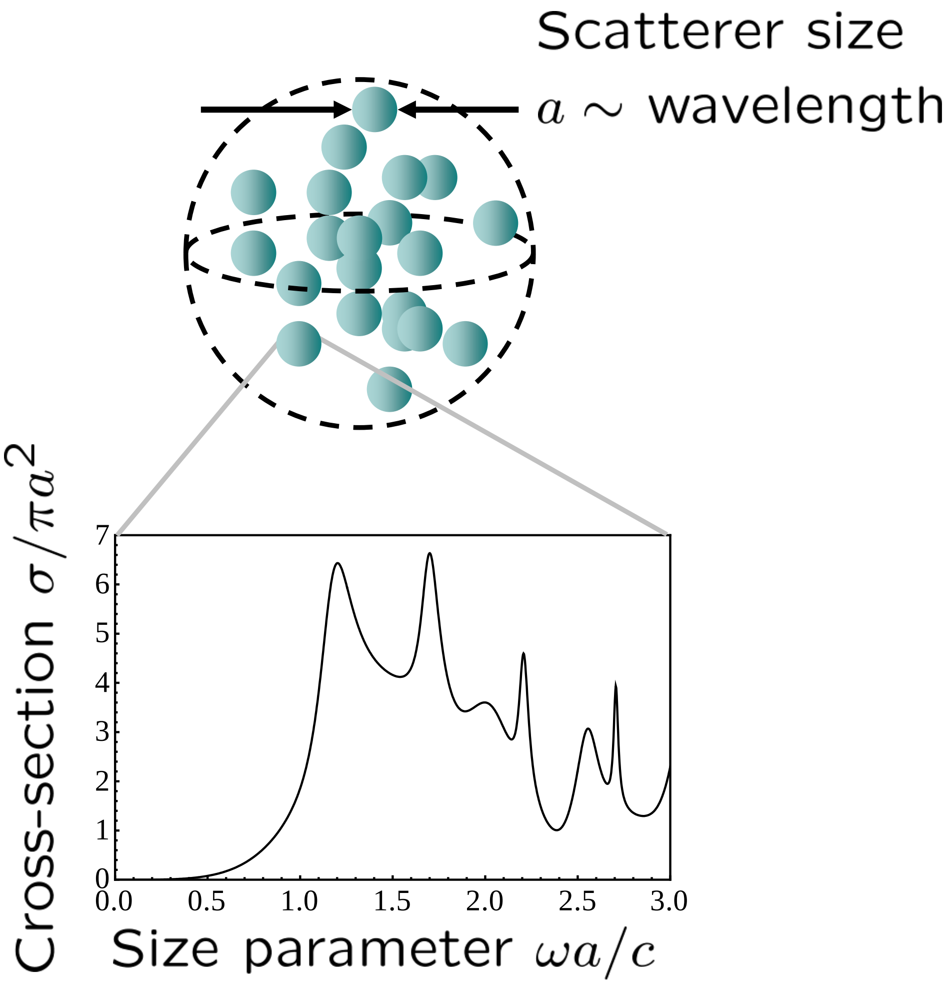
A minimal model of disordered media

Real samples complex

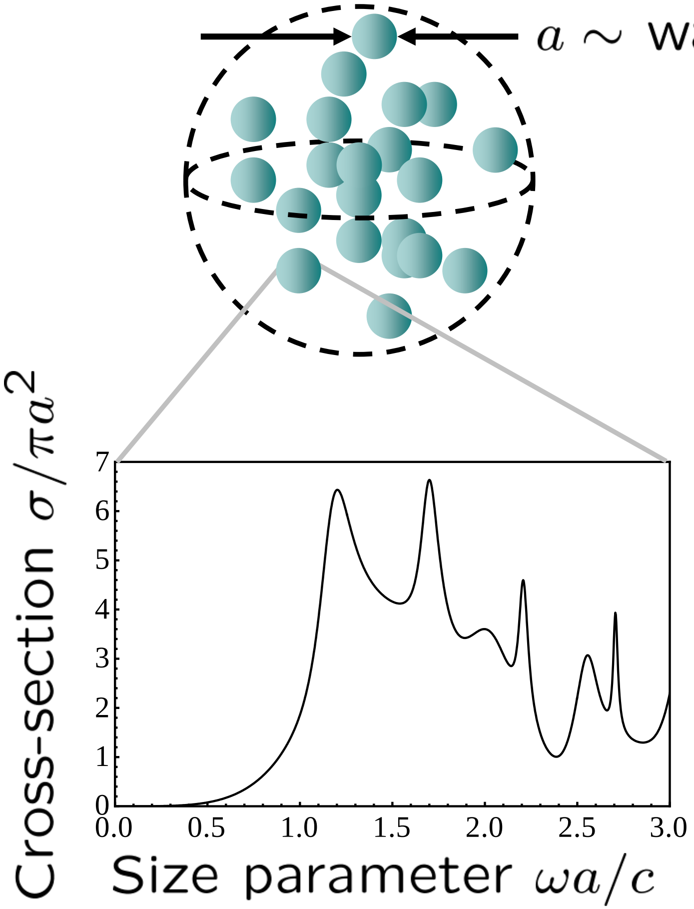


A minimal model of disordered media

Real samples complex

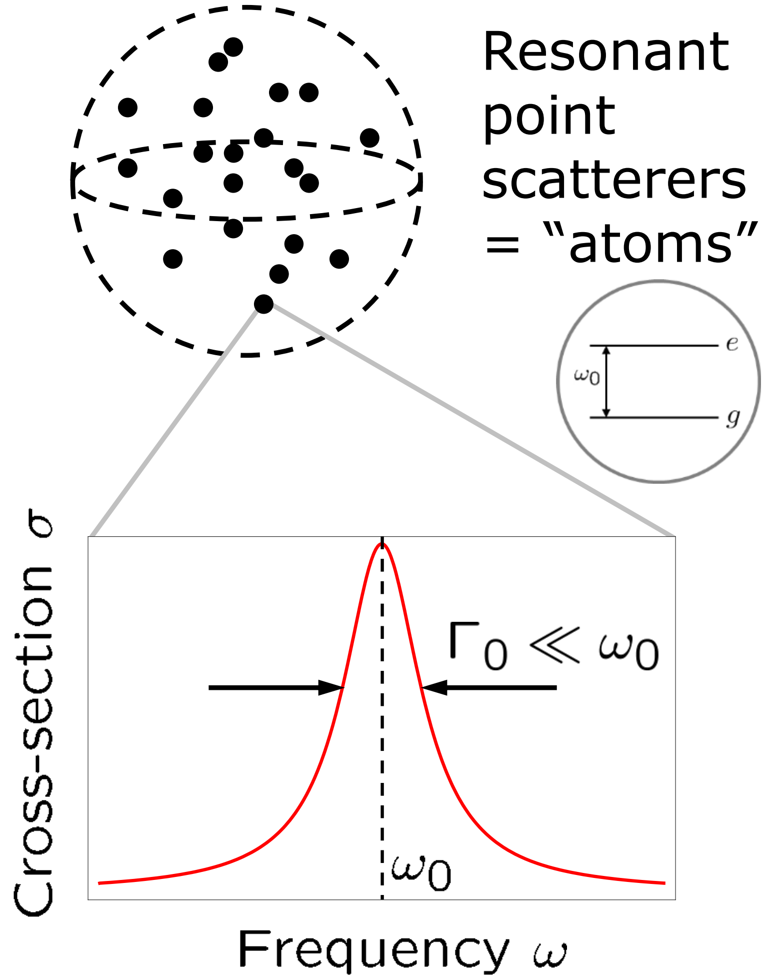
Scatterer size

$a \sim$ wavelength



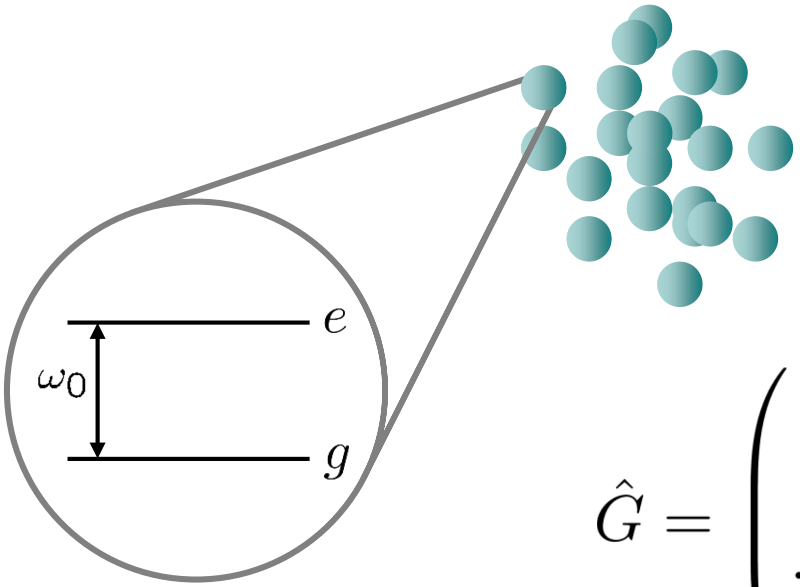
Our model simpler

Resonant point scatterers = “atoms”



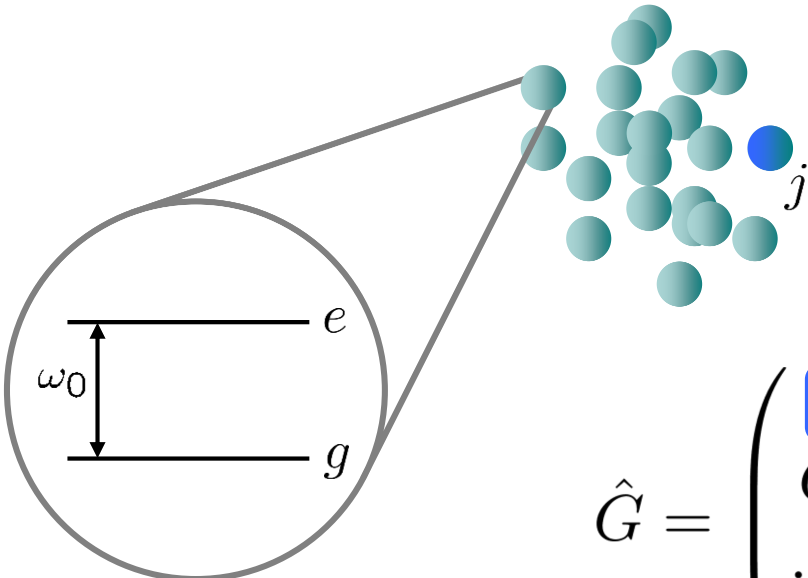
$$\alpha(\omega) = -(\Gamma_0/2)/(\omega - \omega_0 + i\Gamma_0/2)$$

Structure of the Green's matrix



$$\hat{G} = \begin{pmatrix} i & G_{12} & G_{13} & \dots & G_{1N} \\ G_{21} & i & G_{23} & \dots & G_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ G_{N1} & G_{N2} & G_{N3} & \dots & i \end{pmatrix}$$

Structure of the Green's matrix



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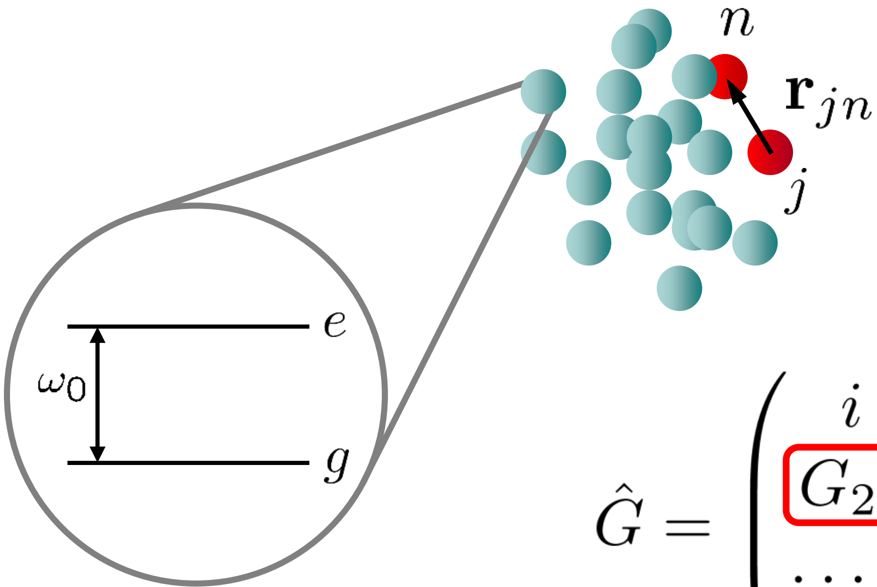
One-atom dynamics:

$G_{jj} = i$ describes the decay $e^{-\Gamma_0 t}$ of the excitation of an isolated excited atom. Deterministic (not random).

Structure of the Green's matrix



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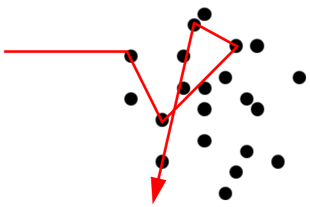
Pairwise coupling between atoms 1 & 2:

$G_{12} = e^{ik_0 r_{12}}/k_0 r_{12}$ is the field at position 2 due to a source at position 1. Random.

Off-diagonal disorder: see, e.g., Eilmes, Römer, Schreiber, Physica B **296**, 46 (2001)

Green's matrix as an effective Hamiltonian

Disordered system



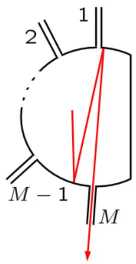
$$\hat{H}_{\text{eff}} = \hat{G}$$

Hermitian Re$\hat{G}$ and anti-Hermitian Im$\hat{G}$ parts of $\hat{H}_{\text{eff}}$ are correlated

$\hat{G}$ has correlated elements

see J. Phys. A: Math. Theor. **44**, 065102 (2011)

Chaotic billiard



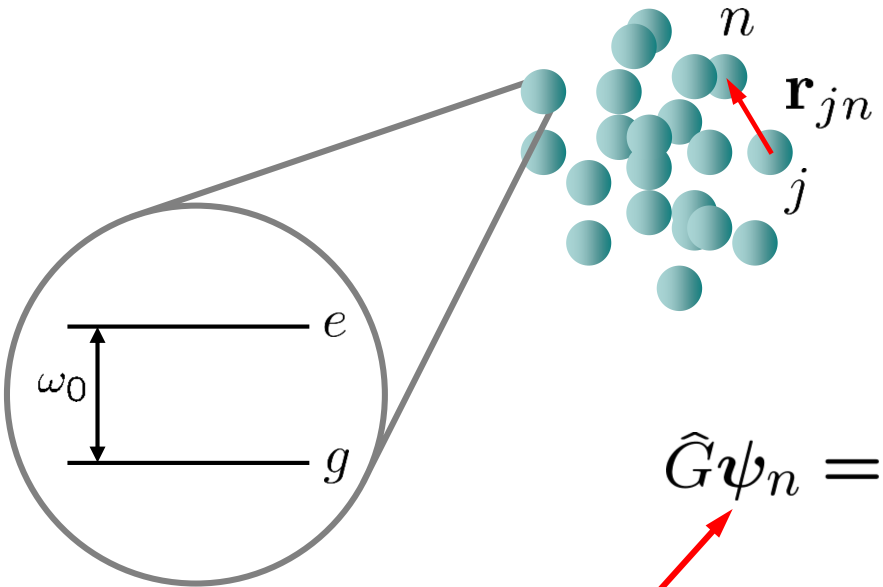
$$\hat{H}_{\text{eff}} = \hat{H}_0 - \frac{i}{2}\hat{V}\hat{V}^\dagger$$

Hermitian ($\hat{H}_0$) and anti-Hermitian ($-\frac{i}{2}\hat{V}\hat{V}^\dagger$) parts of $\hat{H}_{\text{eff}}$ are independent

$\hat{H}_0$ and $\hat{V}$ have i.i.d. elements $\rightarrow$ "simple" theory

see Haake, Izrailev, Lehmann, Saher, Sommers, Z. Phys. B **88**, 359 (1992)

Quasi-modes of the system



Green's matrix G describes propagation of light between pairs of atoms

$r_{jn} = |\mathbf{r}_n - \mathbf{r}_j|$

$$\widehat{G}\boldsymbol{\psi}_n = \Lambda_n \boldsymbol{\psi}_n$$

(Right) eigenvectors:

$\boldsymbol{\psi}_n = \{\psi_n^1, \psi_n^2, \ldots, \psi_n^N\}$

Eigenvalues (resonances):

$\Lambda_n = \mathrm{Re}\Lambda_n + i\mathrm{Im}\Lambda_n$

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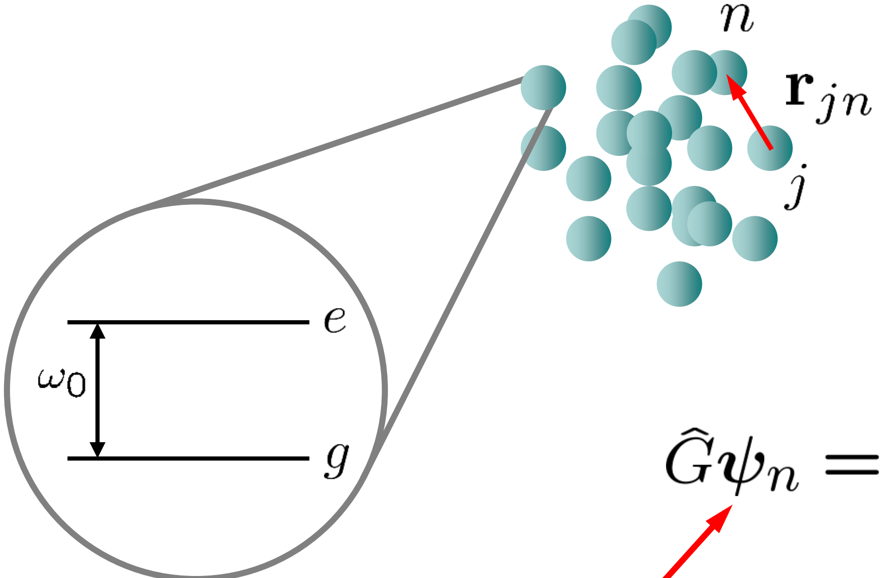

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Frequency of the mode

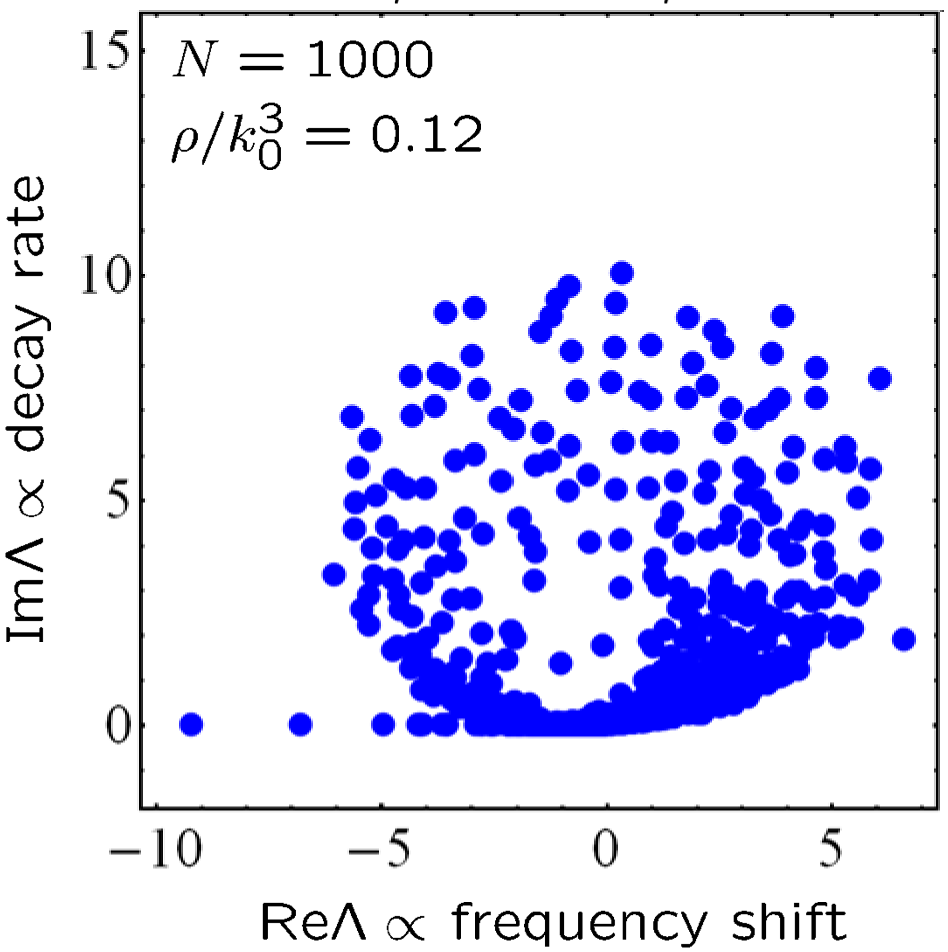
$\omega_n = \omega_0 - \frac{\Gamma_0}{2}\mathrm{Re}\Lambda_n$

Decay rate of the mode

$\Gamma_n = \frac{\Gamma_0}{2}\mathrm{Im}\Lambda_n$

Green's matrix for $N \gg 1$

$$\hat{G}\psi_n = \Lambda_n \psi_n$$



EPL **96**, 34005 (2011)

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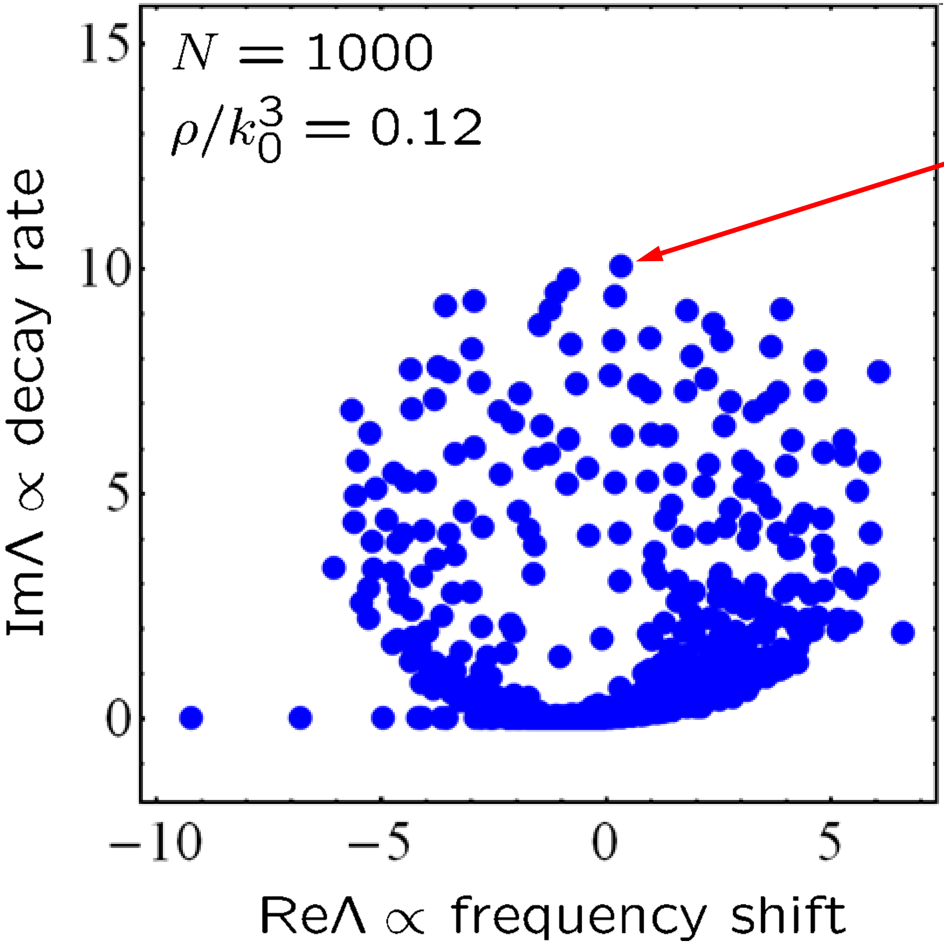
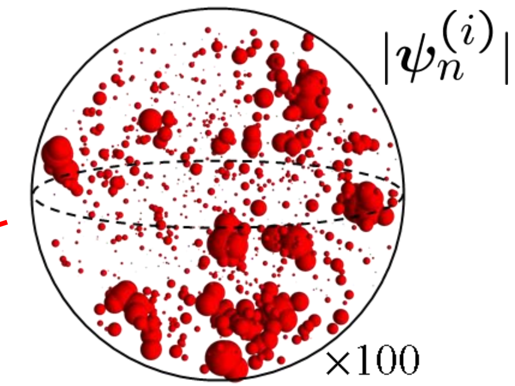


