Many-Body Invariants of

Electric Multipoles and More

Byungmin Kang (KIAS)

Gil Young Cho

POSTECH

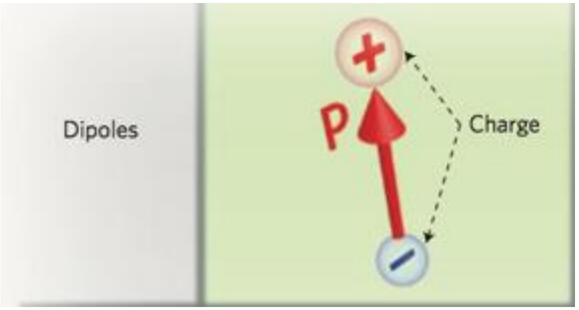
Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

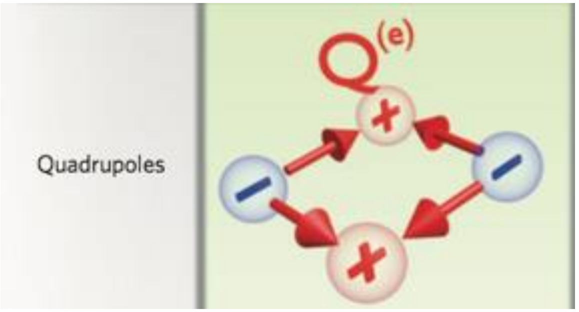
Byungmin Kang, and GYC, in preparation

See also: William Wheeler, Lucas Wagner, and Taylor Hughes, arxiv:1812.06990 (2018)

Proposal:

Generic Definitions of Electric Multipoles in Solids





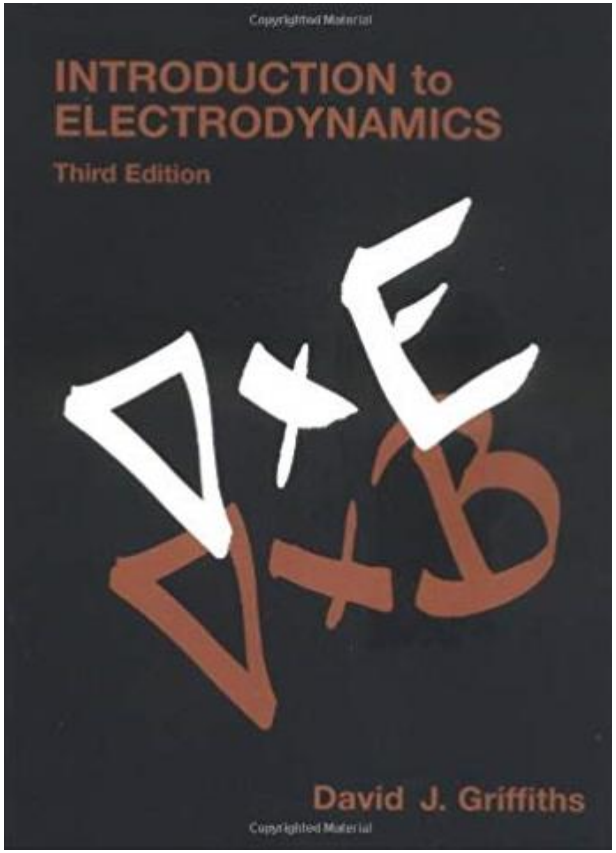
...many-body invariants for various topological states

Contents

1. Introduction

: Why do you want to care about electric multipoles?