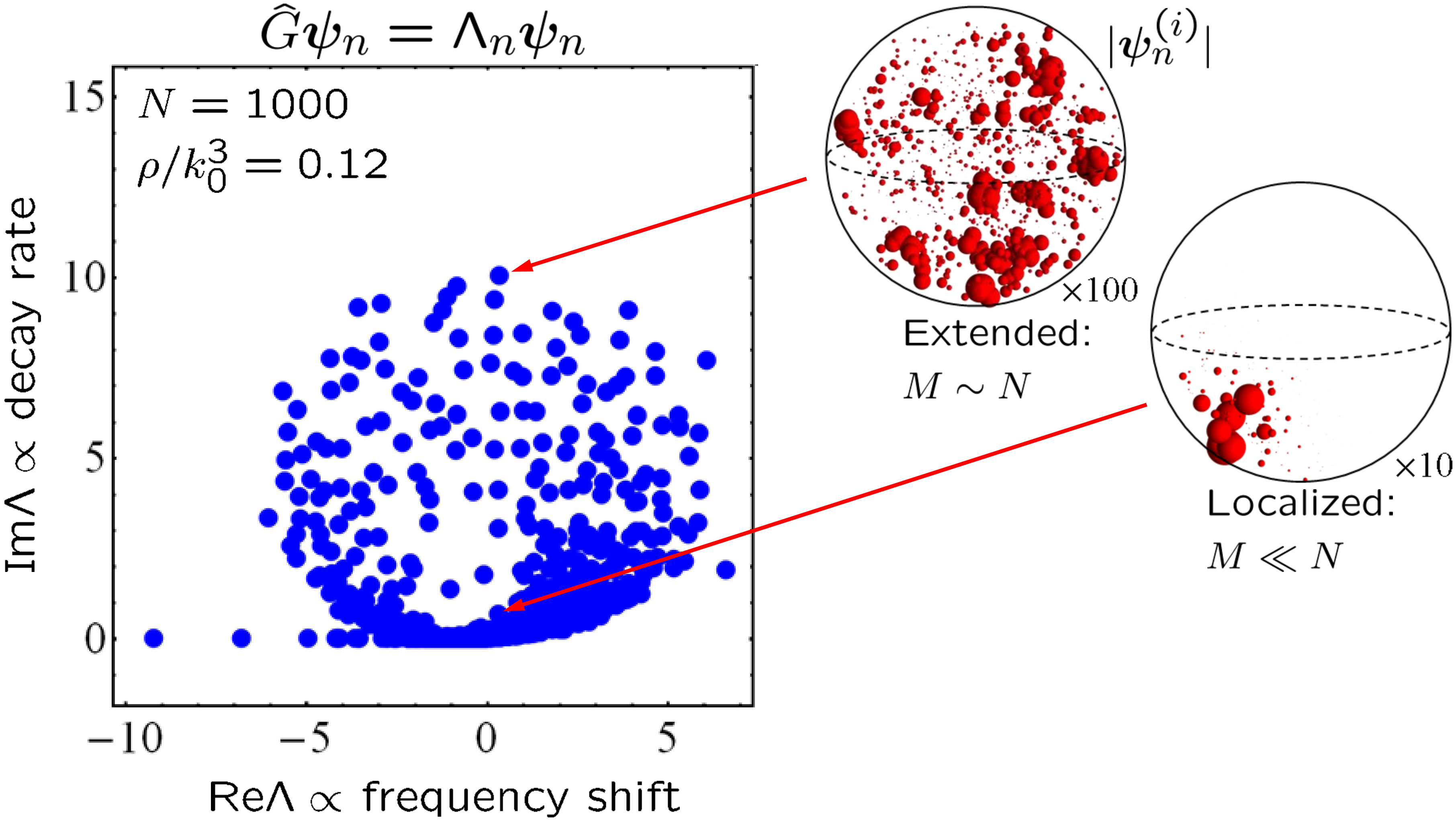


Extended:

$M \sim N$

EPL **96**, 34005 (2011)

Green's matrix for $N \gg 1$



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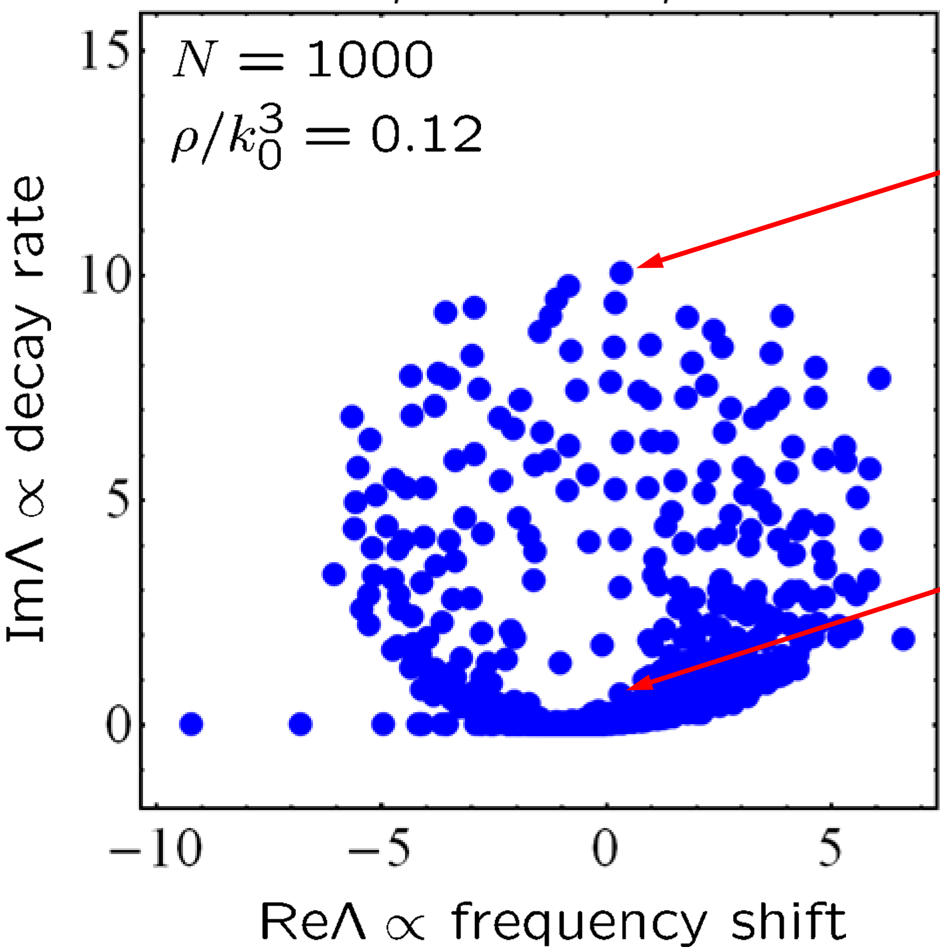
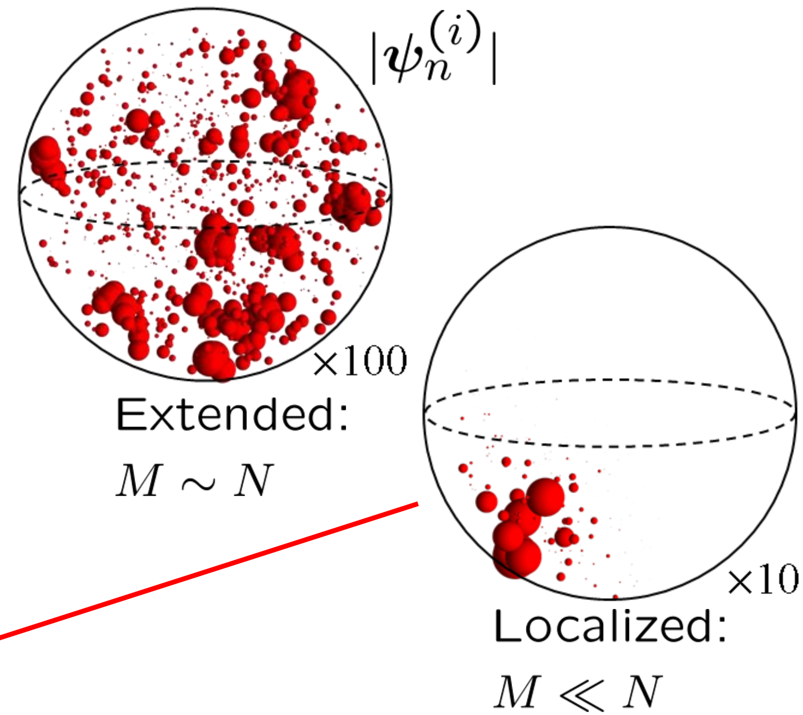


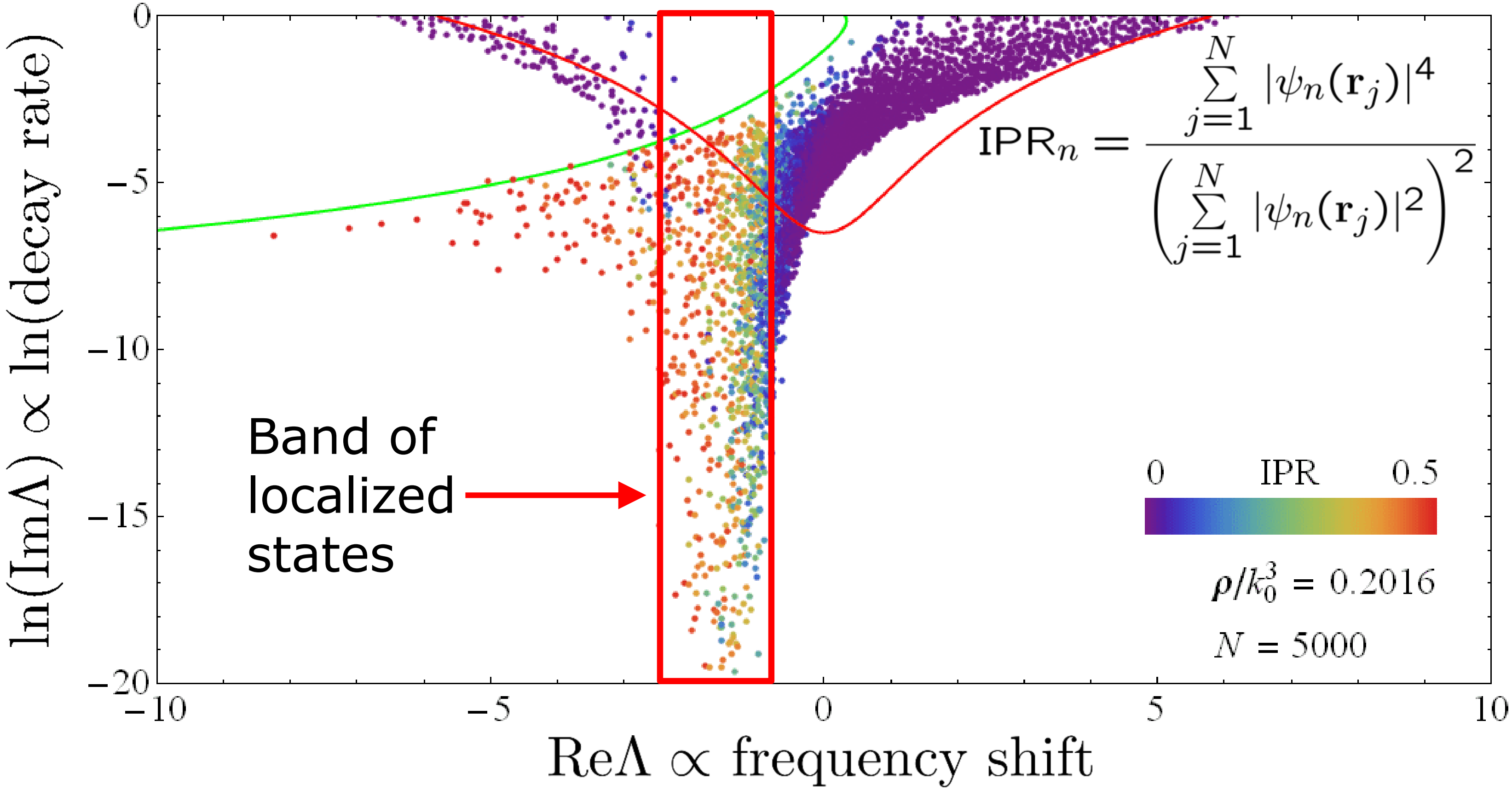


$$\mathrm{IPR}_n = \frac{\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^4}{\left(\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^2\right)^2} \sim \frac{1}{M}$$

for a mode localized on M atoms

EPL **96**, 34005 (2011)

IPR at a sufficiently high density



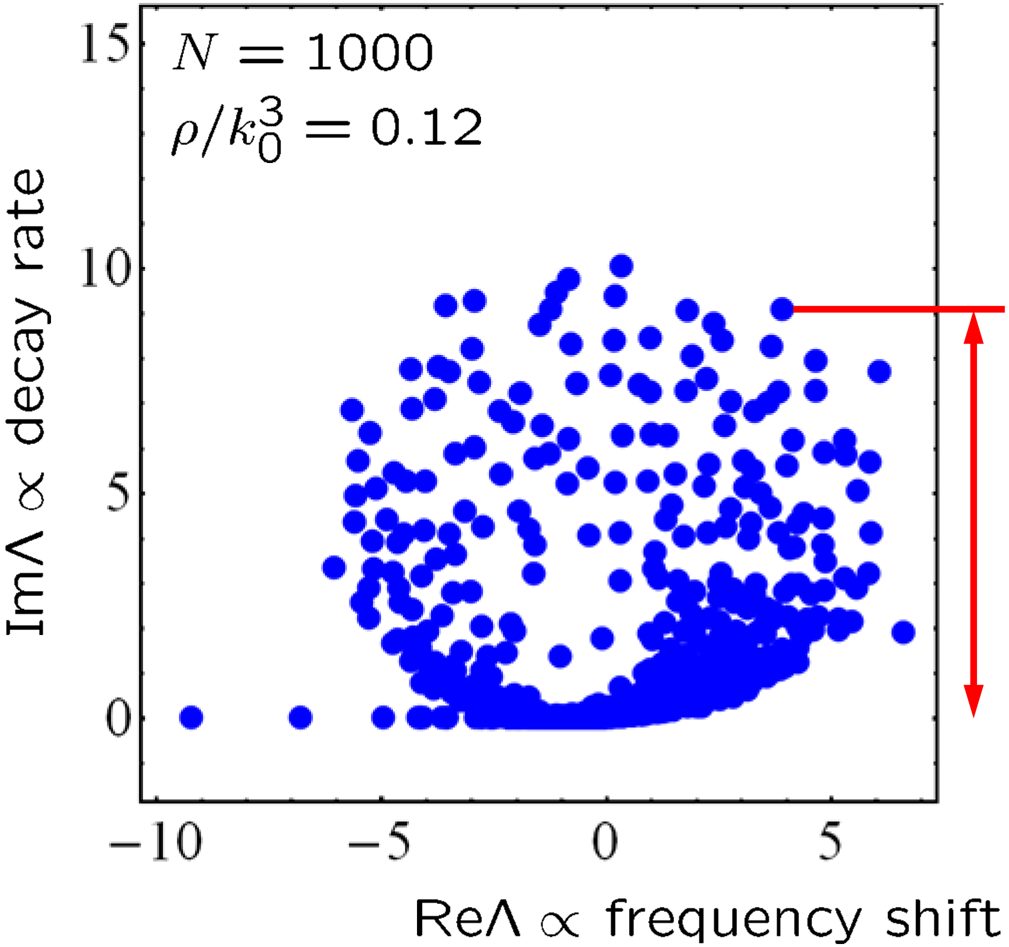
Eigenvalue domain boundary from the diffusion theory

Subradiant states localized on 2 closely located atoms

PRL **112**, 023905 (2014)

Dimensionless conductance = normalized decay rate

$$\widehat{G}\psi_n = \Lambda_n \psi_n$$



$\delta\omega \sim \mathrm{Im}\Lambda$ — "mode width"

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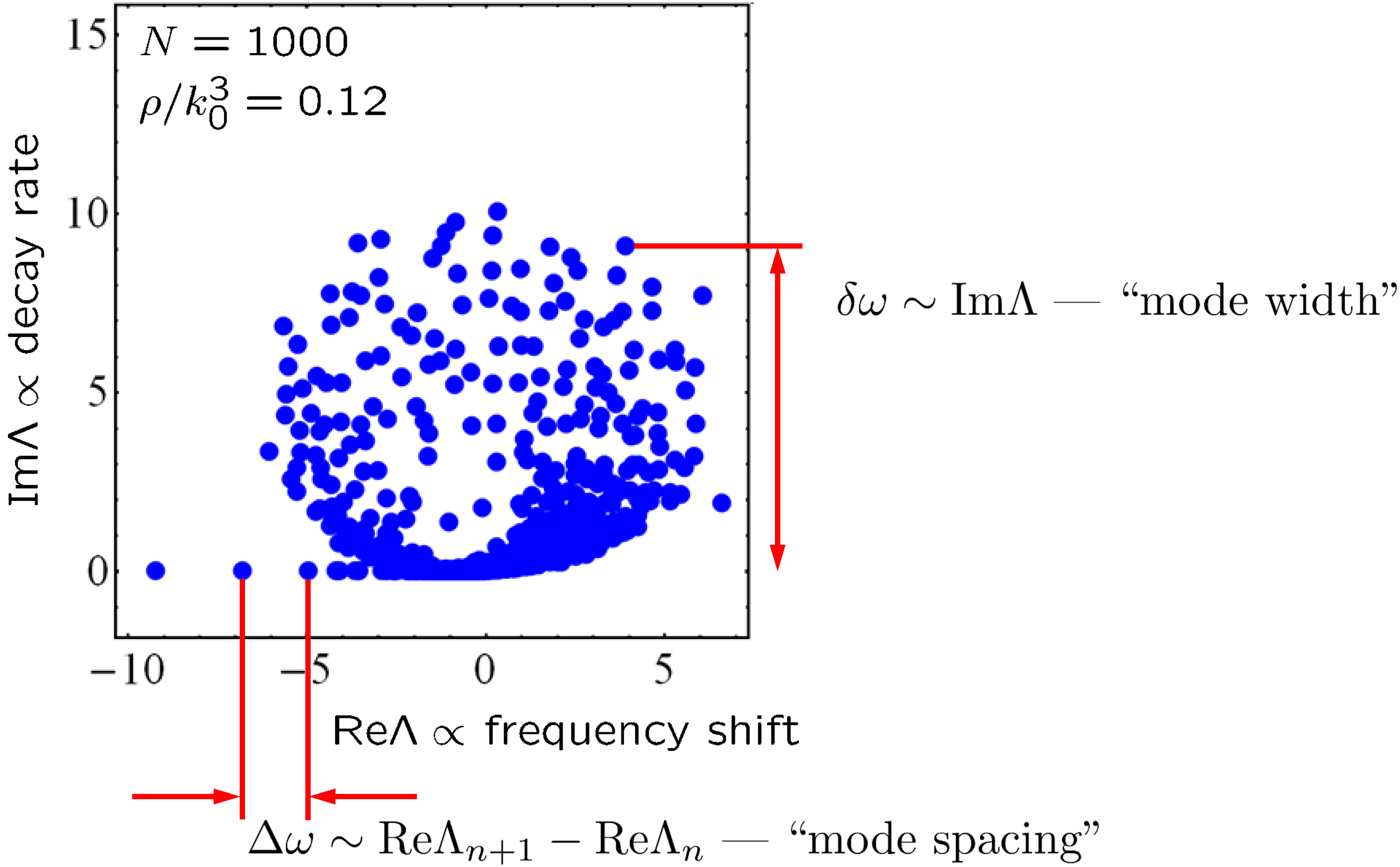

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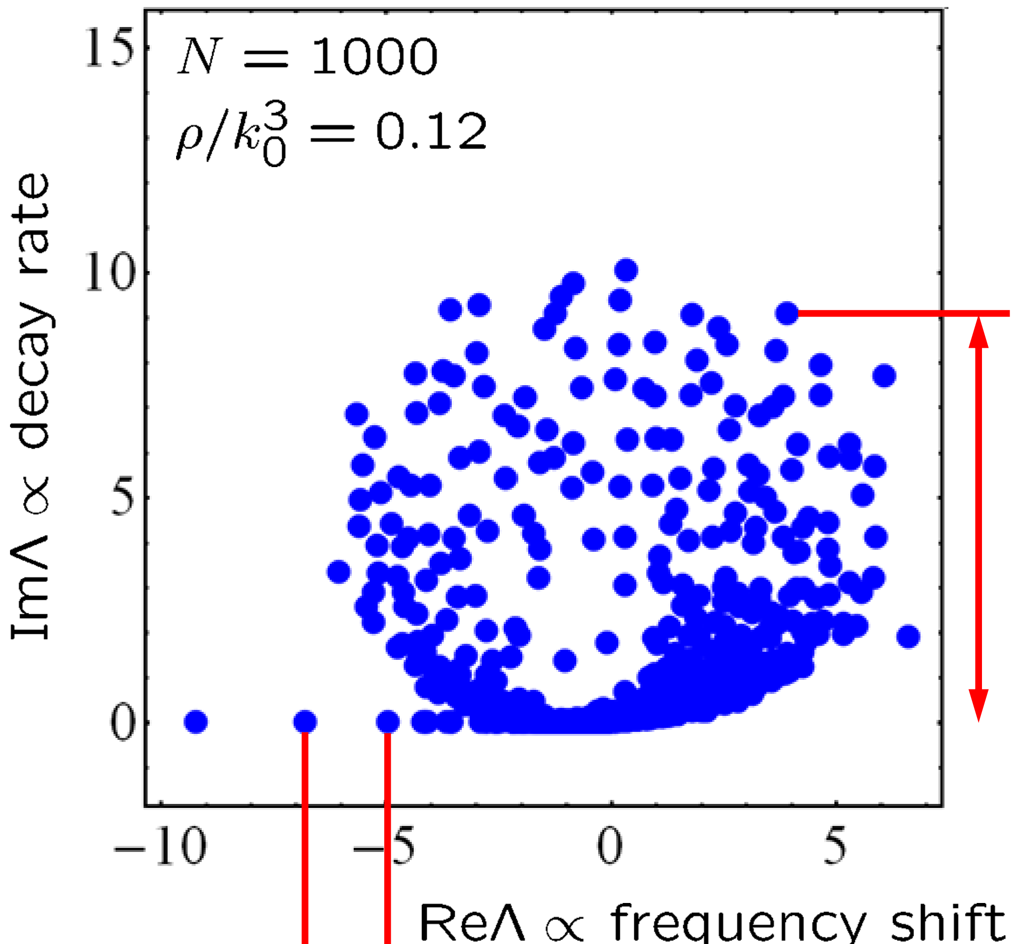


$\delta\omega \sim \mathrm{Im}\Lambda$ — “mode width”

$$g = \frac{\delta\omega}{\langle \Delta\omega \rangle}$$

Thouless parameter
dimensionless conductance

$\Delta\omega \sim \mathrm{Re}\Lambda_{n+1} - \mathrm{Re}\Lambda_n$ — “mode spacing”

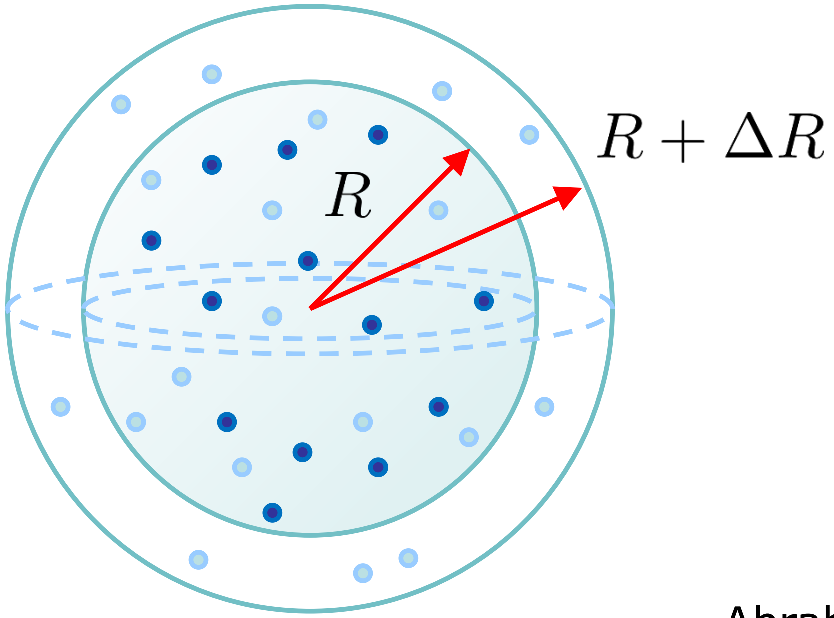
Scaling theory of Anderson localization

Main idea: Study how g evolves with sample size R

If the modes are extended, g grows with R

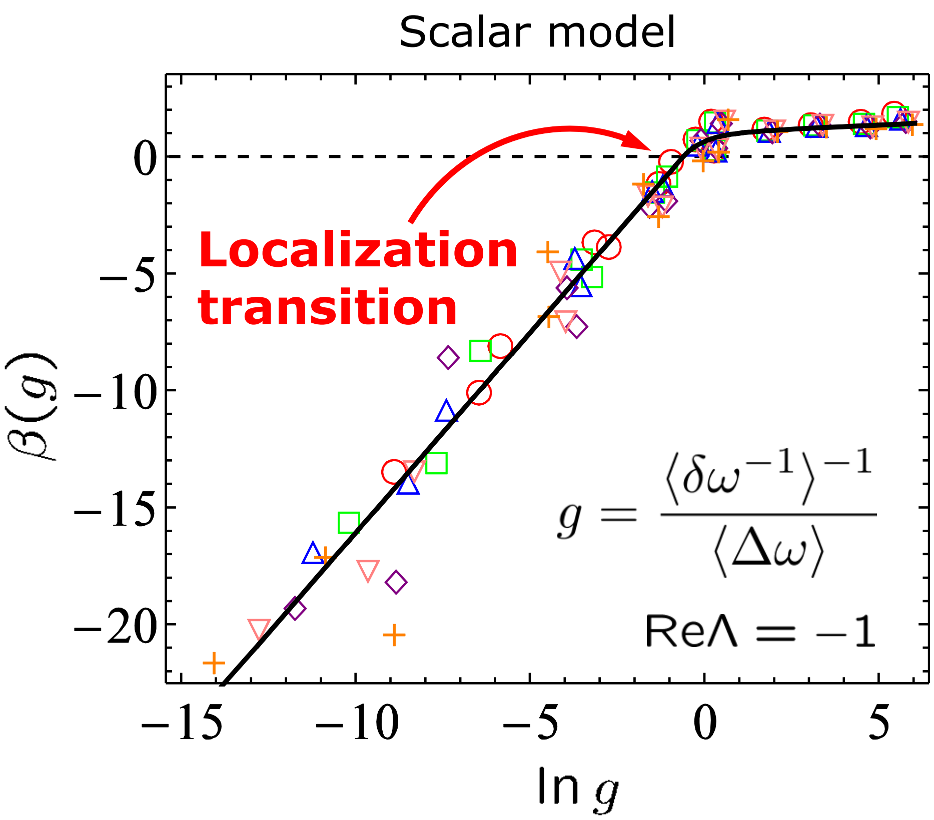
If the modes are localized, g decreases with R

At the critical point $g = g_c$ is independent of R



Abrahams et al., PRL **42**, 673 (1979)

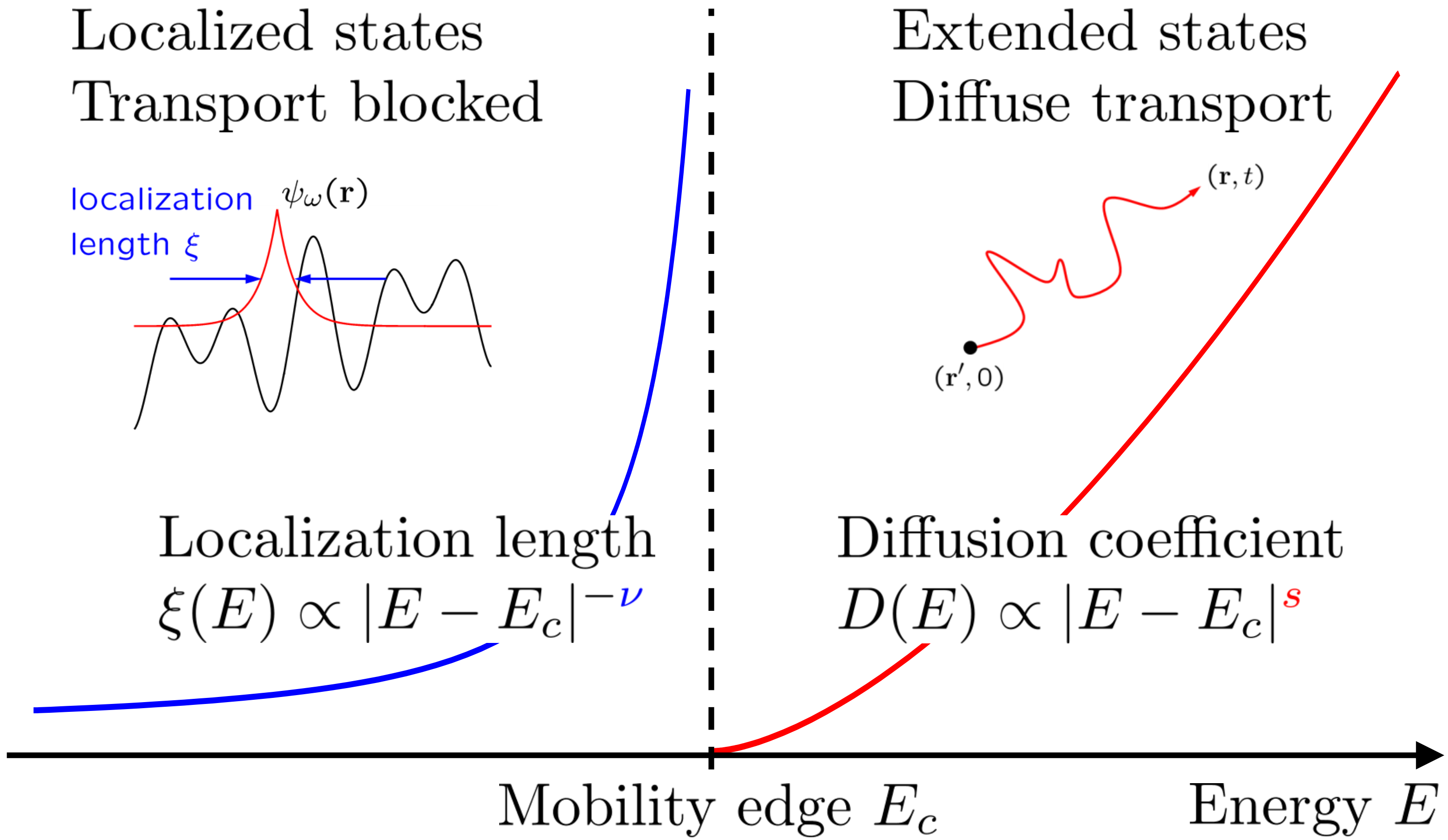
Scaling of dimensionless conductance



$$\beta(g) = \frac{\partial \ln g}{\partial \ln k_0 R}$$

PRL **112**, 023905 (2014)

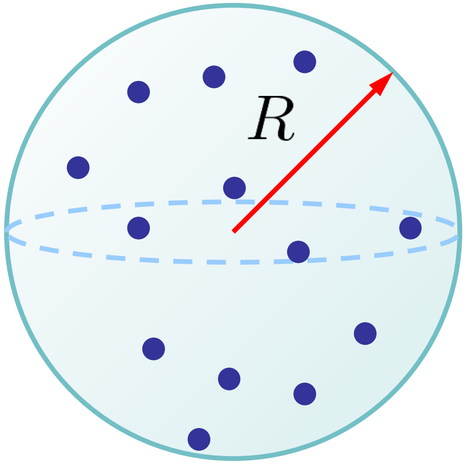
Critical behavior around the mobility edge



Critical exponents $s = \nu$ (in 3D)

Thouless conductance & scaling

N scatterers at density ρ



$$G_{ij} = i\delta_{ij} + (1-\delta_{ij})\frac{e^{ik_0 r_{ij}}}{k_0 r_{ij}}$$

(in the scalar case)

PRB **94**, 064202 (2016)

Thouless conductance & scaling

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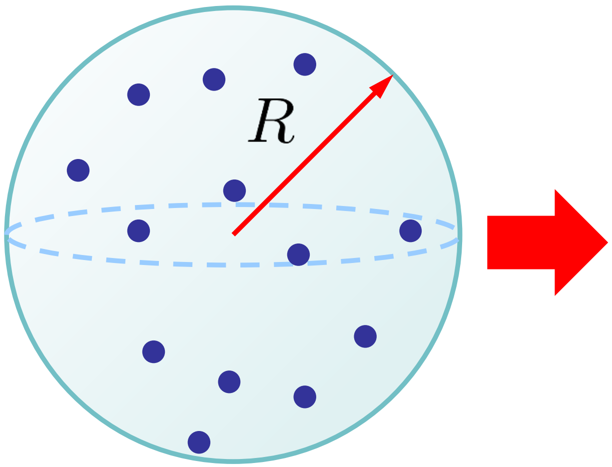

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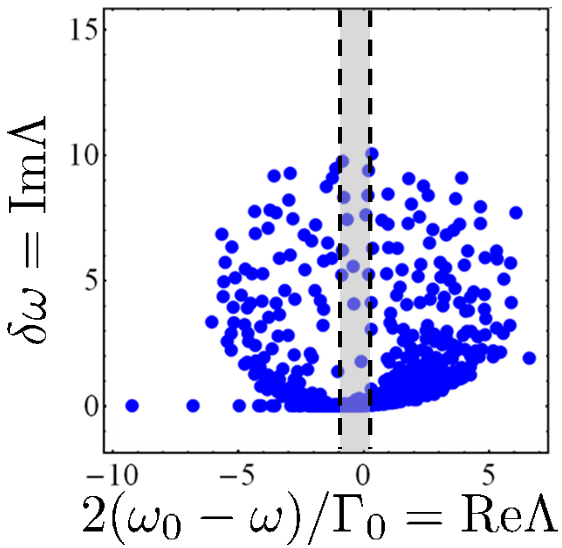
Eigenvalues of the Green's matrix

$G\psi_n = \Lambda_n \psi_n$



PRB **94**, 064202 (2016)

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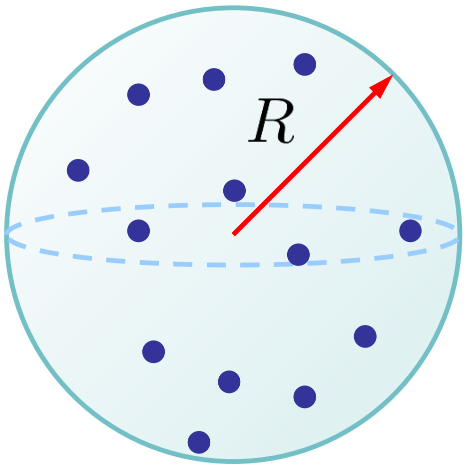

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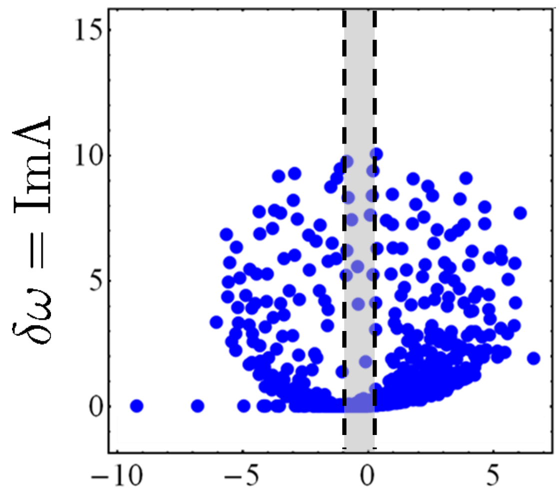


$2(\omega_0 - \omega)/\Gamma_0 = \mathrm{Re}\Lambda$

Thouless conductance

$$\delta\omega = \mathrm{Im}\Lambda$$

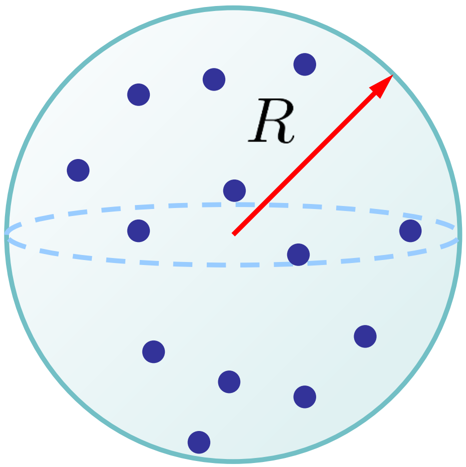
$$g(\omega) = \frac{\delta\omega}{\langle\Delta\omega\rangle}$$

$$\langle\Delta\omega\rangle = \langle\omega_{n+1} - \omega_n\rangle$$

PRB **94**, 064202 (2016)

Thouless conductance & scaling

N scatterers at density ρ



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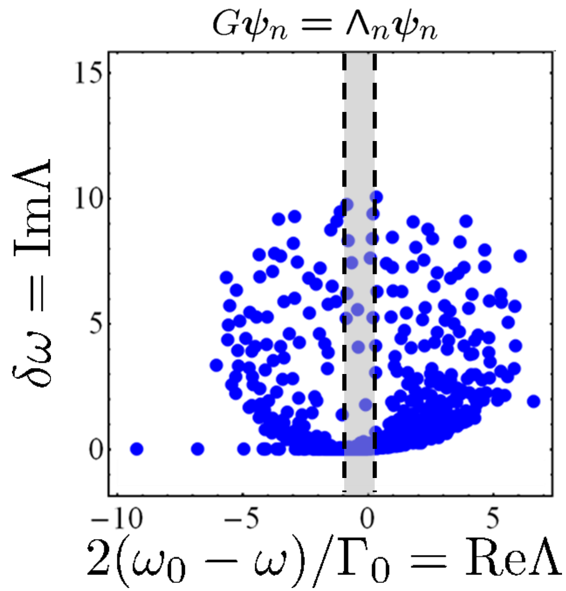


Thouless conductance

$$\delta\omega = \mathrm{Im}\Lambda$$

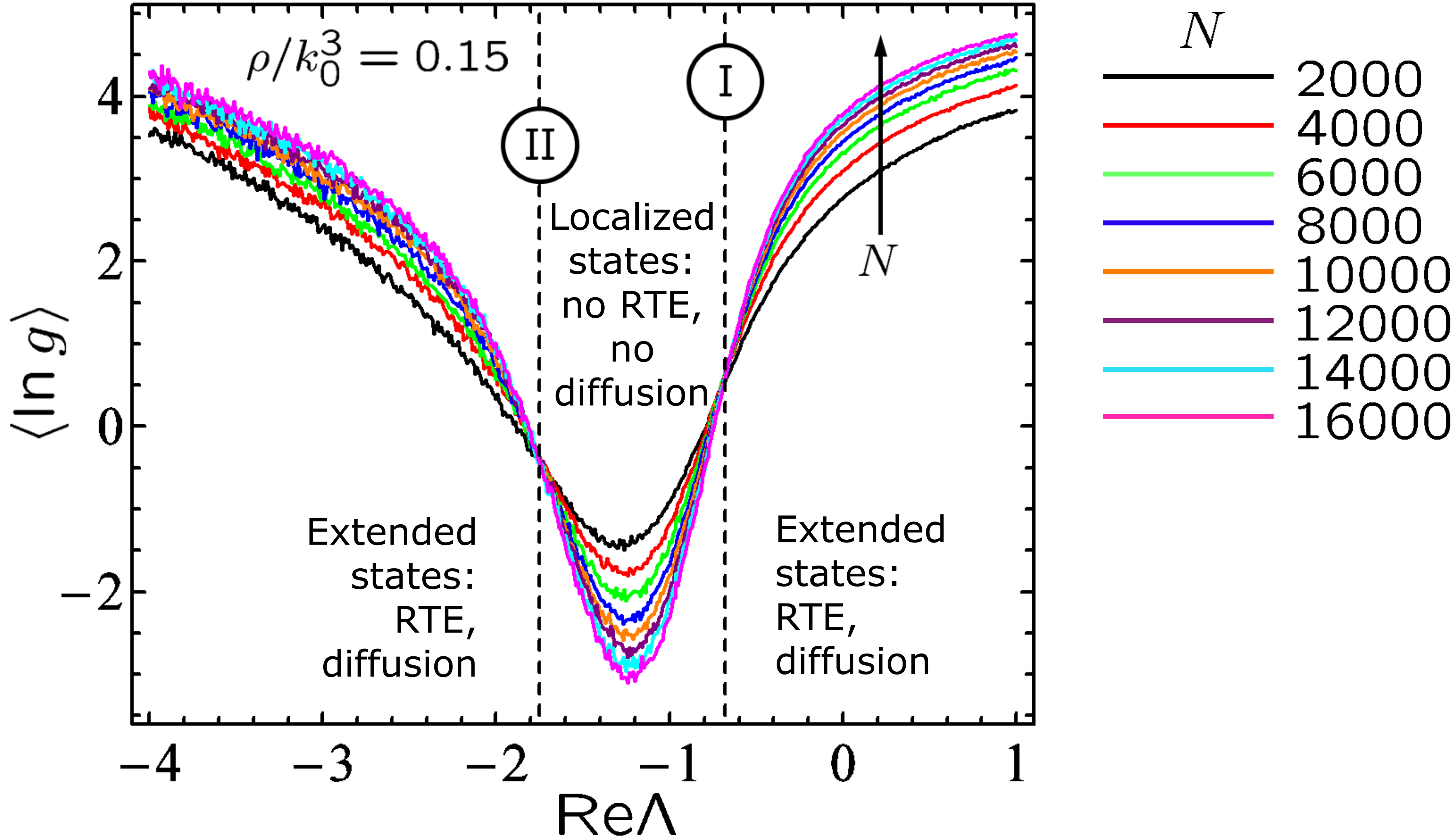
$$g(\omega) = \frac{\delta\omega}{\langle\Delta\omega\rangle}$$

$$\langle\Delta\omega\rangle = \langle\omega_{n+1} - \omega_n\rangle$$

We are going to study statistical properties of $g(\omega)$ at high $\rho > 0.1k_0^3$ at which localized states are expected

PRB **94**, 064202 (2016)

Scaling of the average ln *g*



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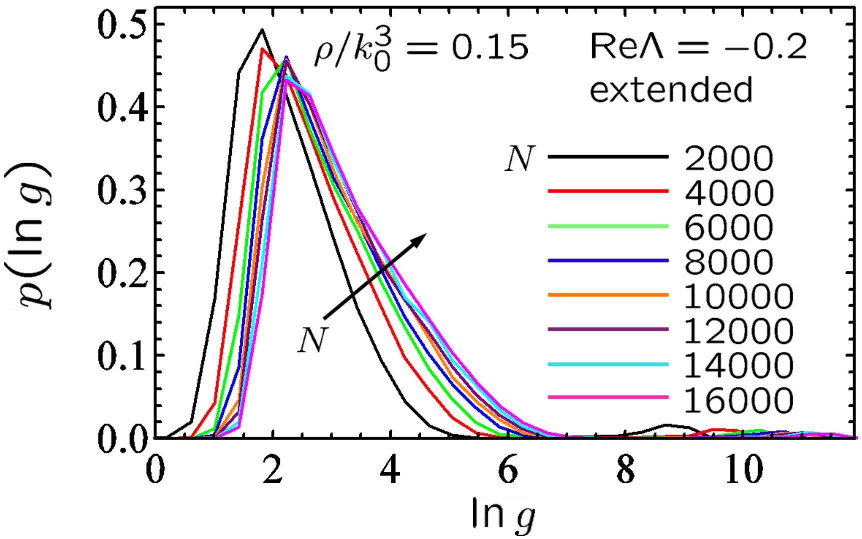
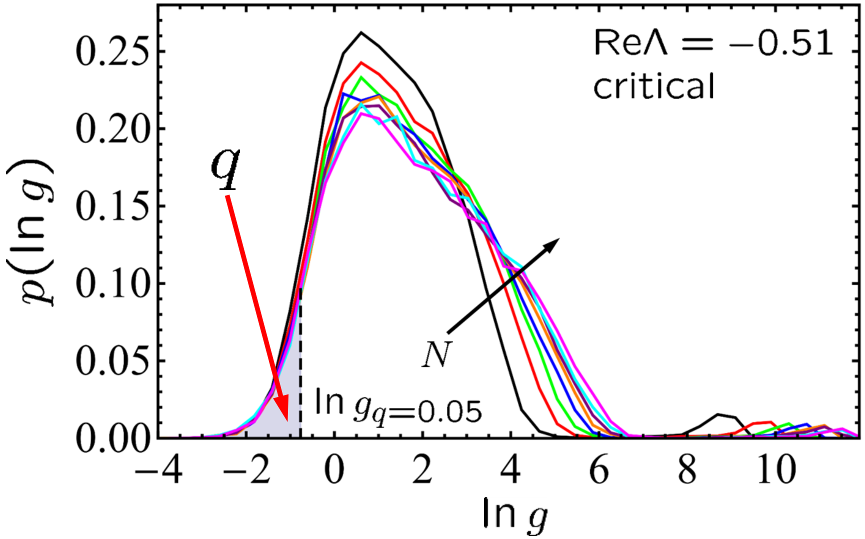
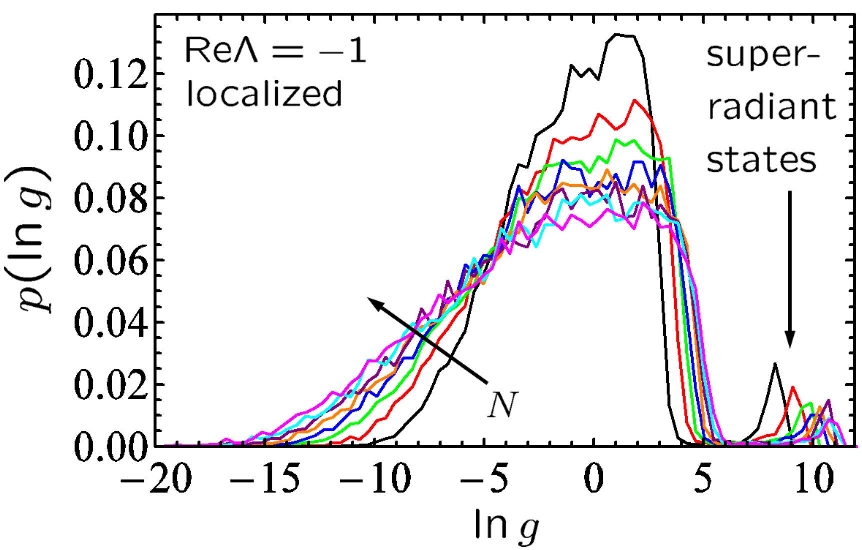
Distribution of conductance







Percentile g_q:

$$q = \int_0^{g_q} p(g) dg$$

PRB **94**, 064202 (2016)

Single-parameter scaling

R — system size

$$\ln g_q = F_q(R/\xi)$$

$$\xi \propto \frac{1}{(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c)^\nu}$$

— localization length

$$\begin{aligned} \ln g_q &= F_q(R/\xi) = F_q[R(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c)^\nu] \\ &= F_q[R^{1/\nu}(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c)] \longrightarrow F_q(\psi, \phi) \end{aligned}$$

Relevant scaling variable:

$$\psi = R^{1/\nu} u(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c), \quad u(x) = u_1 x + u_2 x^2 + \dots$$