Warm-up: classical multipoles



Input: Charge Distribution $\rho(\vec{r})$

Output: Well-defined Multipoles

Ex: $$\mathbf{p}(\mathbf{r}) = \int_V \rho(\mathbf{r}_0)\ (\mathbf{r}_0 - \mathbf{r})\ d^3\mathbf{r}_0$$

Electric Field Lines

Ex: $$\mathbf{E}(\mathbf{R}) = \frac{3\left(\mathbf{p}\cdot\hat{\mathbf{R}}\right)\hat{\mathbf{R}} - \mathbf{p}}{4\pi\varepsilon_0 R^3}$$

Warm-up: classical multipoles

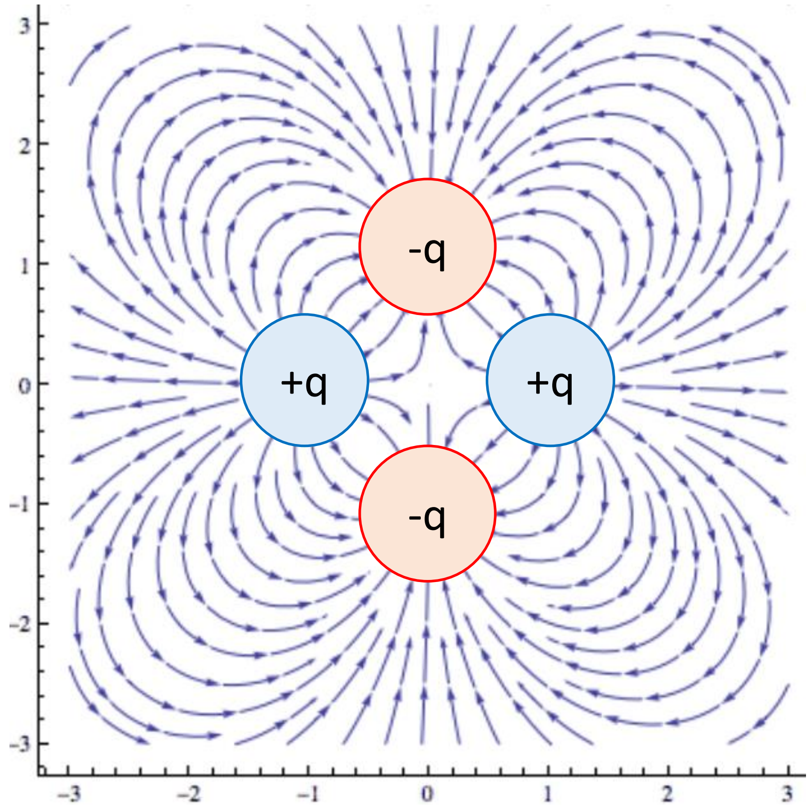
Multipole expansion

From Wikipedia, the free encyclopedia

$$q_{\mathrm{tot}} \equiv \sum_{i=1}^{N} q_i$$

$$P_\alpha \equiv \sum_{i=1}^{N} q_i r_{i\alpha}$$

$$Q_{\alpha\beta} \equiv \sum_{i=1}^{N} q_i \left(3 r_{i\alpha} r_{i\beta} - \delta_{\alpha\beta} r_i^2\right)$$



$$4\pi\varepsilon_0 V(\mathbf{R}) = \frac{q_{\mathrm{tot}}}{R} + \frac{1}{R^3} \sum_{\alpha=x,y,z} P_\alpha R_\alpha + \frac{1}{2R^5} \sum_{\alpha,\beta=x,y,z} Q_{\alpha\beta} R_\alpha R_\beta + \cdots$$

Classical Multipoles seem to be Done

: Why do I still care about Electric Multipoles?

Difficulties in Crystals [at low temperature]:

(1) Quantum-mechanical Electrons

[may well be highly-correlated]

(2) Lattice (Periodicity)

Difficulties in Crystals [at low temperature]:

(1) Quantum-mechanical Electrons

[may well be highly-correlated]