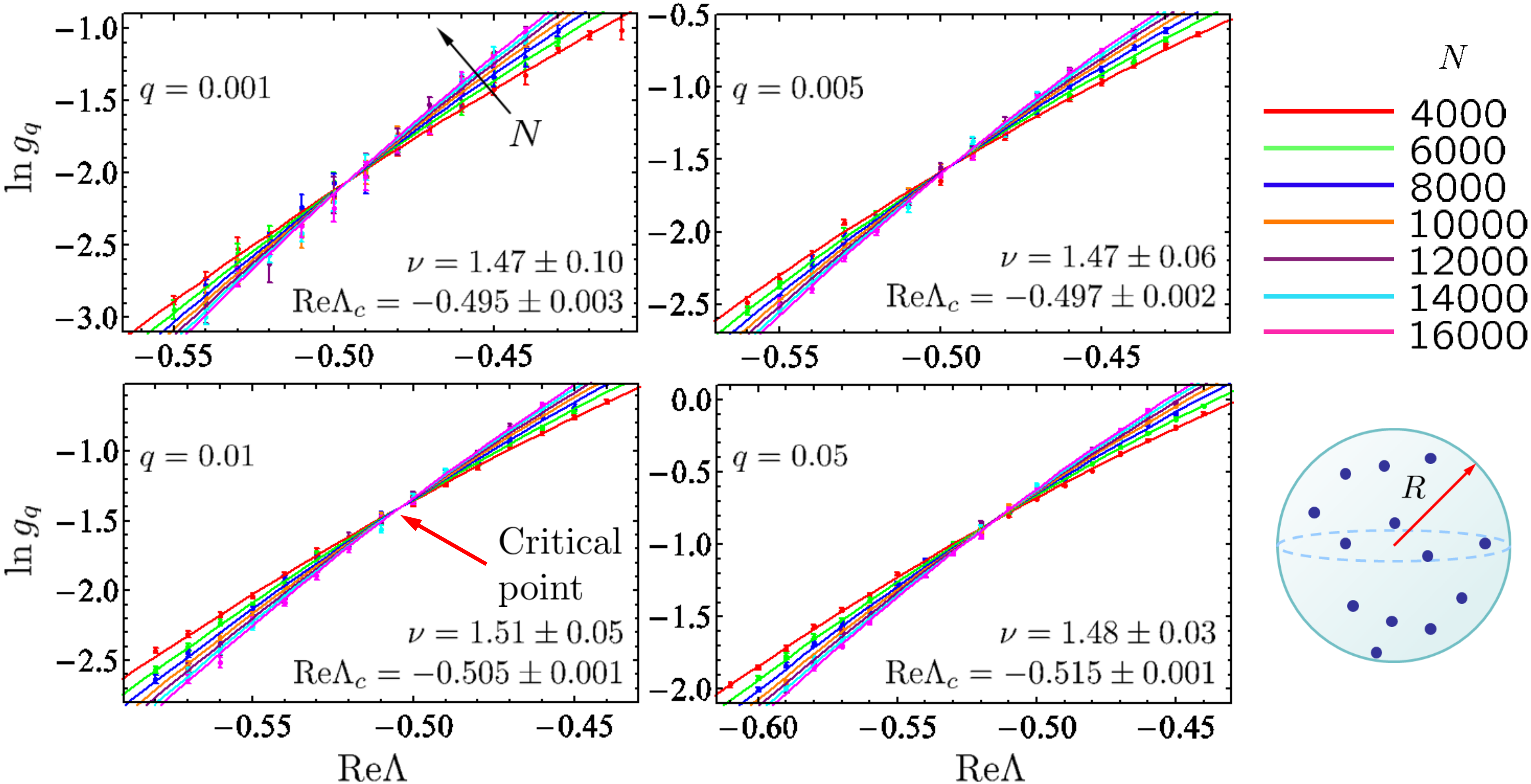
Irrelevant scaling variable:

$$\phi = R^{-y} v(\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c), \quad v(x) = v_0 + v_1 x + v_2 x^2 + \dots$$

Slevin, Markos, Ohtsuki, PRB **67**, 155106 (2003)

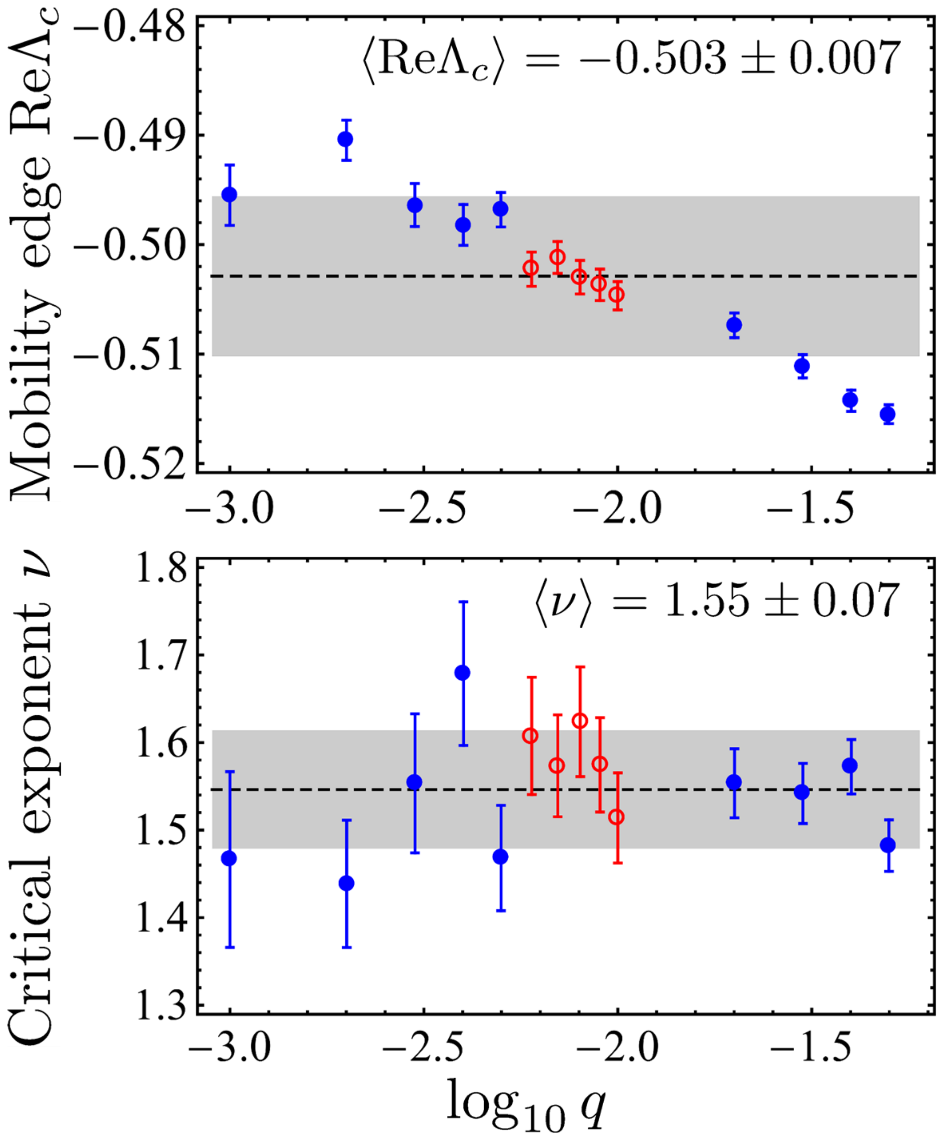
Finite-size scaling of percentiles

$$\ln g_q = F_q(R/\xi), \quad \xi \propto (\mathrm{Re}\Lambda - \mathrm{Re}\Lambda_c)^{-\nu}$$



PRB **94**, 064202 (2016)

Best-fit parameters

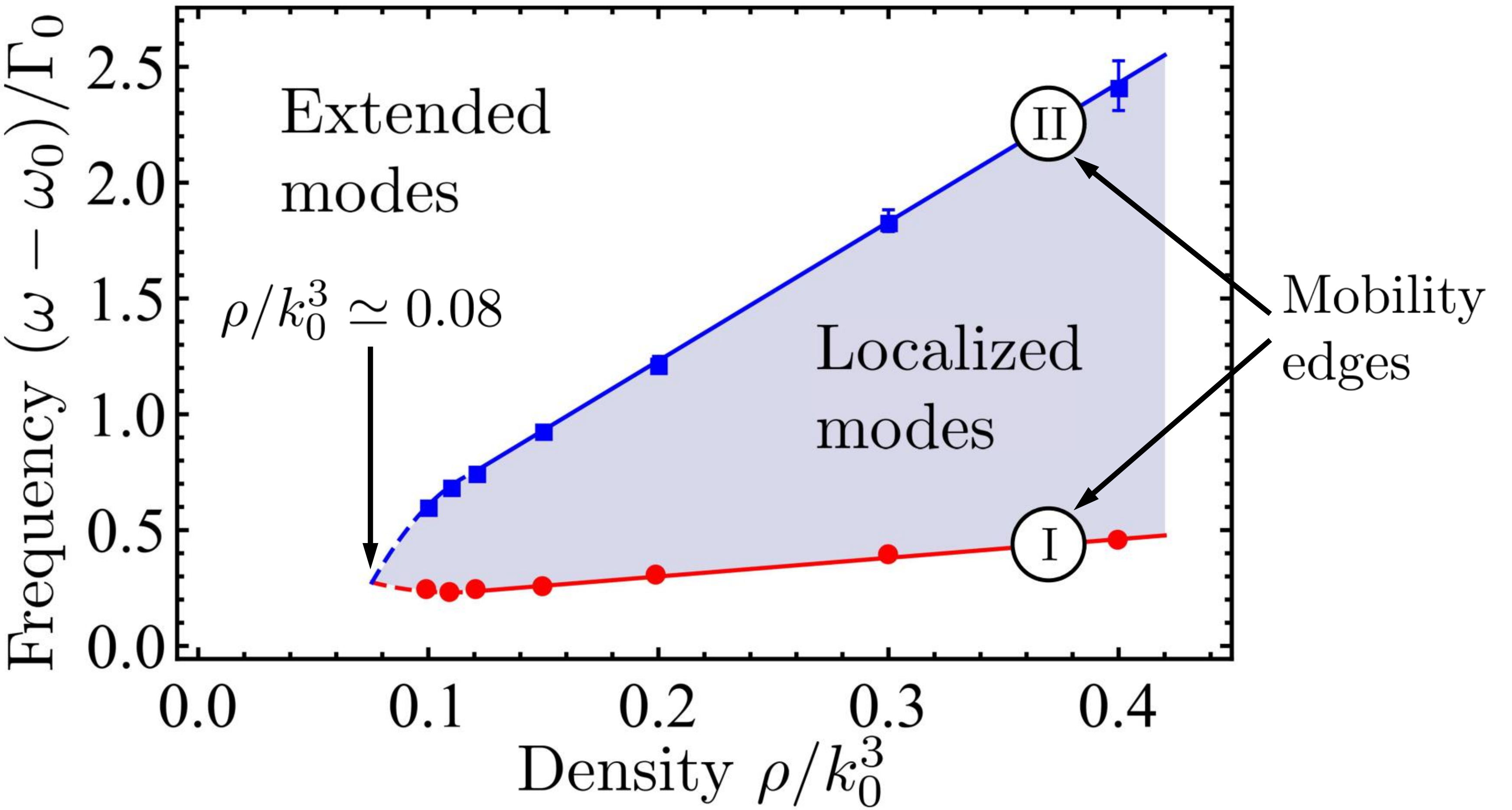


The value of critical exponent following from the fits is close to $\nu \simeq 1.57$ expected for the 3D orthogonal symmetry class.

We conclude that the observed transition is likely to belong to the same symmetry class as the **Anderson transition in a system of spinless electrons.**

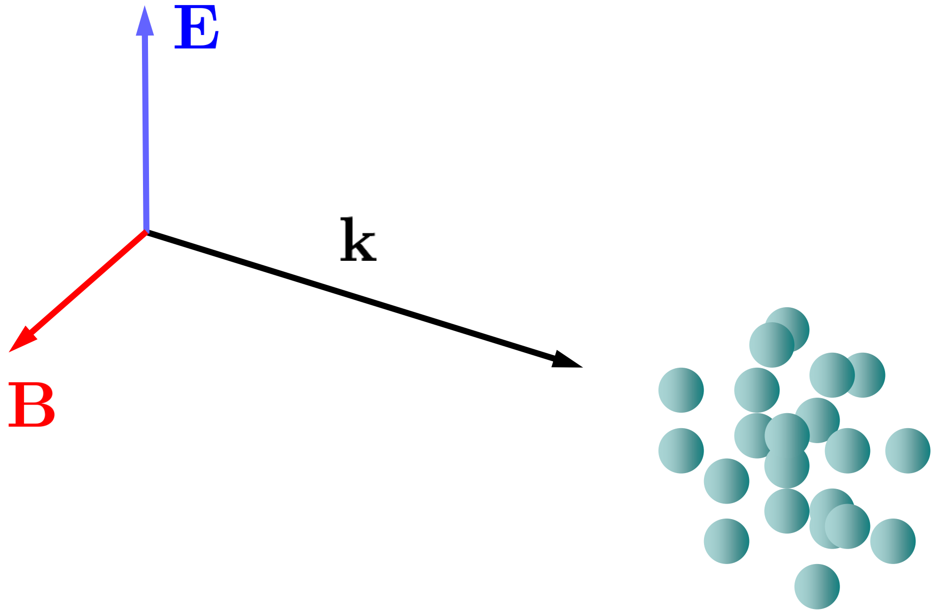
PRB **94**, 064202 (2016)

Phase diagram for scalar waves



PRB **98**, 064207 (2018)

Anderson localization of light?

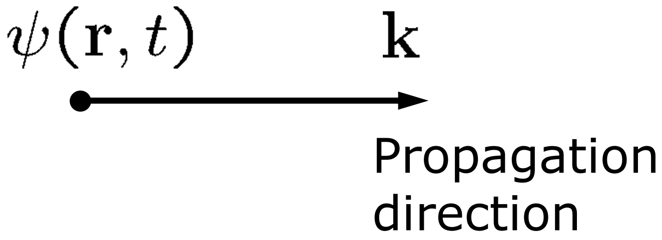


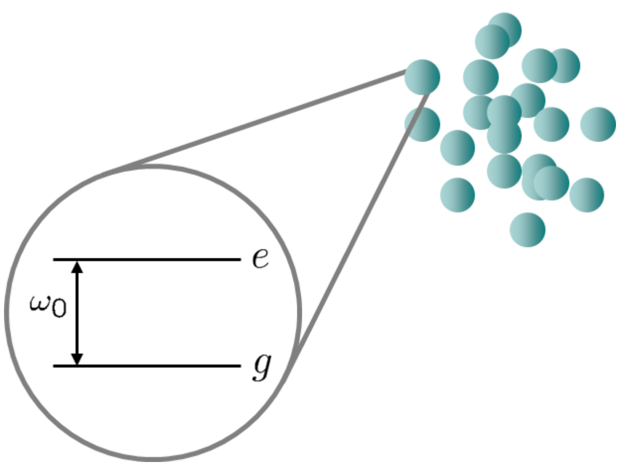
Pioneering theoretical works: John, PRL **53**, 2169 (1984)
Anderson, Philos. Mag. B **52**, 505 (1985)

Experiments inconclusive: Wiersma et al., Nature **390**, 671 (1997)
Sperling et al., Nat. Photonics **7**, 48 (2013)
Sperling et al., New J. Phys. **18**, 013039 (2016)

Light is a vector wave

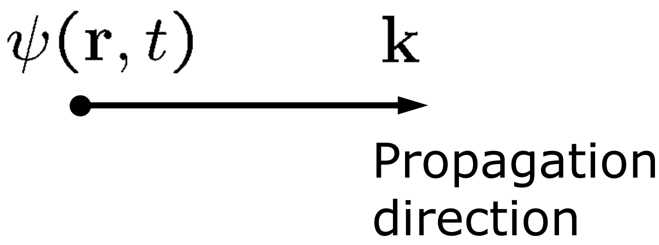
“Schrödinger waves” or sound



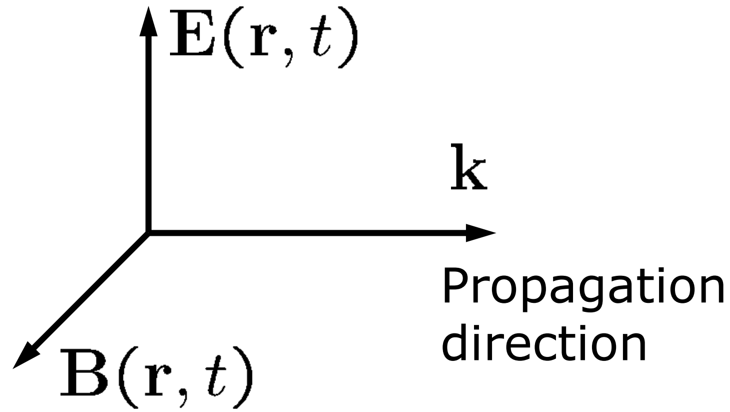


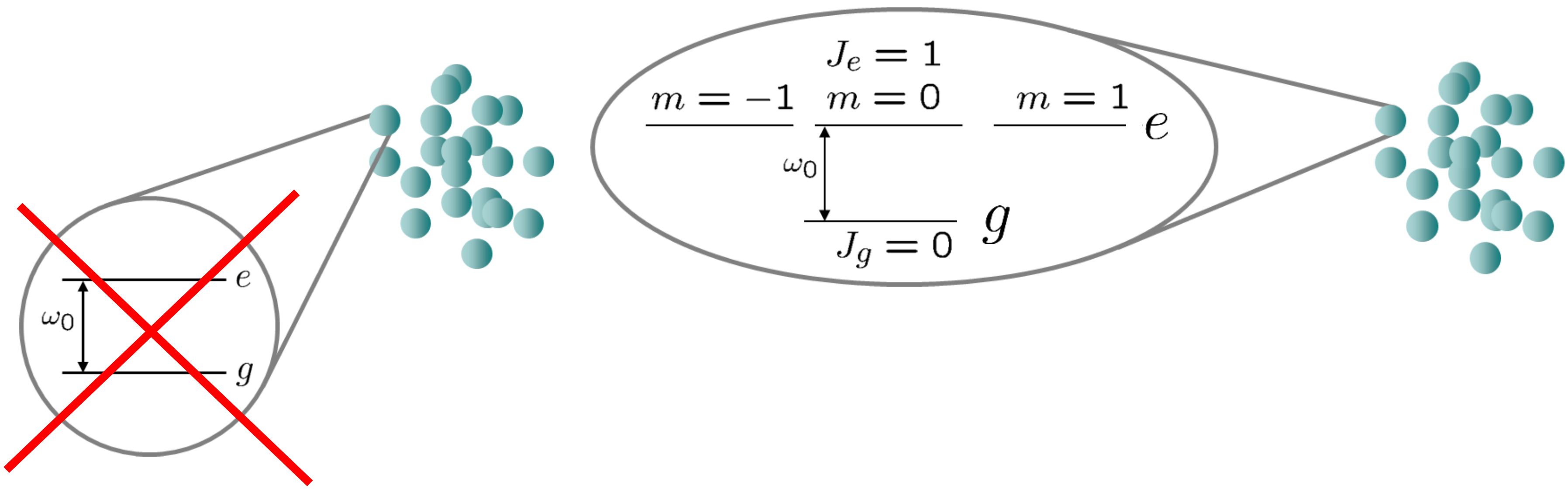
Light is a vector wave

“Schrödinger waves” or sound

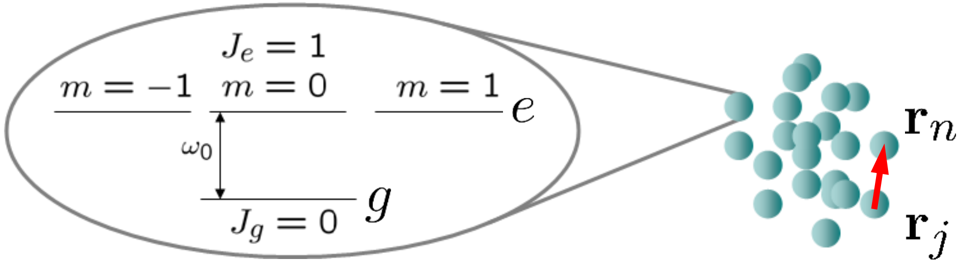


Electromagnetic waves





Green's matrix for light



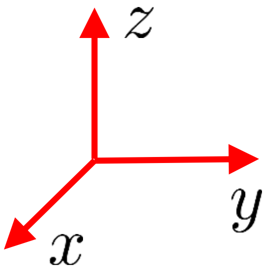
Green's matrix G describes propagation of light between pairs of atoms
$\mathbf{r}_{jn} = \mathbf{r}_n - \mathbf{r}_j$

$$G_{jn}^{\mu\nu} = i\delta_{jn}\delta_{\mu\nu} + (1-\delta_{jn})\frac{3}{2}\frac{e^{ik_0 r_{jn}}}{k_0 r_{jn}}\left[P(ik_0 r_{jn})\delta_{\mu\nu} + Q(ik_0 r_{jn})\frac{r_{jn}^{\mu} r_{jn}^{\nu}}{r_{jn}^2}\right]$$

$\mu,\ \nu = x,\ y,\ z$

natural basis

$$P(x) = 1 - 1/x + 1/x^2,\ Q(x) = -1 + 3/x - 3/x^2$$



Structure of the Green's matrix

$$\begin{pmatrix}
i & 0 & 0 & G_{12}^{xx} & G_{12}^{xy} & G_{12}^{xz} & \cdots & G_{1N}^{xx} & G_{1N}^{xy} & G_{1N}^{xz} \\
0 & i & 0 & G_{12}^{yx} & G_{12}^{yy} & G_{12}^{yz} & \cdots & G_{1N}^{yx} & G_{1N}^{yy} & G_{1N}^{yz} \\
0 & 0 & i & G_{12}^{zx} & G_{12}^{zy} & G_{12}^{zz} & \cdots & G_{1N}^{zx} & G_{1N}^{zy} & G_{1N}^{zz} \\
G_{21}^{xx} & G_{21}^{xy} & G_{21}^{xz} & i & 0 & 0 & \cdots\cdots & \cdots & \cdots & \cdots \\
G_{21}^{yx} & G_{21}^{yy} & G_{21}^{yz} & 0 & i & 0 & \cdots\cdots & \cdots & \cdots & \cdots \\
G_{21}^{zx} & G_{21}^{zy} & G_{21}^{zz} & 0 & 0 & i & \cdots\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
G_{N1}^{xx} & G_{N1}^{xy} & G_{N1}^{xz} & \cdots & \cdots & \cdots & \cdots & i & 0 & 0 \\
G_{N1}^{yx} & G_{N1}^{yy} & G_{N1}^{yz} & \cdots & \cdots & \cdots & \cdots & 0 & i & 0 \\
G_{N1}^{zx} & G_{N1}^{zy} & G_{N1}^{zz} & \cdots & \cdots & \cdots & \cdots & 0 & 0 & i
\end{pmatrix}$$

Structure of the Green's matrix

$$
\begin{pmatrix}
i & 0 & 0 & G_{12}^{xx} & G_{12}^{xy} & G_{12}^{xz} & \cdots & G_{1N}^{xx} & G_{1N}^{xy} & G_{1N}^{xz} \\
0 & i & 0 & G_{12}^{yx} & G_{12}^{yy} & G_{12}^{yz} & \cdots & G_{1N}^{yx} & G_{1N}^{yy} & G_{1N}^{yz} \\
0 & 0 & i & G_{12}^{zx} & G_{12}^{zy} & G_{12}^{zz} & \cdots & G_{1N}^{zx} & G_{1N}^{zy} & G_{1N}^{zz} \\
G_{21}^{xx} & G_{21}^{xy} & G_{21}^{xz} & i & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
G_{21}^{yx} & G_{21}^{yy} & G_{21}^{yz} & 0 & i & 0 & \cdots & \cdots & \cdots & \cdots \\
G_{21}^{zx} & G_{21}^{zy} & G_{21}^{zz} & 0 & 0 & i & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
G_{N1}^{xx} & G_{N1}^{xy} & G_{N1}^{xz} & \cdots & \cdots & \cdots & \cdots & i & 0 & 0 \\
G_{N1}^{yx} & G_{N1}^{yy} & G_{N1}^{yz} & \cdots & \cdots & \cdots & \cdots & 0 & i & 0 \\
G_{N1}^{zx} & G_{N1}^{zy} & G_{N1}^{zz} & \cdots & \cdots & \cdots & \cdots & 0 & 0 & i
\end{pmatrix}
$$

One-atom dynamics:

Excitation of an isolated excited atom decays as $e^{-\Gamma_0 t}$

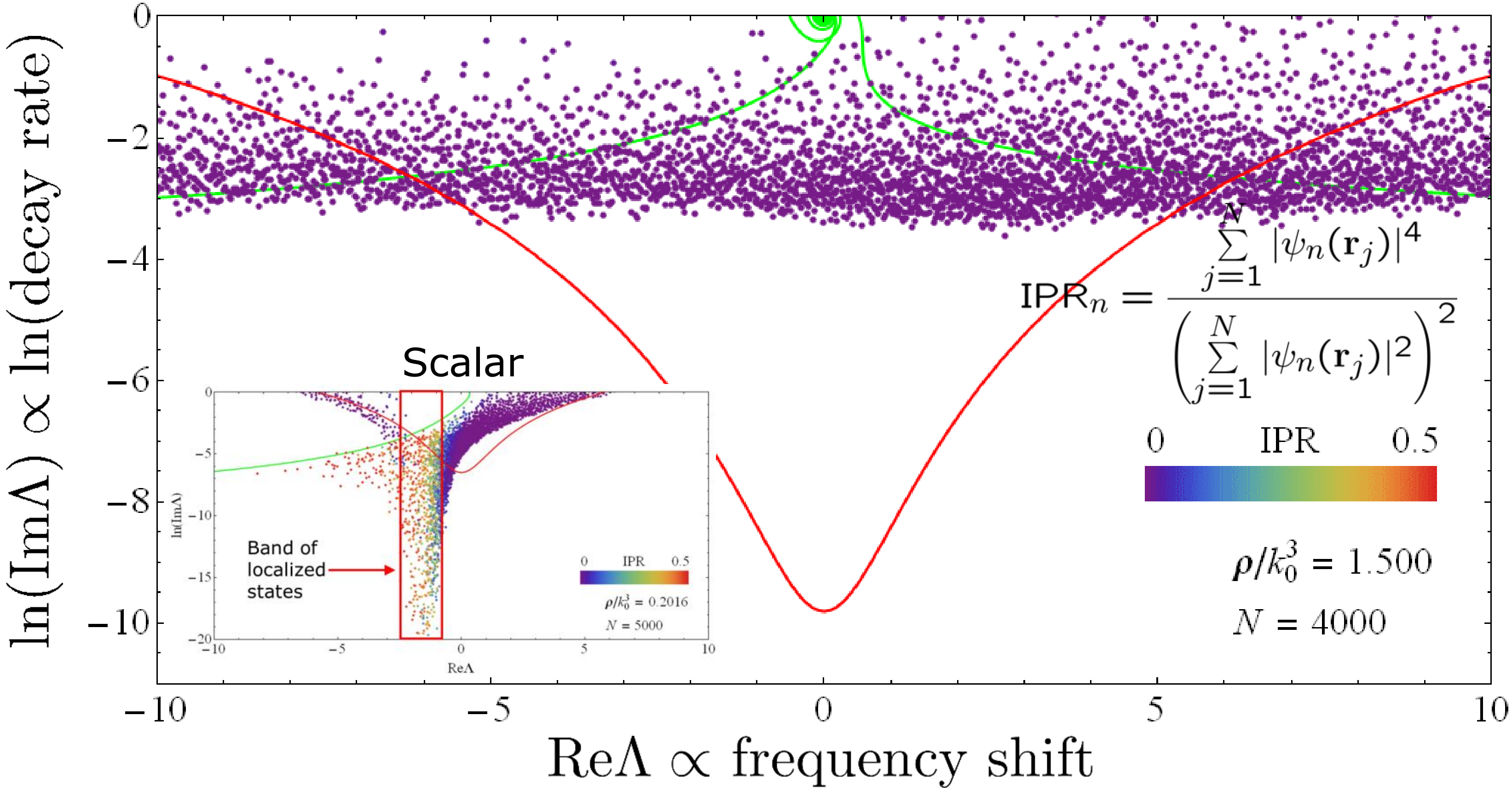
Structure of the Green's matrix

$$
\begin{pmatrix}
i & 0 & 0 & G^{xx}_{12} & G^{xy}_{12} & G^{xz}_{12} & \cdots & G^{xx}_{1N} & G^{xy}_{1N} & G^{xz}_{1N} \\
0 & i & 0 & G^{yx}_{12} & G^{yy}_{12} & G^{yz}_{12} & \cdots & G^{yx}_{1N} & G^{yy}_{1N} & G^{yz}_{1N} \\
0 & 0 & i & G^{zx}_{12} & G^{zy}_{12} & G^{zz}_{12} & \cdots & G^{zx}_{1N} & G^{zy}_{1N} & G^{zz}_{1N} \\
G^{xx}_{21} & G^{xy}_{21} & G^{xz}_{21} & i & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
G^{yx}_{21} & G^{yy}_{21} & G^{yz}_{21} & 0 & i & 0 & \cdots & \cdots & \cdots & \cdots \\
G^{zx}_{21} & G^{zy}_{21} & G^{zz}_{21} & 0 & 0 & i & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
G^{xx}_{N1} & G^{xy}_{N1} & G^{xz}_{N1} & \cdots & \cdots & \cdots & \cdots & i & 0 & 0 \\
G^{yx}_{N1} & G^{yy}_{N1} & G^{yz}_{N1} & \cdots & \cdots & \cdots & \cdots & 0 & i & 0 \\
G^{zx}_{N1} & G^{zy}_{N1} & G^{zz}_{N1} & \cdots & \cdots & \cdots & \cdots & 0 & 0 & i
\end{pmatrix}
$$

Pairwise coupling between atoms 1 & 2:

G^{xy}_{12} is the y component of the field at position 2 due to a dipole oscillating along x at position 1

Inverse participation ratio for light

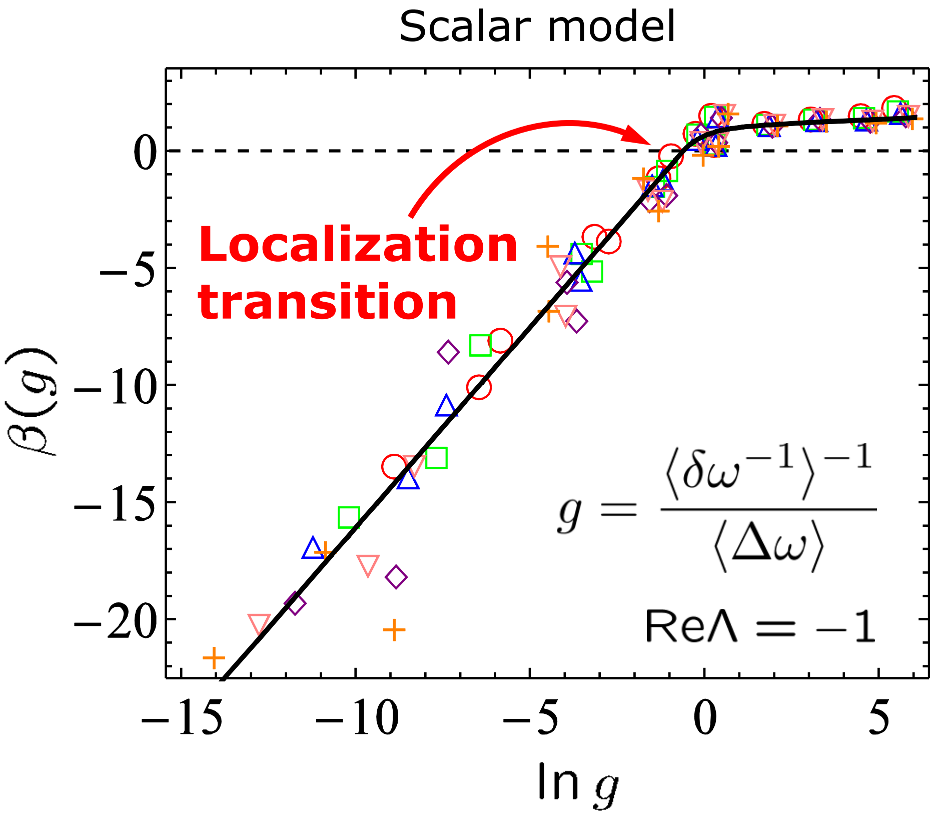


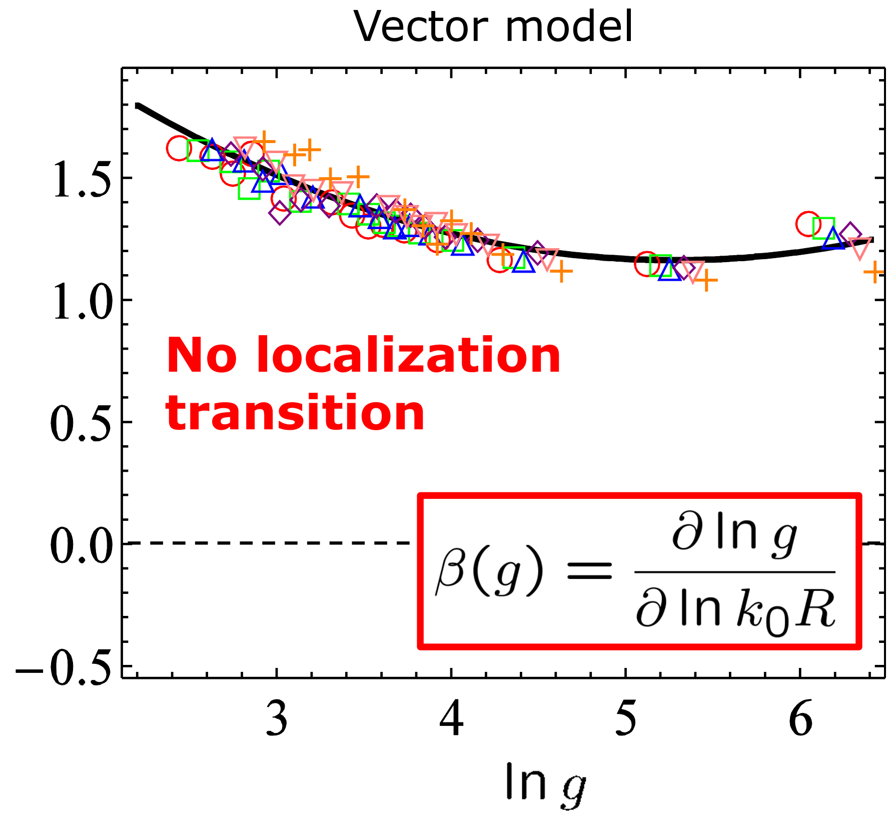
Eigenvalue domain boundary from the diffusion theory

Subradiant states localized on 2 closely located atoms

PRL **112**, 023905 (2014)

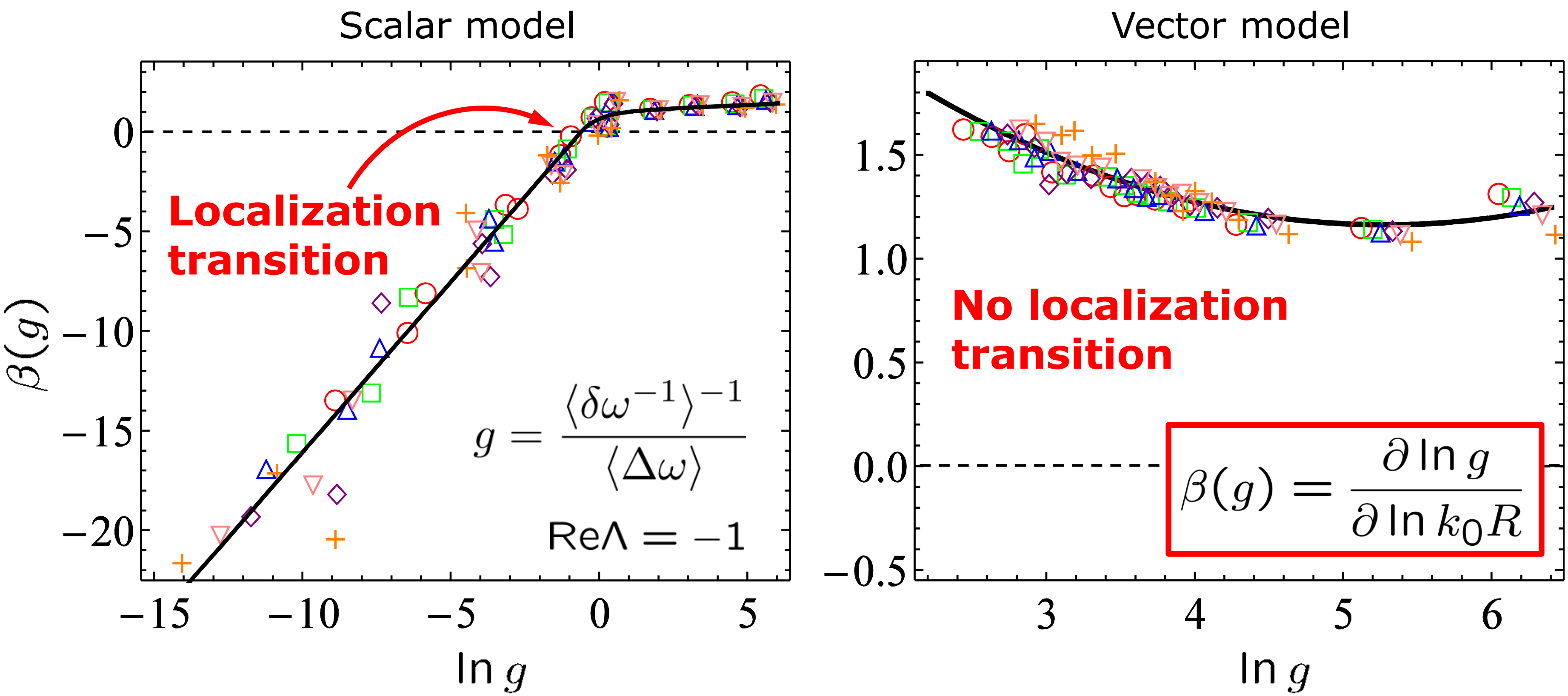
No Anderson localization for light in 3D





PRL **112**, 023905 (2014)

No Anderson localization for light in 3D

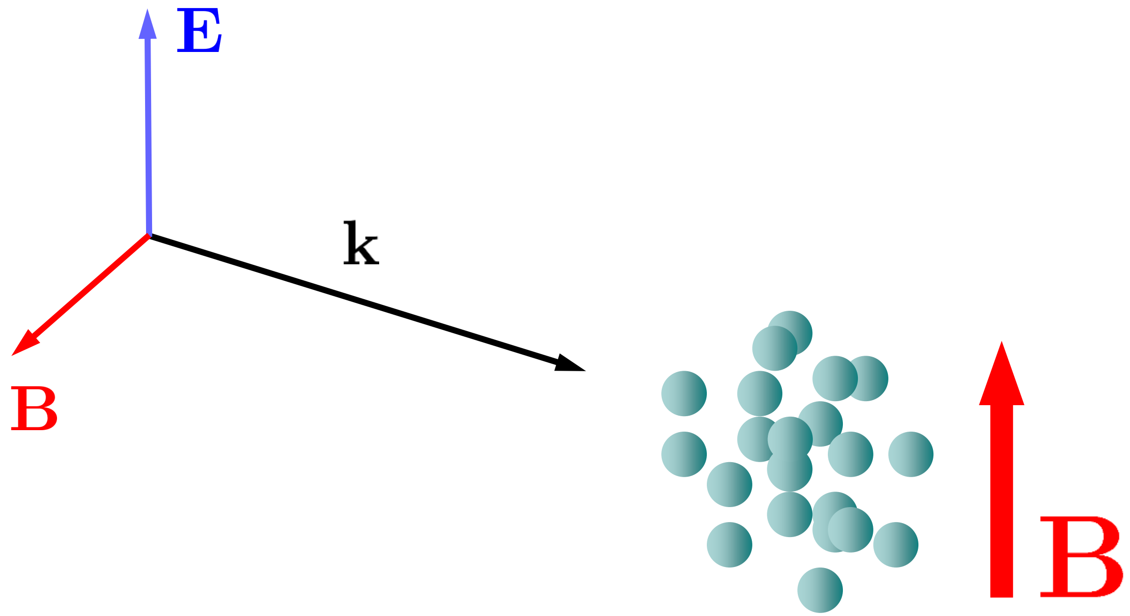


Explanation stems from near-field effects (dipole-dipole coupling):

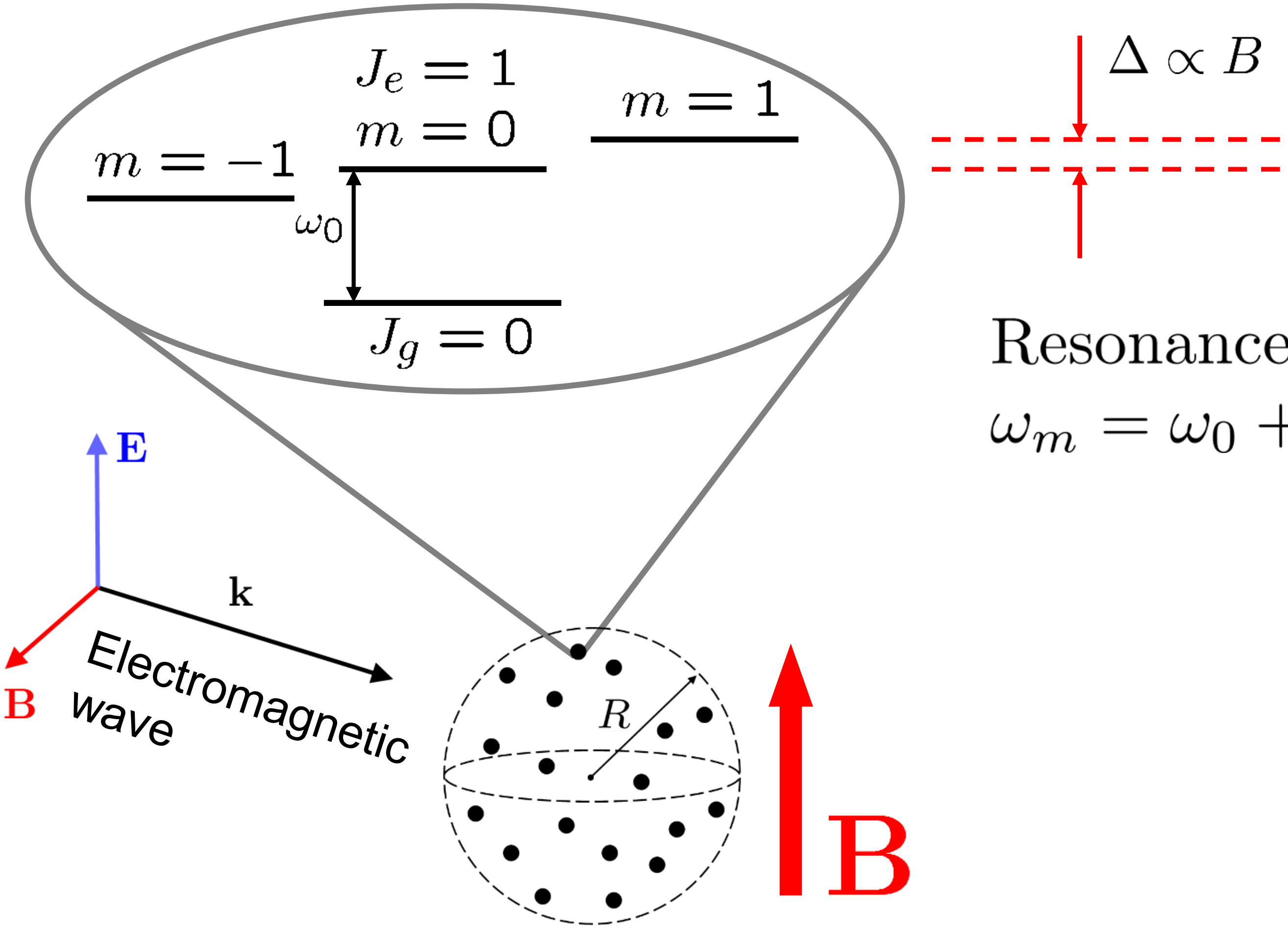
$$G_{\text{scalar}}(\mathbf{r})\big|_{r\to 0} \propto \frac{1}{r} \qquad \hat{G}_{\text{EM}}(\mathbf{r})\Big|_{r\to 0} \propto \frac{1}{r^3}$$

PRL **112**, 023905 (2014)

Anderson localization of light in a magnetic field



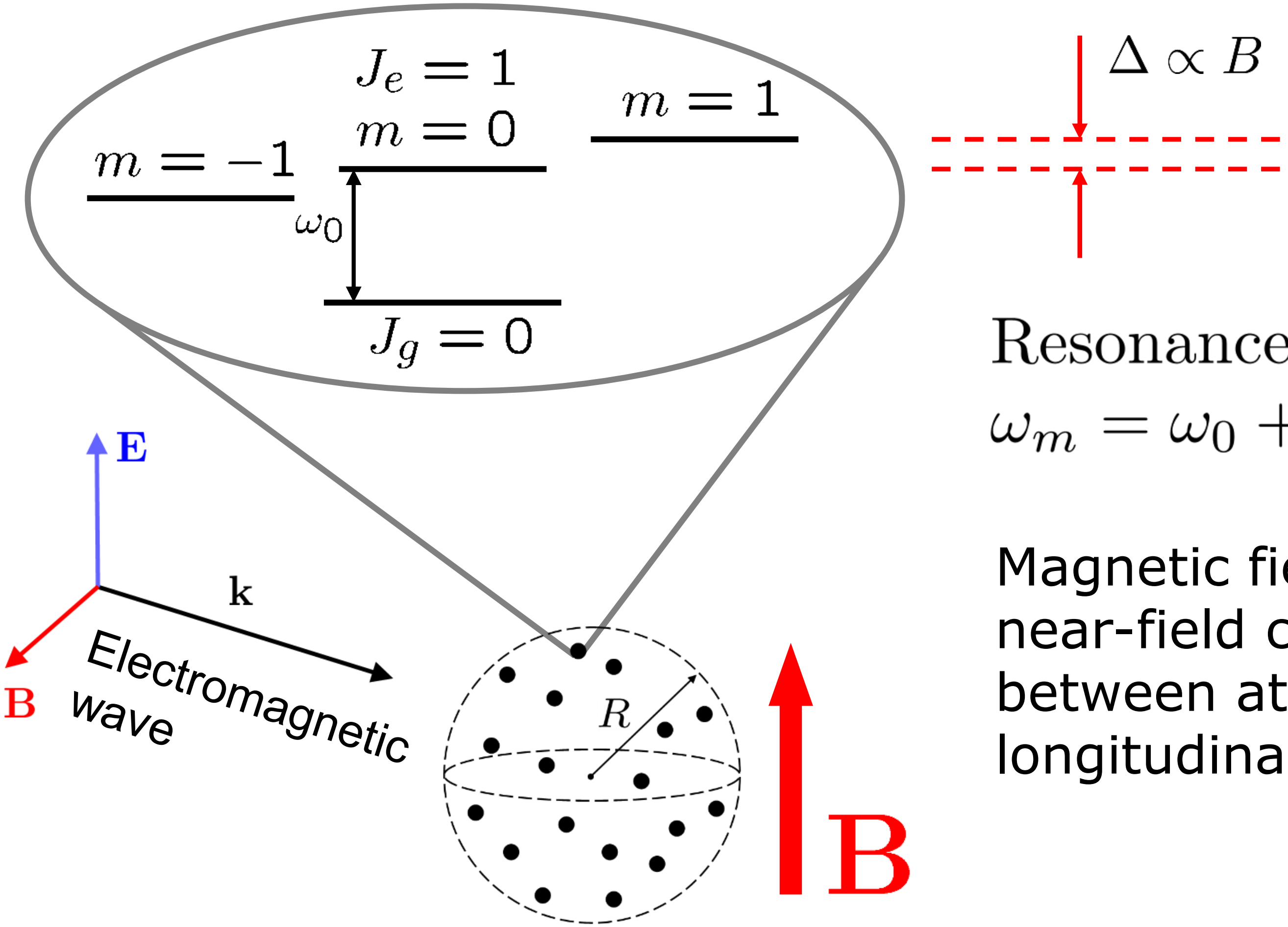
Atoms in a magnetic field



Resonance frequencies:

$$\omega_m = \omega_0 + m\Gamma_0\Delta$$

Atoms in a magnetic field



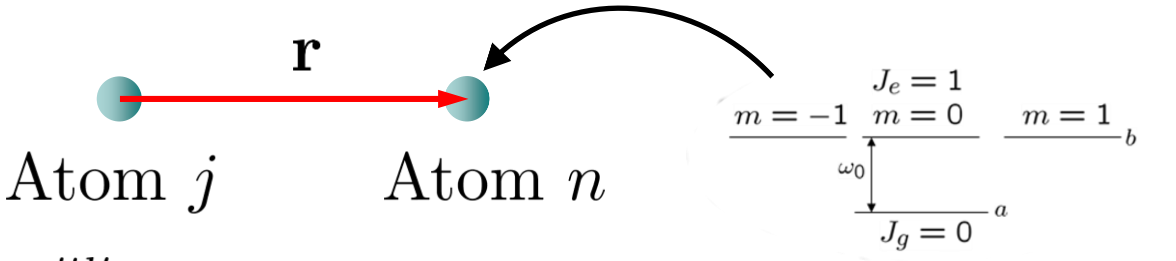
Resonance frequencies:

$$\omega_m = \omega_0 + m\Gamma_0\Delta$$

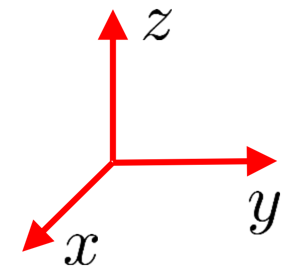
Magnetic field suppresses near-field coupling between atoms by longitudinal fields

Afrousheh et al., PRA **73**, 063403 (2006)

From natural to spiral basis



natural basis

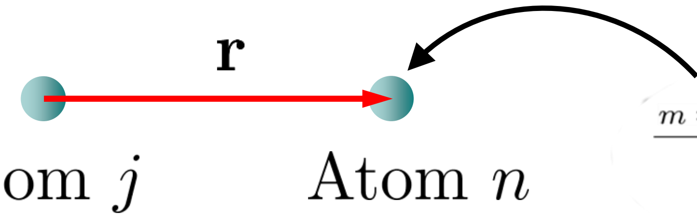


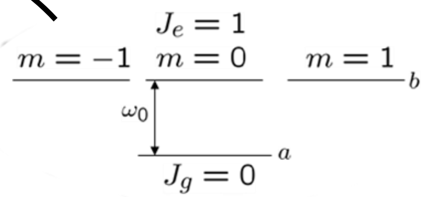
$$G_{jn}^{\mu\nu} = i\delta_{jn}\delta_{\mu\nu} + (1-\delta_{jn})\frac{3}{2}\frac{e^{ik_0 r_{jn}}}{k_0 r_{jn}}\left[P(ik_0 r_{jn})\delta_{\mu\nu} + Q(ik_0 r_{jn})\frac{r_{jn}^{\mu} r_{jn}^{\nu}}{r_{jn}^2}\right]$$