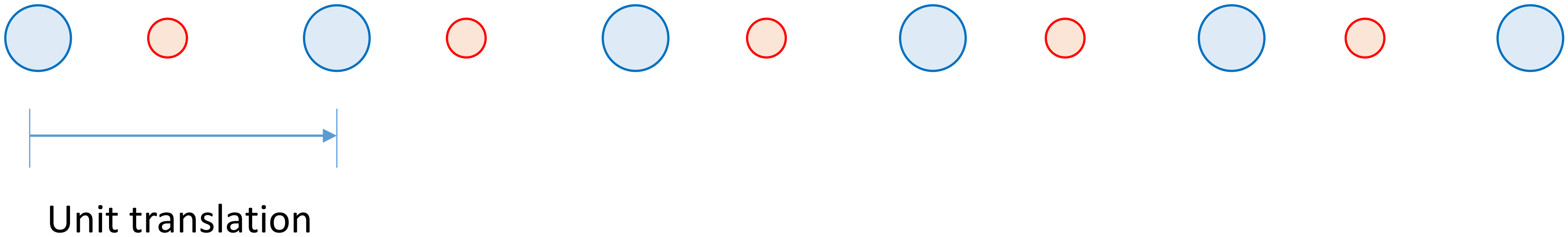
(2) Lattice (Periodicity)

Bad: Non-trivial to define multipoles

Good: Links to Topological Band Insulators

Warm-up: Dipole/Polarization

Infinite Lattice (Crystal):



Dipole moment: $P_x = \sum \color{red}{x} q_x \rightarrow \sum \langle \psi | \color{red}{x} | \psi \rangle q_x\,?$

Warm-up: Dipole/Polarization

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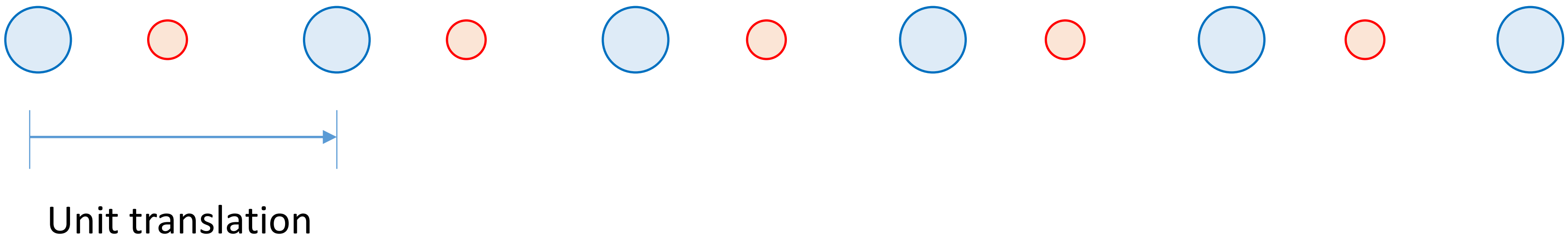
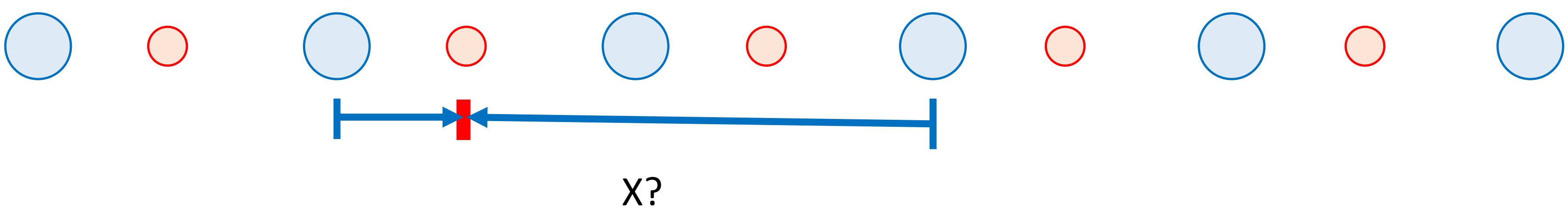


Dipole moment: $P_x = \sum \boldsymbol{x} q_x \to \sum \langle\psi|\boldsymbol{x}|\psi\rangle q_x\,?$

Position x of Electrons ?