$\mu,\ \nu = x,\ y,\ z$

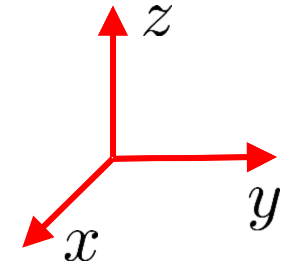
$$P(x) = 1 - 1/x + 1/x^2,\ \ Q(x) = -1 + 3/x - 3/x^2$$

From natural to spiral basis





natural basis



$$G^{\mu\nu}_{jn} = i\delta_{jn}\delta_{\mu\nu} + (1-\delta_{jn})\frac{3}{2}\frac{e^{ik_0 r_{jn}}}{k_0 r_{jn}}\left[P(ik_0 r_{jn})\delta_{\mu\nu} + Q(ik_0 r_{jn})\frac{r^{\mu}_{jn} r^{\nu}_{jn}}{r^2_{jn}}\right]$$

$\mu,\ \nu = x,\ y,\ z$

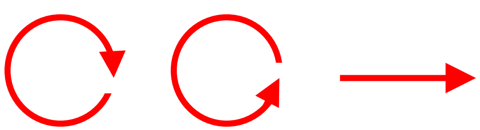
$$P(x) = 1 - 1/x + 1/x^2,\ \ Q(x) = -1 + 3/x - 3/x^2$$

$$G_{mm'}(\mathbf{r}) = -\frac{4k_0^3}{3\hbar\Gamma_0}\sum_{\mu,\nu} d_{m\mu} G_{\mu\nu}(\mathbf{r}) d_{m'\nu}$$

$m,\ m' = -1,\ 0,\ 1$

$\mathbf{d}_m = \langle J_e m | \hat{\mathbf{D}} | J_g 0 \rangle$

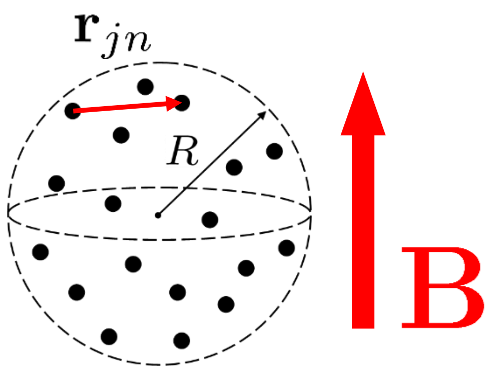
spiral basis



$m = -1 \quad +1 \quad 0$

Green's matrix in a magnetic field

$$
\begin{aligned}
G_{e_{jm}e_{nm'}} \quad = \quad & (i - 2m\Delta)\,\delta_{e_{jm}e_{nm'}} - \frac{2}{\hbar\Gamma_0}(1 - \delta_{e_{jm}e_{nm'}}) \\
\times \quad & \sum_{\mu,\nu} d^{\mu}_{e_{jm}g_j} d^{\nu}_{g_n e_{nm'}} \frac{e^{ik_0 r_{jn}}}{r^3_{jn}} \\
\times \quad & \left\{ \delta_{\mu\nu} \left[1 - ik_0 r_{jn} - (k_0 r_{jn})^2\right] \right. \\
- \quad & \left. \frac{r^{\mu}_{jn} r^{\nu}_{jn}}{r^2_{jn}} \left[3 - 3ik_0 r_{jn} - (k_0 r_{jn})^2\right] \right\}
\end{aligned}
$$

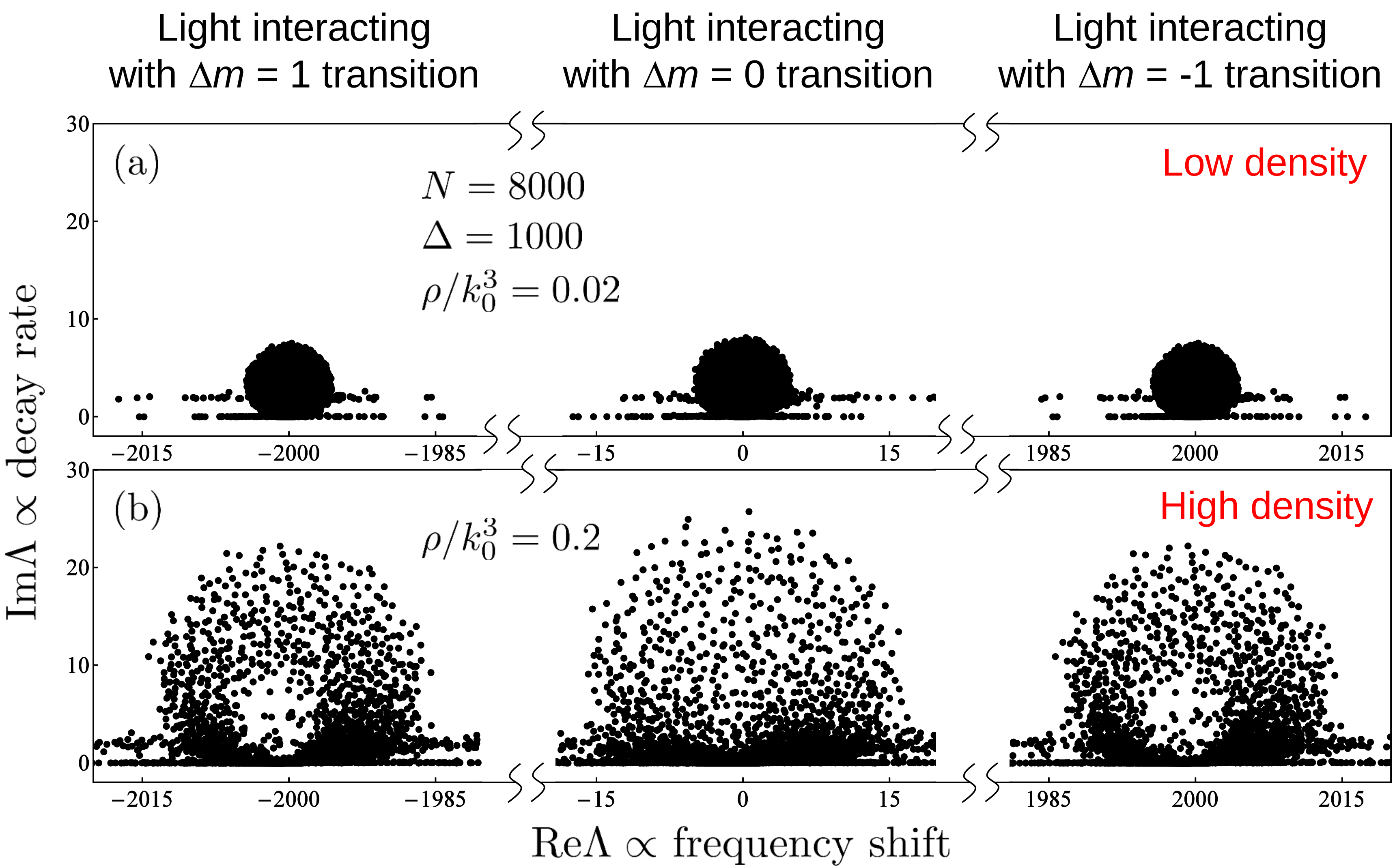


$$\Delta = g_e \mu_B B / \hbar\Gamma_0$$

$$\mathbf{d}_{e_{jm}g_j} = \langle J_e m | \hat{\mathbf{D}}_j | J_g 0 \rangle$$

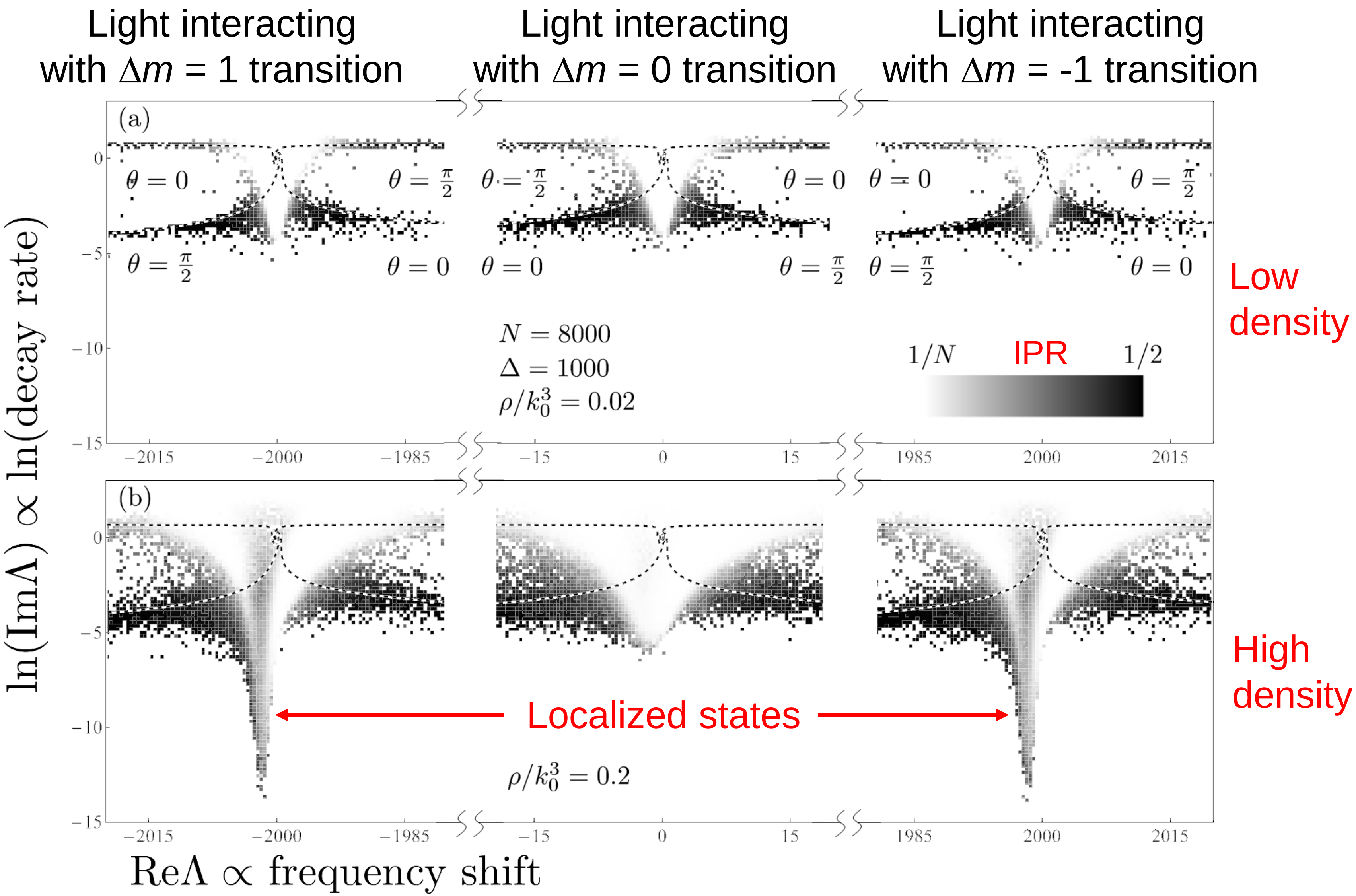
see also Pinheiro et al., Acta. Phys. Pol. A **105**, 339 (2004)

Eigenvalues in a strong magnetic field



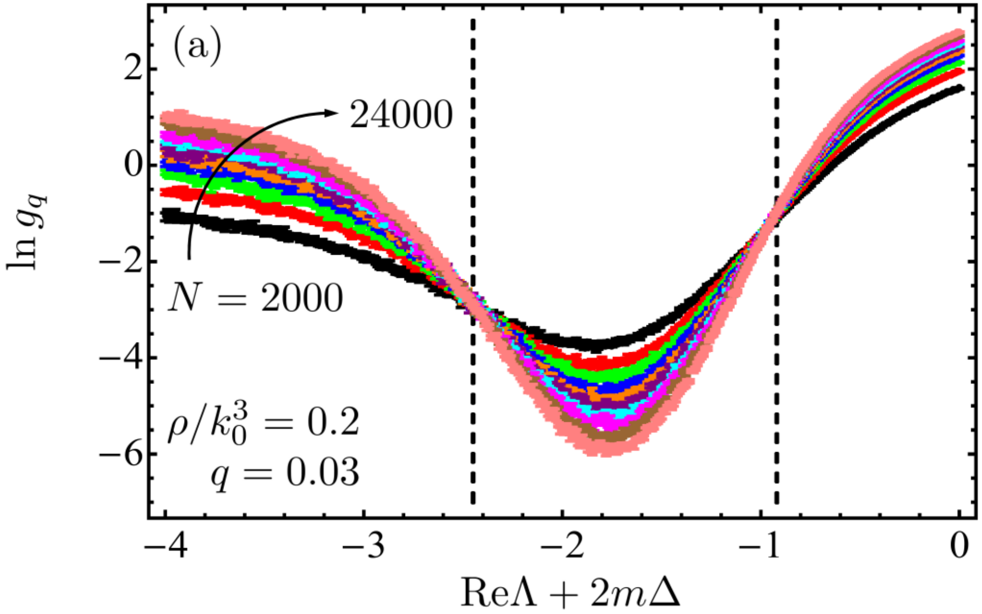
PRL **114**, 053902 (2015)

Average inverse participation ratio



PRL **114**, 053902 (2015)

Finite-size scaling of percentiles

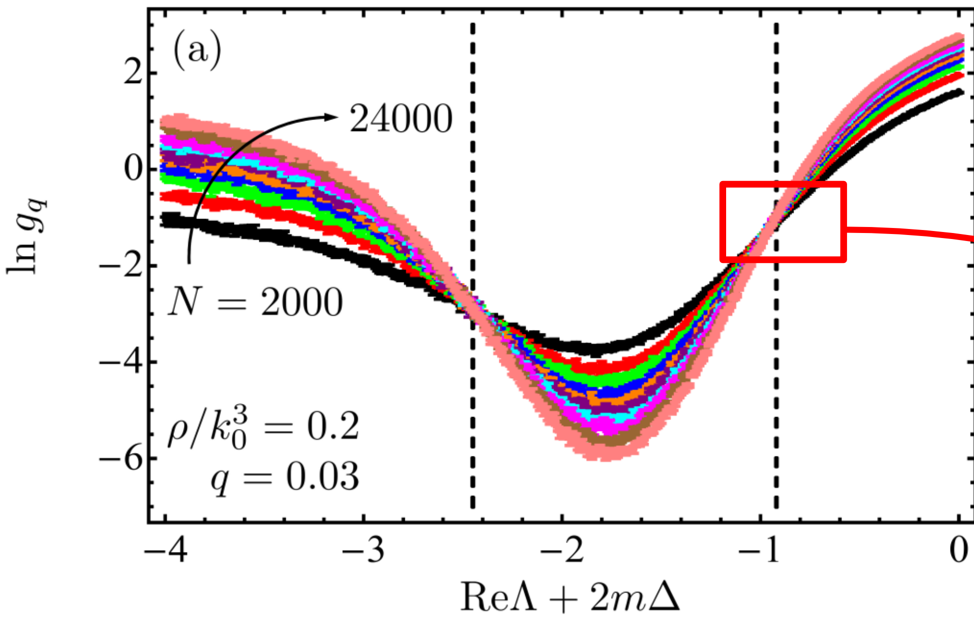


Percentile g_q:

$$q = \int_0^{g_q} p(g)dg$$

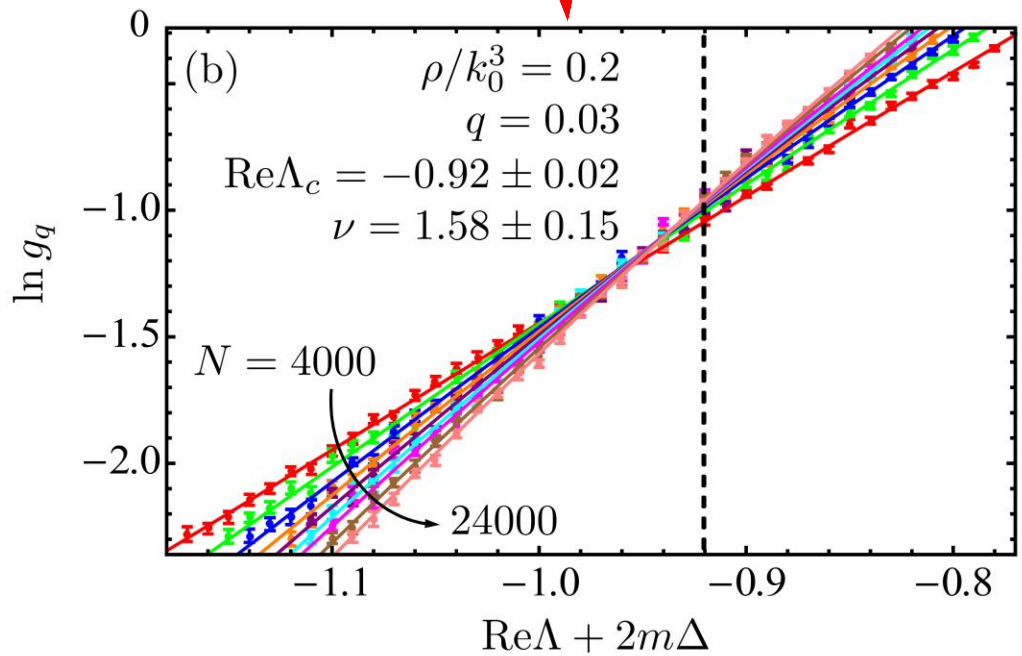
PRL **121**, 093601 (2018)

Finite-size scaling of percentiles



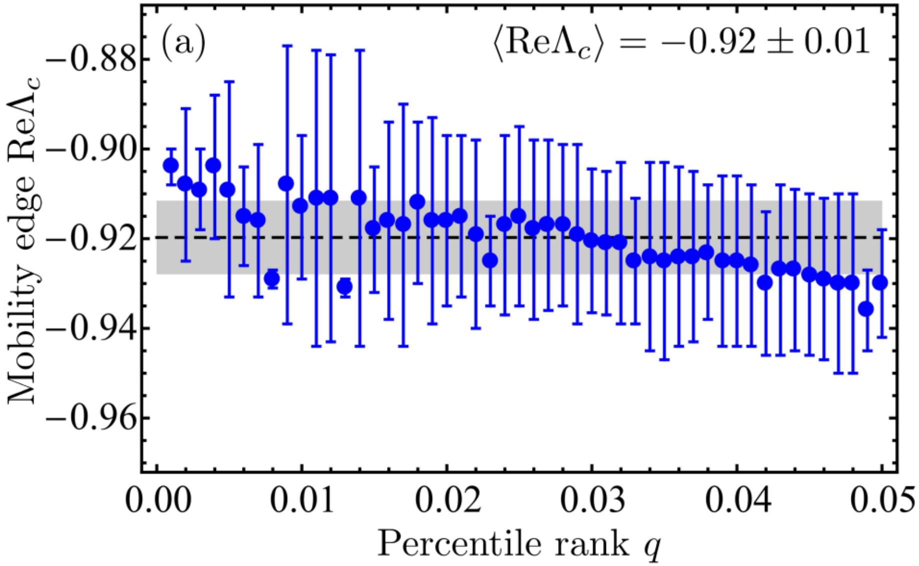
Percentile g_q:

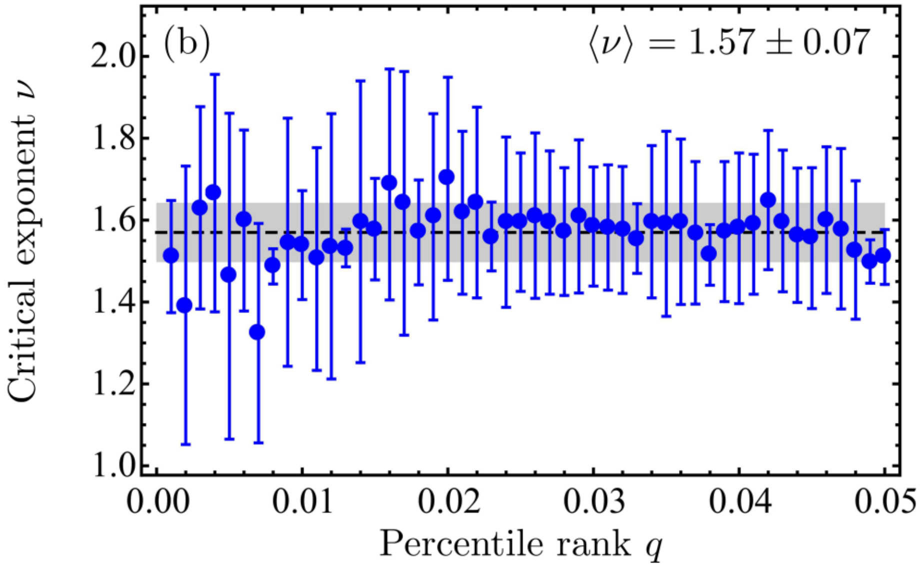
$$q = \int_0^{g_q} p(g)\,dg$$



PRL **121**, 093601 (2018)

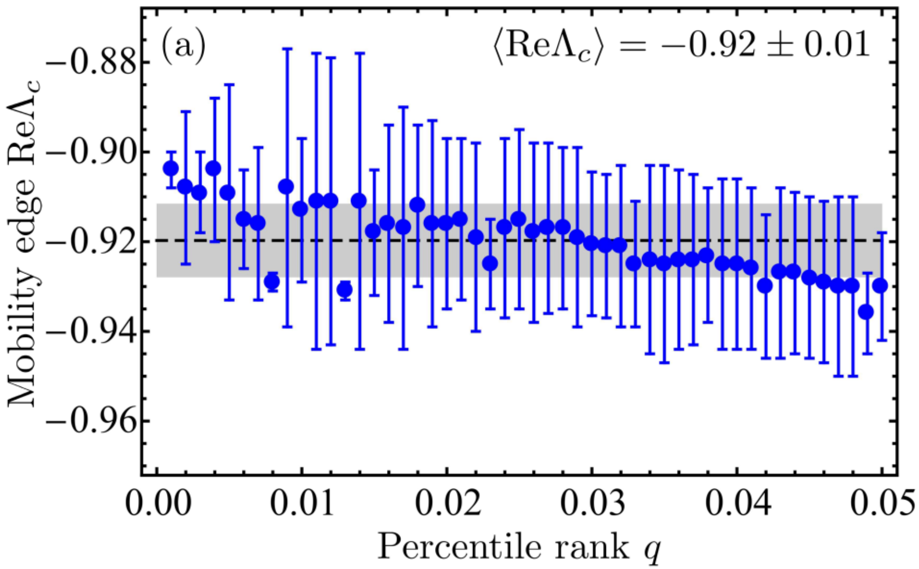
Critical parameters





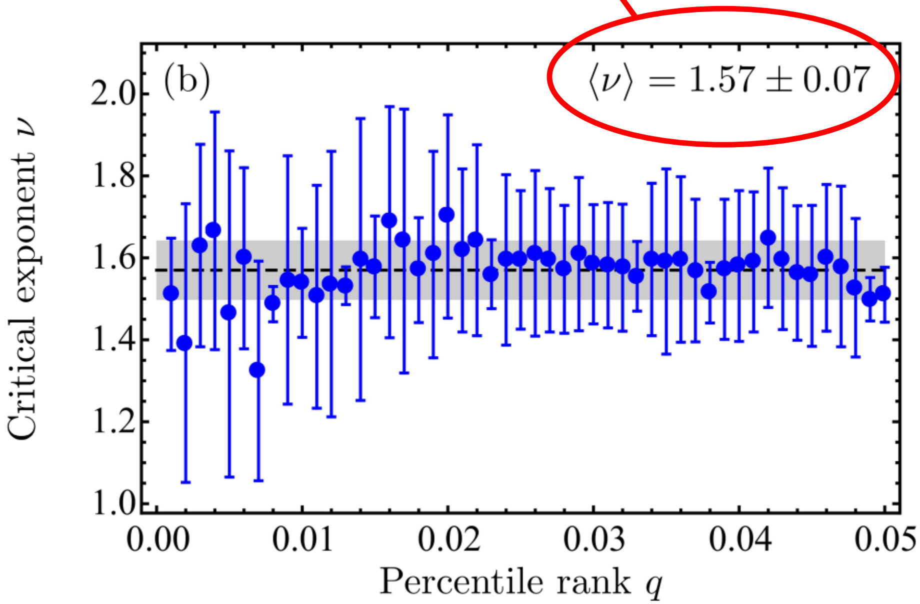
PRL **121**, 093601 (2018)