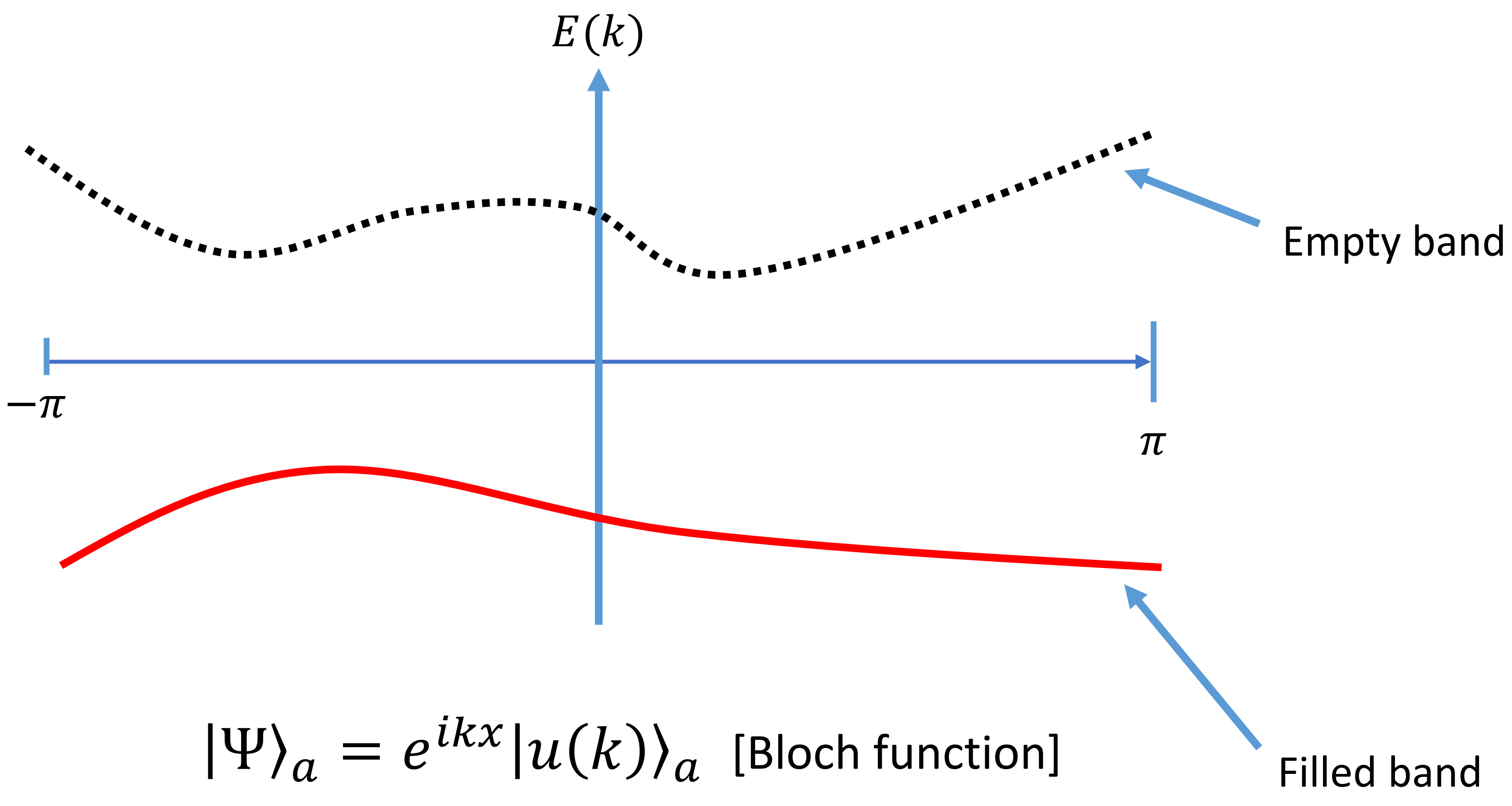


Warm-up: Dipole/Polarization

Momentum Space:

$$\exp(2\pi i \boldsymbol{P}_x) = \exp\left[i \oint \mathrm{Tr}\, A_k \, dk\right] \text{ and } A_k = \langle u(k)|\, i\partial_k \,|u(k)\rangle$$

[Berry connection of filled states]