Critical parameters



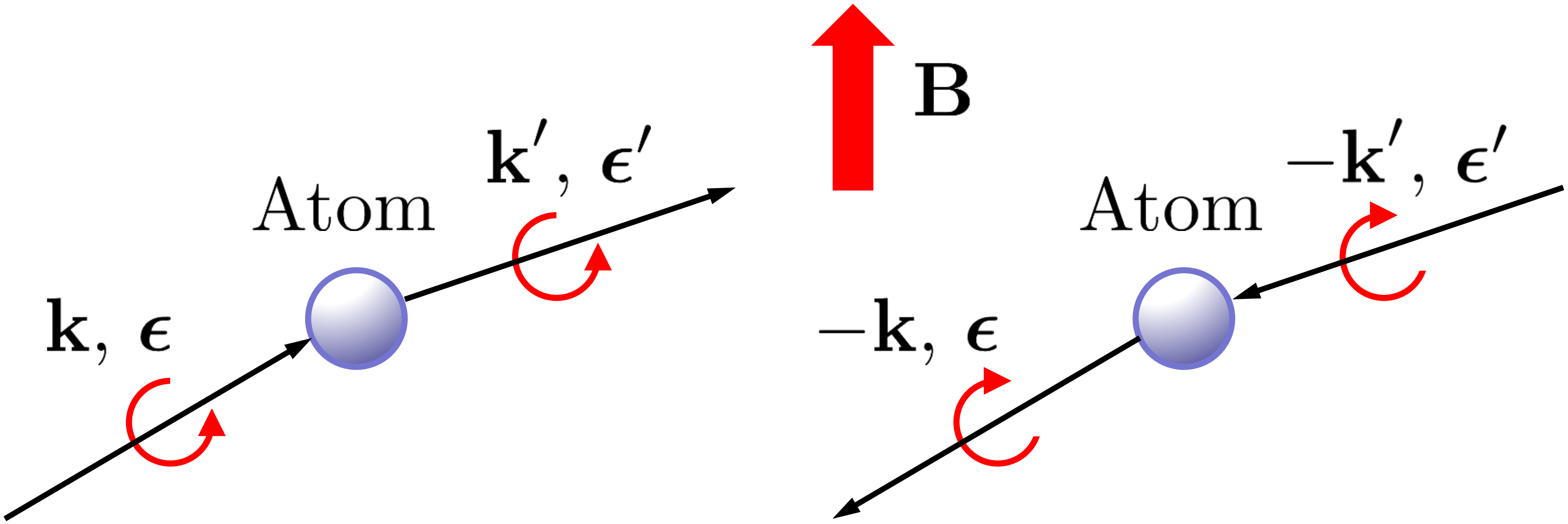
Transition of 3D orthogonal universality class

(= spinless electrons in the presence of time-reversal invariance)



PRL **121**, 093601 (2018)

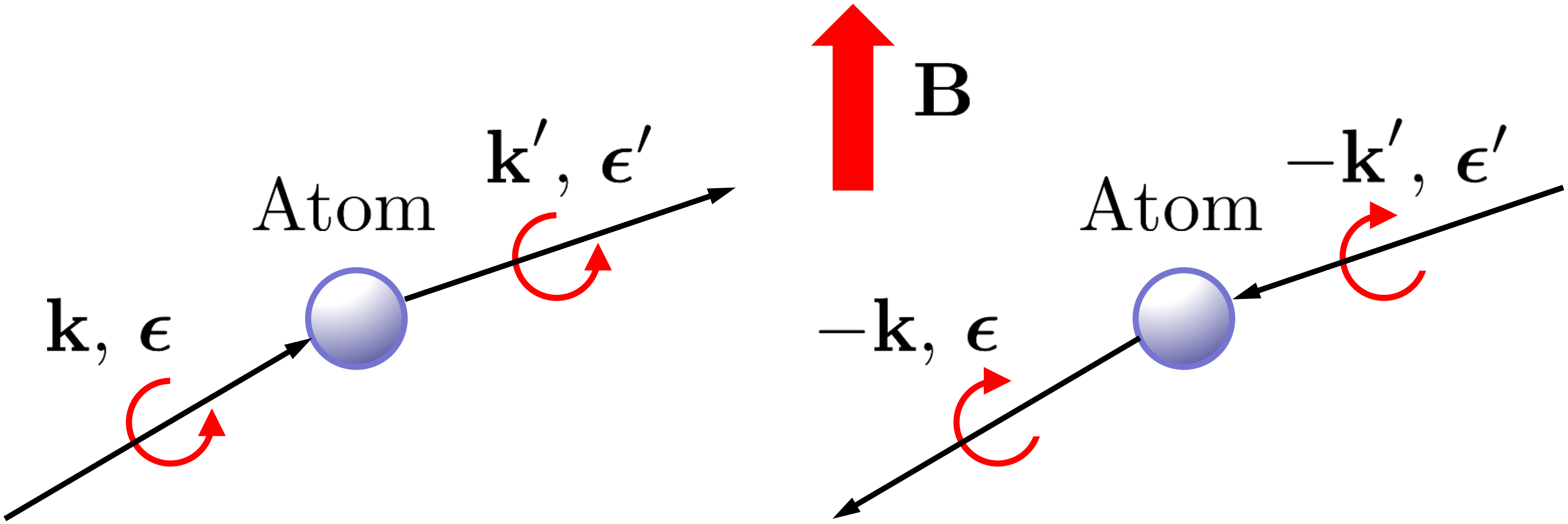
Breakdown of time-reversal invariance



$$t_{\omega,\mathbf{B}}(\mathbf{k},\boldsymbol{\epsilon} \rightarrow \mathbf{k}',\boldsymbol{\epsilon}') \neq t_{\omega,\mathbf{B}}(-\mathbf{k}',\boldsymbol{\epsilon}' \rightarrow -\mathbf{k},\boldsymbol{\epsilon})$$

Van Tiggelen & Maynard (1998)

Breakdown of time-reversal invariance

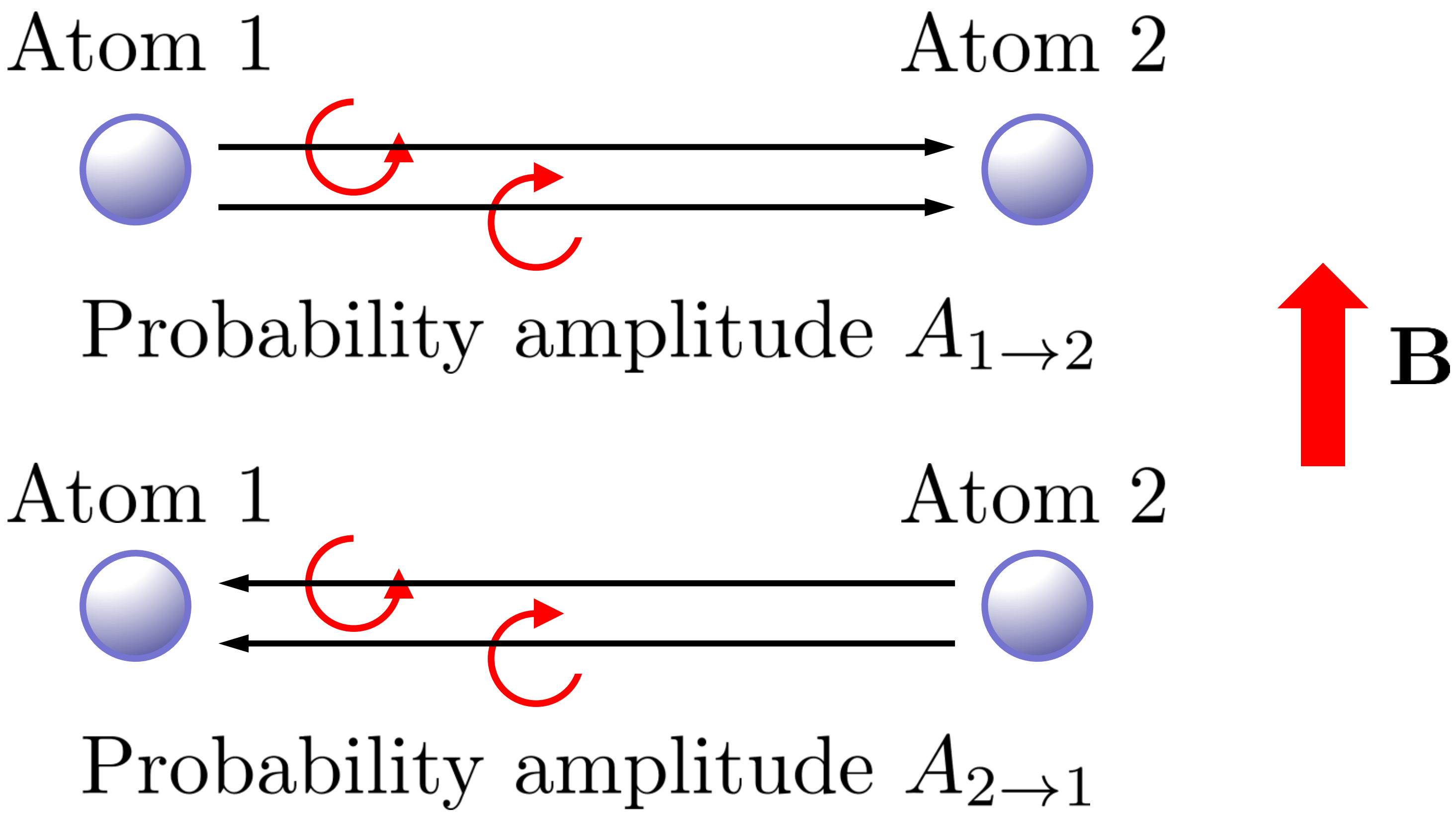


$$t_{\omega,\mathbf{B}}(\mathbf{k}, \epsilon \to \mathbf{k}', \epsilon') \neq t_{\omega,\mathbf{B}}(-\mathbf{k}', \epsilon' \to -\mathbf{k}, \epsilon)$$

$$t_{\omega,\mathbf{B}}(\mathbf{k}, \epsilon \to \mathbf{k}', \epsilon') = t_{\omega,-\mathbf{B}}(-\mathbf{k}', \epsilon' \to -\mathbf{k}, \epsilon)$$

Van Tiggelen & Maynard (1998)

Breakdown of time-reversal invariance



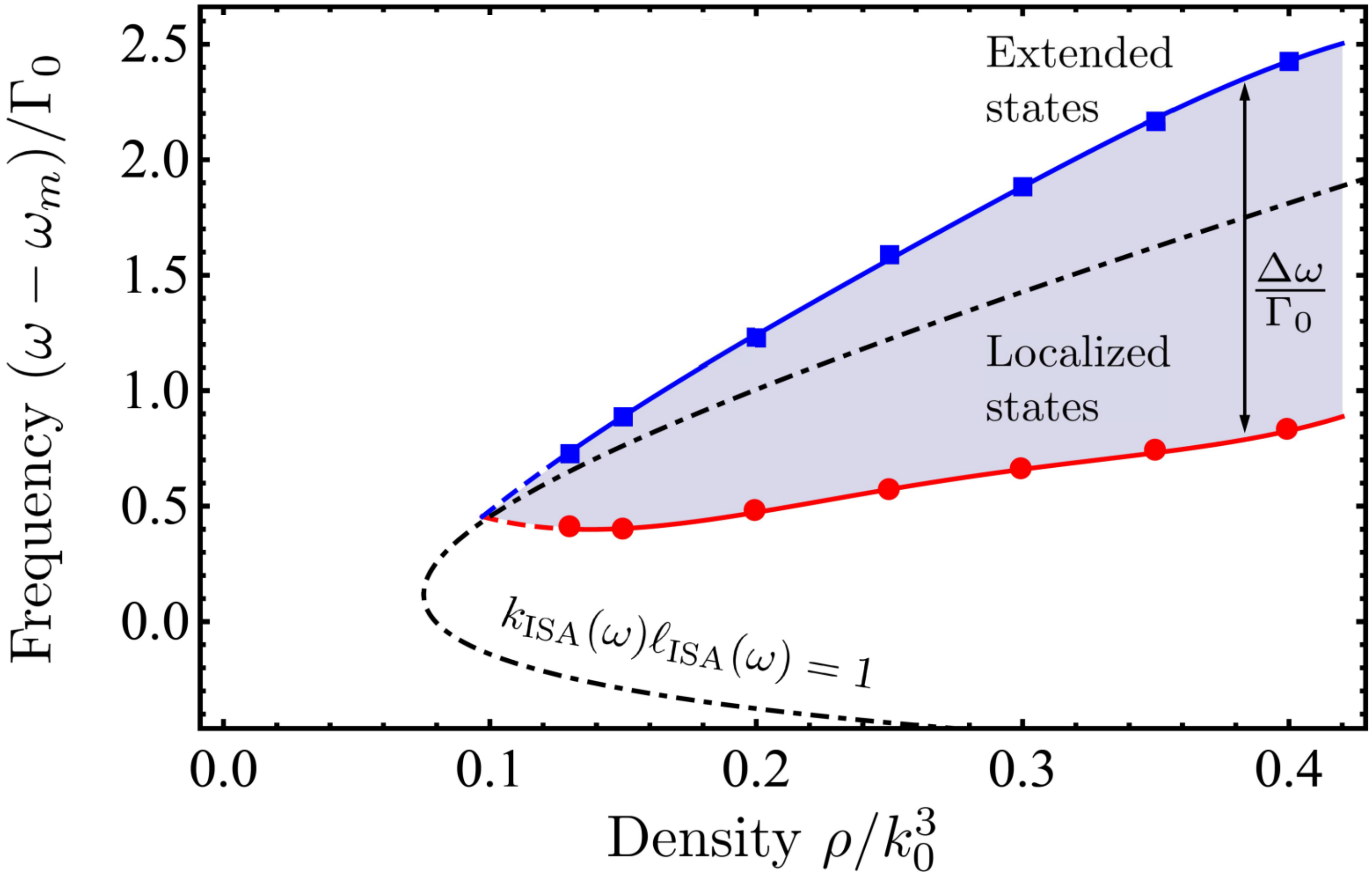
$$A_{1\to 2} = A_{2\to 1}$$

PRL **121**, 093601 (2018)

Phase diagram for light in magnetic field

$$\omega_m = \omega_0 + m\Gamma_0\Delta$$

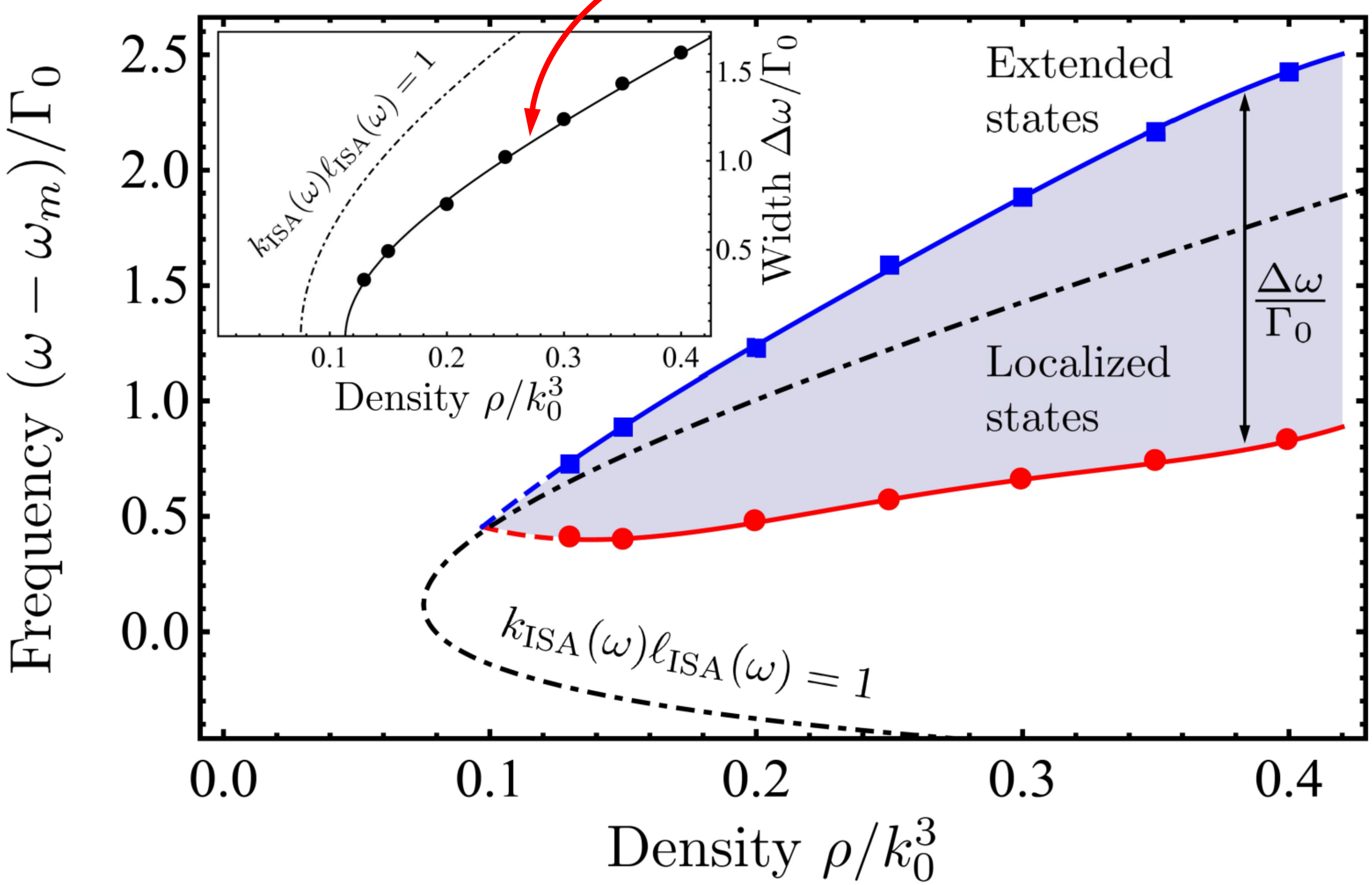
$$m = \pm 1$$



PRL **121**, 093601 (2018)

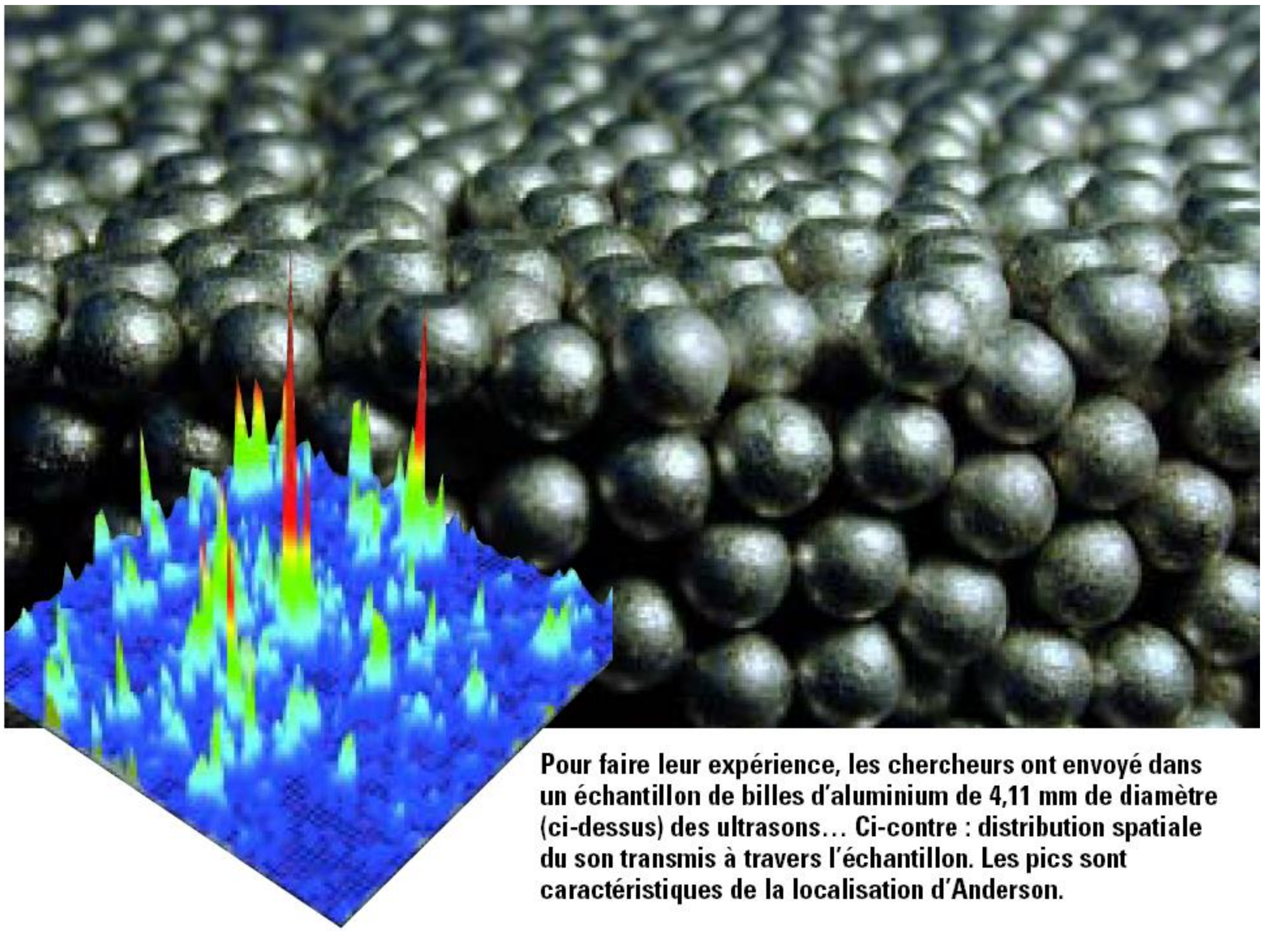
Phase diagram for light in magnetic field

$$\frac{\Delta\omega}{\Gamma_0} = \frac{\pi}{k_0^3}\sqrt{(\rho - \rho_c)(\rho + \rho^*)}$$



PRL **121**, 093601 (2018)

Anderson localization of elastic waves



Pour faire leur expérience, les chercheurs ont envoyé dans un échantillon de billes d'aluminium de 4,11 mm de diamètre (ci-dessus) des ultrasons… Ci-contre : distribution spatiale du son transmis à travers l'échantillon. Les pics sont caractéristiques de la localisation d'Anderson.