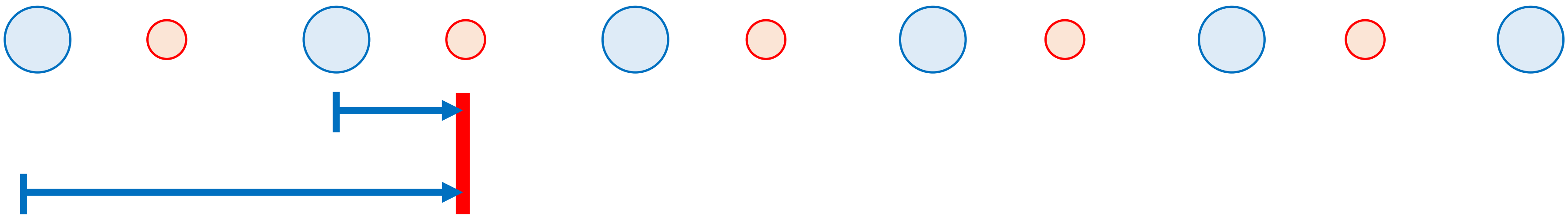


$$|\Psi\rangle_a = e^{ikx}|u(k)\rangle_a$$ [Bloch function]

Warm-up: Dipole/Polarization

Remarks on $\exp(2\pi i \boldsymbol{P_x}) = \exp\left[i \oint \mathrm{Tr}\, A_k\, dk \right]$

1. Polarization: $\boldsymbol{P_x} = \boldsymbol{P_x} \ \mathbf{mod\ 1}$ (symmetry-independent)



Warm-up: Dipole/Polarization

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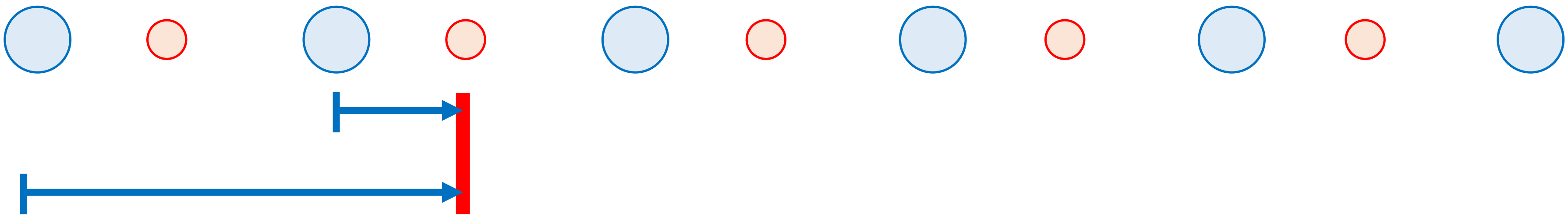


2. Symmetry $\boldsymbol{R}_x\!: \boldsymbol{x} \rightarrow -\boldsymbol{x}$

$: \boldsymbol{P}_x \rightarrow -\boldsymbol{P}_x$

$\boldsymbol{P}_x = \mathbf{0},\ \frac{1}{2}$ **mod 1** (discrete values only)