Image from *Le Journal du CNRS* (December 2008)

Elastic wave equation

$$\rho\omega^2 u_i + \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial}{\partial x_k}u_l\right) = -f_i$$

mechanical displacement in the direction *i*

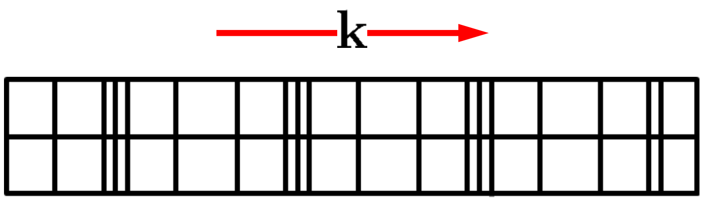
elasticity tensor

excitation

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

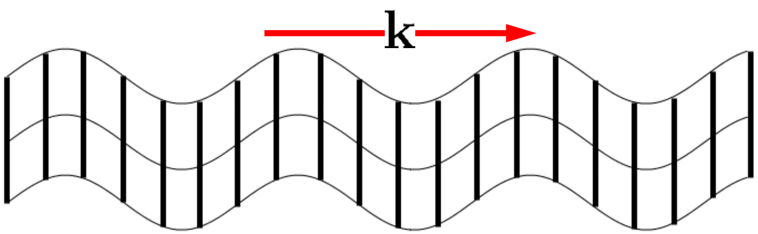
Lamé parameters

compressional waves:



velocity $\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$

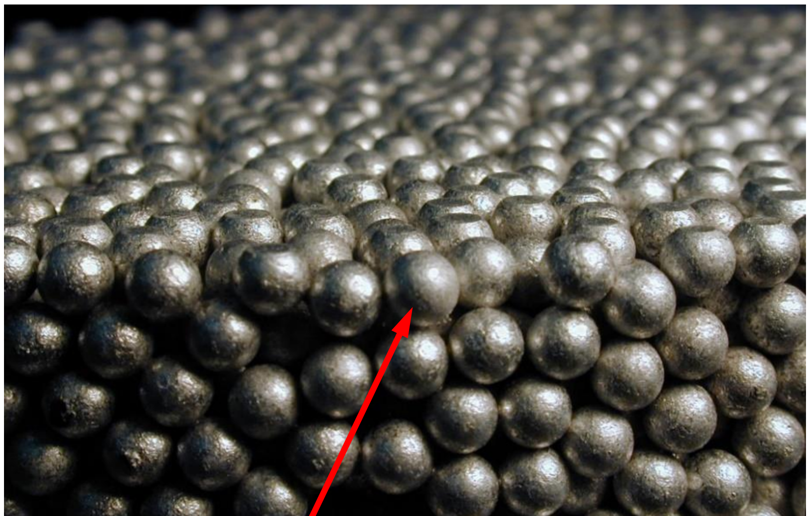
shear waves:



velocity $\beta = \sqrt{\frac{\mu}{\rho}}$

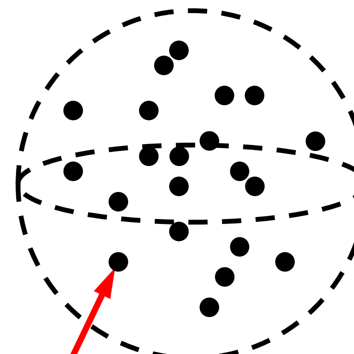
Point-scatterer model

Real sample



Hu et al., Nature Physics **4**, 945 (2008)

Model



Identical aluminum beads
with many resonances

Identical point scatterers
with a single resonance

Elastic Green's function

$$
\begin{aligned}
\widehat{G}(\mathbf{r}) \;&=\; \frac{k_\alpha}{4\pi(\lambda+2\mu)} \times \left\{ \frac{e^{ik_\alpha r}}{3k_\alpha r}\left[\mathbb{1} + (\mathbb{1} - 3\widehat{r}\otimes\widehat{r})\right]\left(-1 - \frac{3i}{k_\alpha r} + \frac{3}{(k_\alpha r)^2}\right)\right. \\
&\quad - \left.\left(\frac{\alpha}{\beta}\right)^3 \frac{e^{ik_\beta r}}{3k_\beta r}\left[-2\mathbb{1} + (\mathbb{1} - 3\widehat{r}\otimes\widehat{r})\right]\left(-1 - \frac{3i}{k_\beta r} + \frac{3}{(k_\beta r)^2}\right)\right\}
\end{aligned}
$$

$k_\alpha = \frac{\omega}{\alpha}$, $k_\beta = \frac{\omega}{\beta}$, $\widehat{r} = \frac{\mathbf{r}}{r}$

Typically, $\alpha > \beta$ ($\alpha/\beta \simeq 2$ for aluminium)

Equipartition principle:

$$
\frac{\langle \text{Energy of shear waves} \rangle}{\langle \text{Enerfy of compressional waves} \rangle} = 2\left(\frac{\alpha}{\beta}\right)^3 > 1
$$

Elastic Green's function in the near field

$$\hat{G}(\mathbf{r}) = \frac{k_\alpha}{4\pi(\lambda+2\mu)} \times \left\{ \frac{e^{ik_\alpha r}}{3k_\alpha r} [\mathbb{1} + (\mathbb{1} - 3\hat{r}\otimes\hat{r})] \left(-1 - \frac{3i}{k_\alpha r} + \frac{3}{(k_\alpha r)^2}\right) \right.$$
$$\left. - \left(\frac{\alpha}{\beta}\right)^3 \frac{e^{ik_\beta r}}{3k_\beta r} [-2\mathbb{1} + (\mathbb{1} - 3\hat{r}\otimes\hat{r})] \left(-1 - \frac{3i}{k_\beta r} + \frac{3}{(k_\beta r)^2}\right) \right\}$$

$\Downarrow$

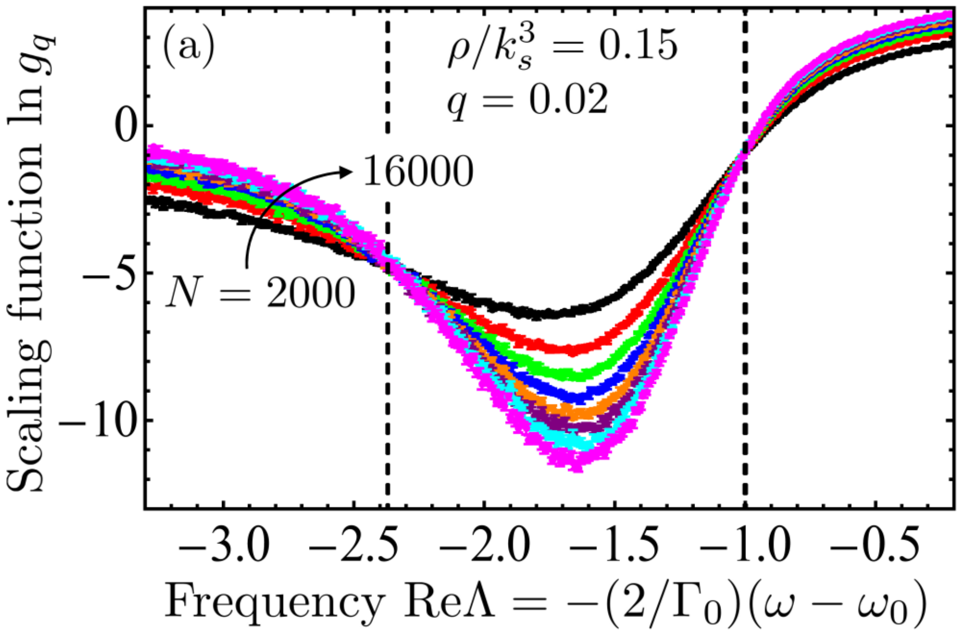
Near-field behavior:

$$\hat{G}(\mathbf{r})\Big|_{r\to 0} = \frac{1}{8\pi\rho_0\beta^2 r} \left\{ \left[1 + \left(\frac{\beta}{\alpha}\right)^2\right] \mathbb{1} + \left[1 - \left(\frac{\beta}{\alpha}\right)^2\right] \hat{r}\otimes\hat{r} \right\} \propto \frac{1}{r}$$

Similar to the scalar case and different from the electromagnetic one:

$$\hat{G}_{\text{EM}}(\mathbf{r})\Big|_{r\to 0} \propto \frac{1}{r^3}$$

Finite-size scaling of percentiles



Percentile g_q:

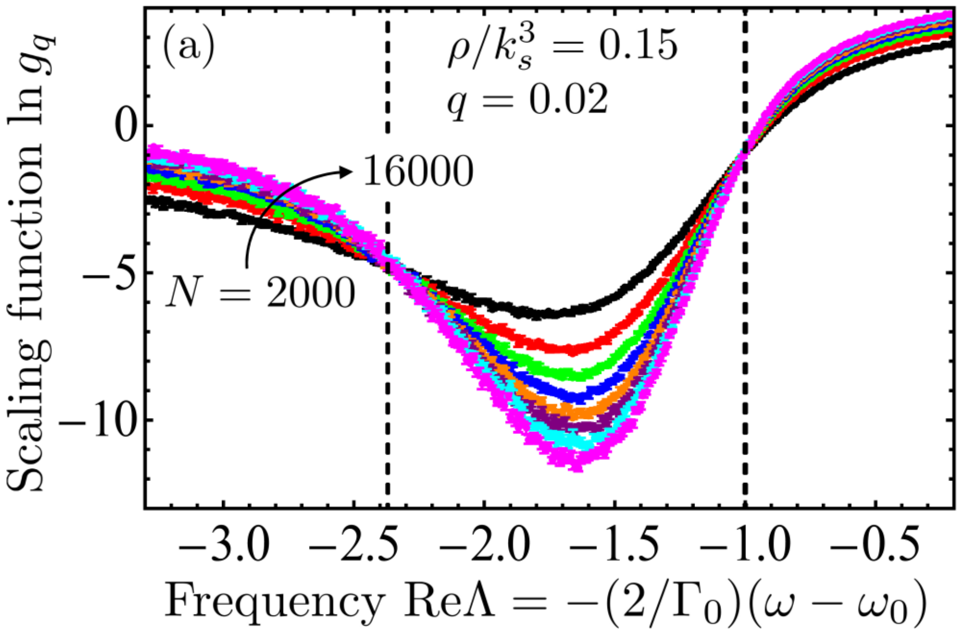
$$q = \int_0^{g_q} p(g)\,dg$$

$\rho/k_0^3 = 0.15$

$\alpha/\beta = 2$

PRB **98**, 064206 (2018)

Finite-size scaling of percentiles



Percentile g_q:

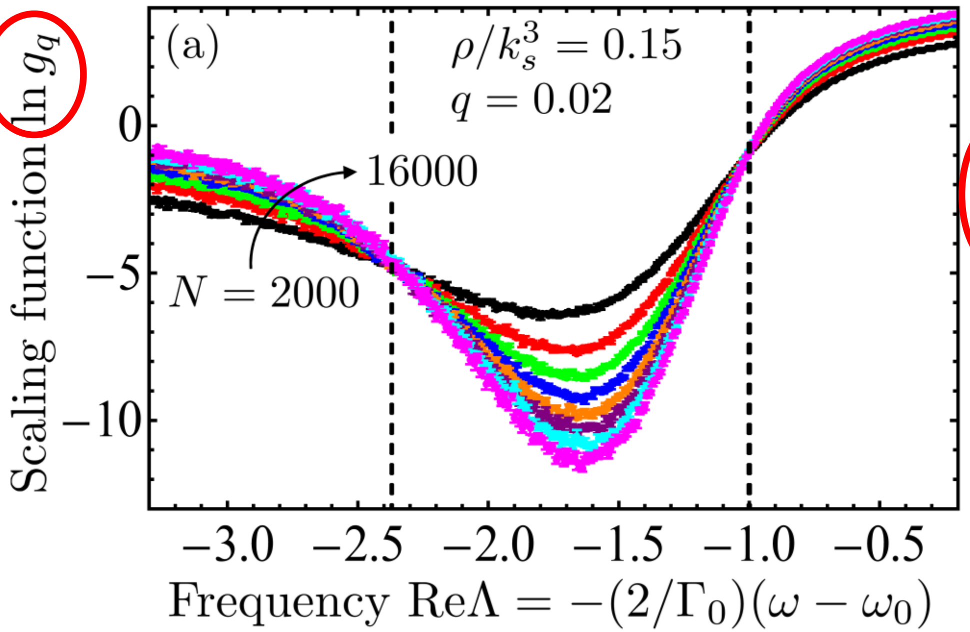
$$q = \int_0^{g_q} p(g)dg$$

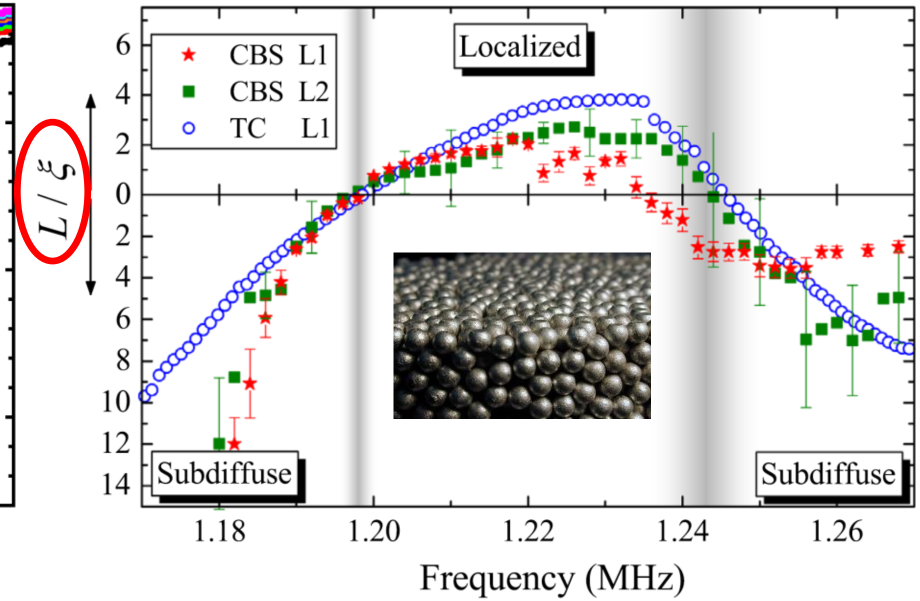
$$g \propto \exp(-L/\xi) \quad \Longrightarrow \quad \ln g = -L/\xi + \text{const}$$

$\rho/k_0^3 = 0.15$
$\alpha/\beta = 2$

PRB **98**, 064206 (2018)

Finite-size scaling of percentiles





Cobus et al., PRL **116**, 193901 (2016)

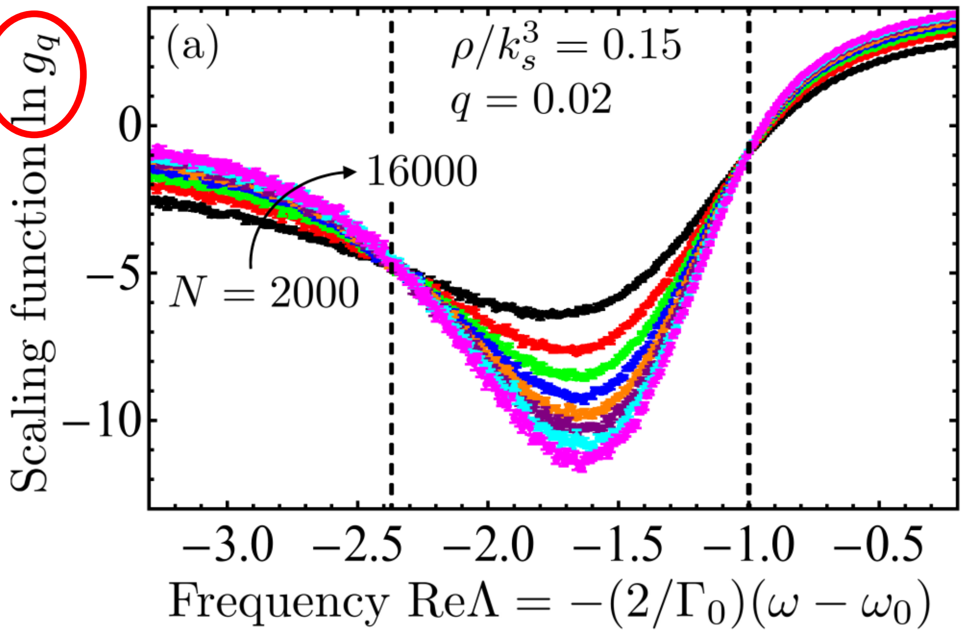
$$g \propto \exp(-L/\xi) \Longrightarrow \ln g = -L/\xi + \text{const}$$

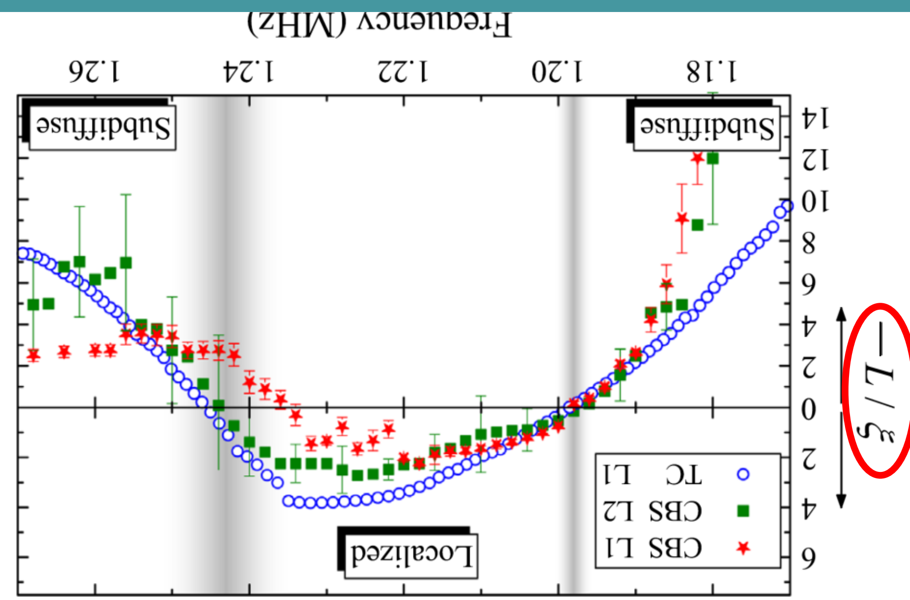
$\rho/k_0^3 = 0.15$

$\alpha/\beta = 2$

PRB **98**, 064206 (2018)

Finite-size scaling of percentiles





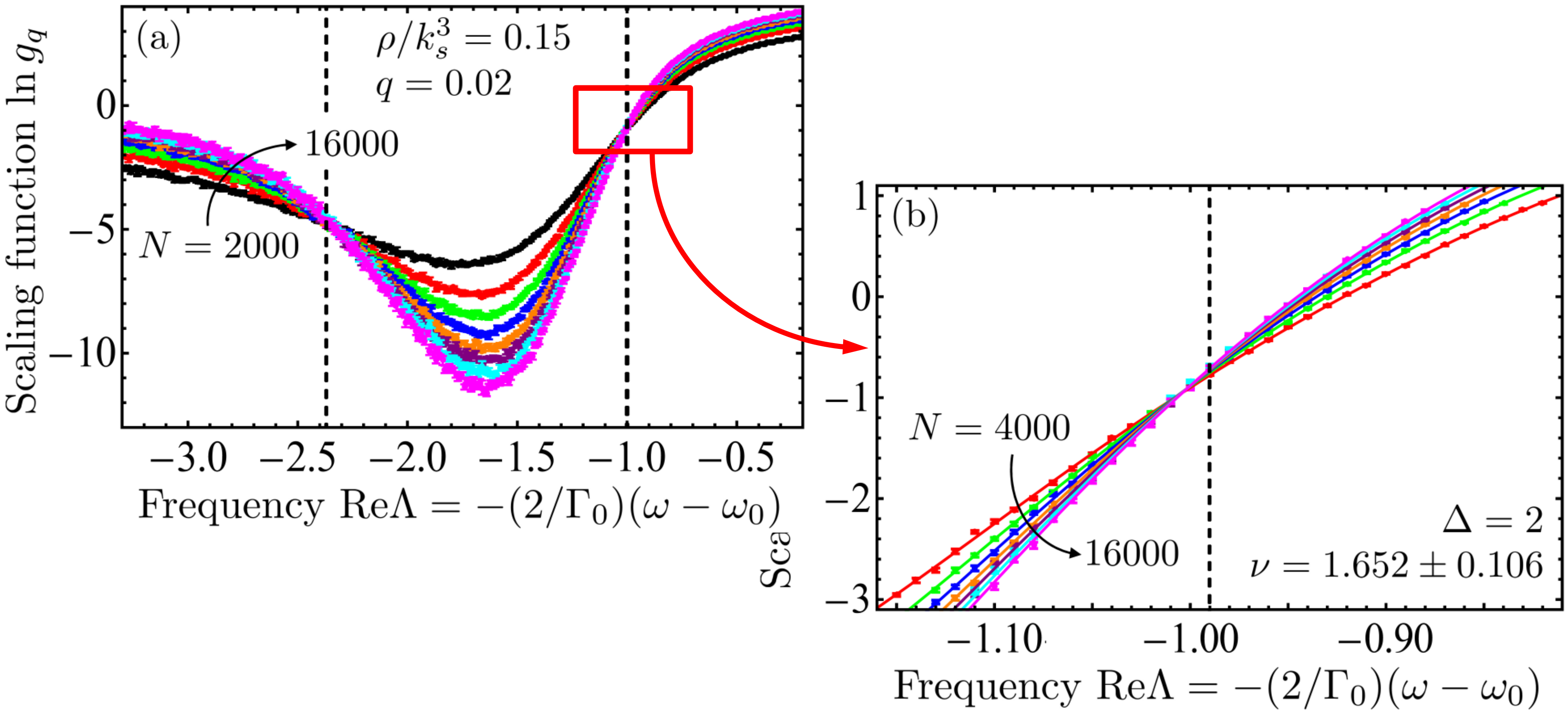
Cobus et al., PRL **116**, 193901 (2016)

$$g \propto \exp(-L/\xi) \Rightarrow \ln g = -L/\xi + \text{const}$$

$$\rho/k_0^3 = 0.15$$
$$\alpha/\beta = 2$$

PRB **98**, 064206 (2018)

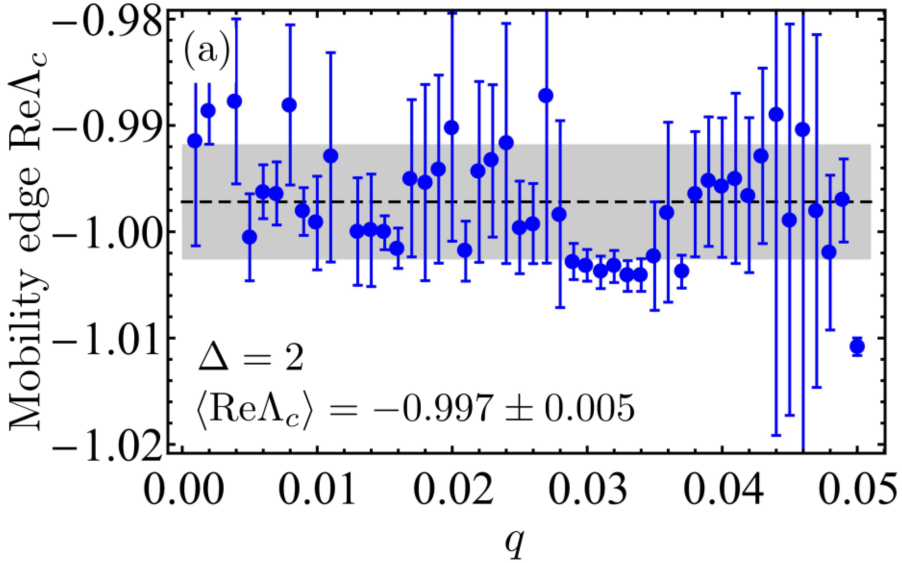
Finite-size scaling of percentiles

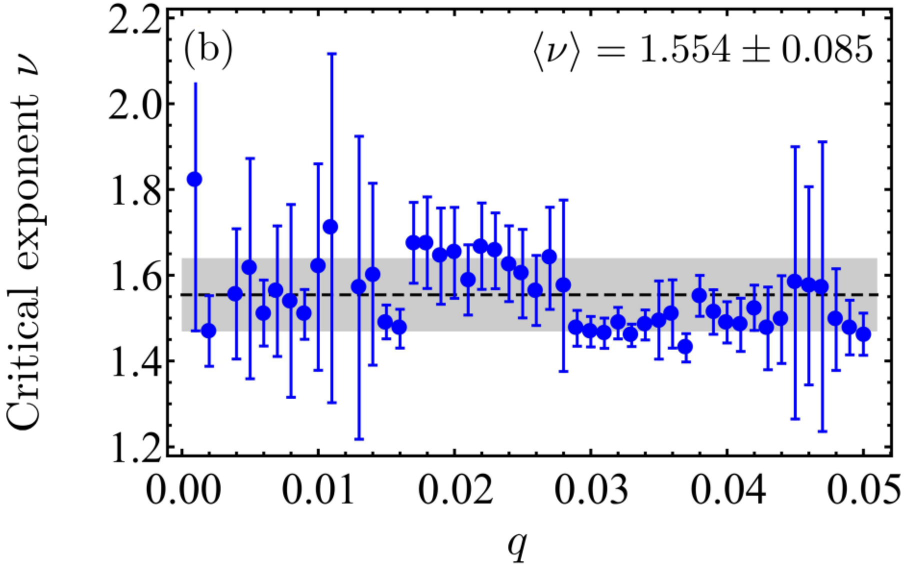


$\rho/k_0^3 = 0.15$
$\alpha/\beta = 2$

PRB **98**, 064206 (2018)

Mobility edge and critical exponent





$\rho/k_0^3 = 0.15$

$\alpha/\beta = 2$

PRB **98**, 064206 (2018)

Anderson transition for vector waves in 3D

	Scalar waves[2]	Light $\mathbf{B}=0$[1]	Light large $\mathbf{B}$[4]	Elastic waves[3]
Critical exponent	$\nu \approx 1.6$	No localization	$\nu \approx 1.6$	$\nu \approx 1.6$
Universality class	orthogonal		**orthogonal** despite broken TR invariance	orthogonal

1. PRL **112**, 023905 (2014)
2. PRB **94**, 064202 (2016)
3. PRB **98**, 064206 (2018)
4. PRL **121**, 093601 (2018)

Conclusions

Vector nature of wave excitations turns out to be important for the Anderson localization problem

In a strongly scattering medium, **different vector waves can behave in qualitatively different ways**

Anderson localization of light should be observable for light scattering by atoms in a strong magnetic field

Anderson localization of elastic waves is similar to that of scalar waves

Thank you for your attention

관심을 가져 주셔서 감사합니다.