3. Not clear how to write in real space (later)

4. Not clear how to apply to interacting electrons (later)

Warm-up: Dipole/Polarization

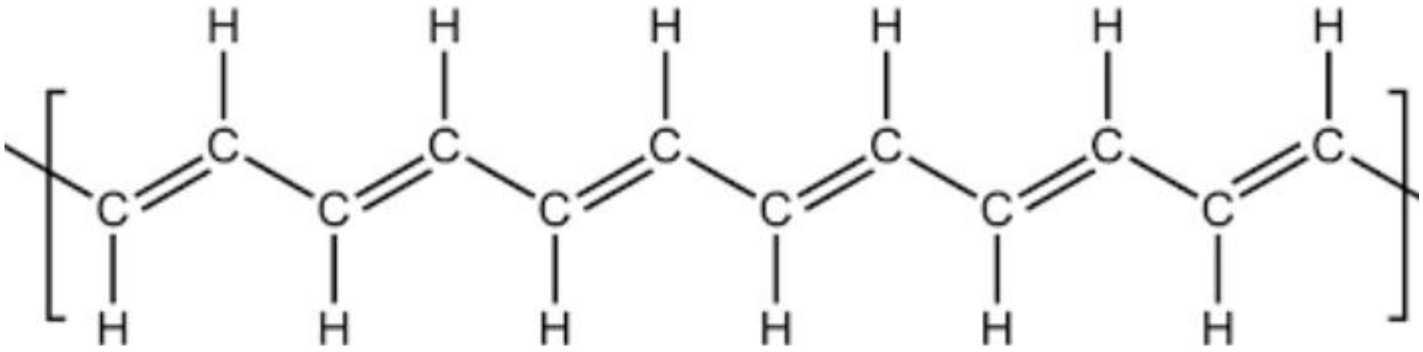
5. Topology: $R_x: x \to -x$

$: P_x \to -P_x$

$P_x = 0, \frac{1}{2}$ **mod 1** (discrete values only)

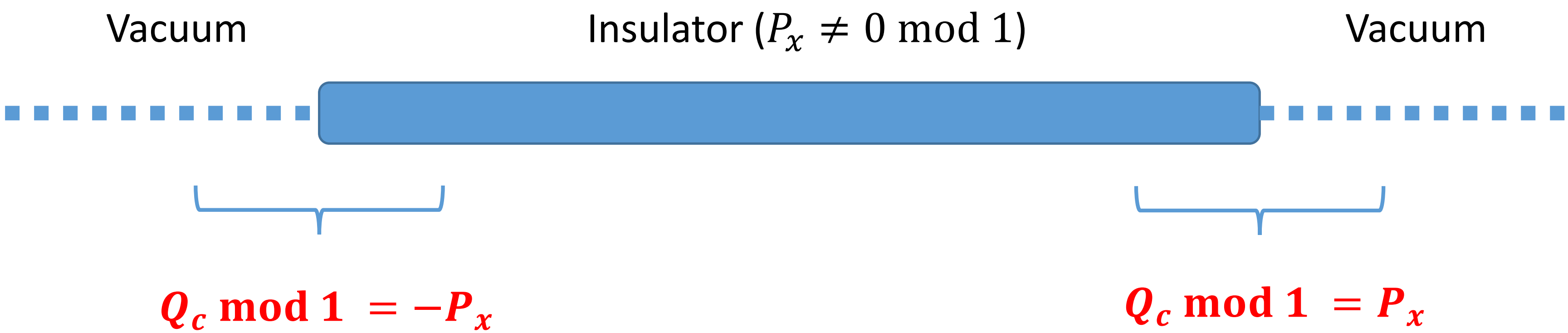
"Symmetry-protected Topology"

Relevant for, e.g., **Su-Schrieffer-Heeger model**



Warm-up: Dipole/Polarization

6. Boundary Charge (Physical Consequence of Polarization)



…consistent with $Q_{\mathrm{bdry}} = \vec{P} \cdot \vec{n}$ (classical electromagnetism)

Warm-up: Dipole/Polarization

7. Many-body real-space formula (Resta's formula)

VOLUME 80, NUMBER 9 PHYSICAL REVIEW LETTERS 2 MARCH 1998

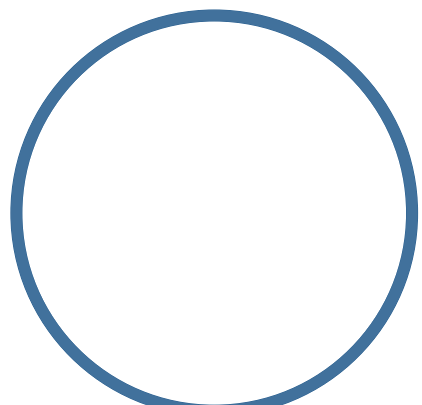
Quantum-Mechanical Position Operator in Extended Systems

Raffaele Resta

Consider…

If band Insulator

$$U_1 = \exp\left(\frac{2\pi i}{L}\sum x\,\widehat{N}(x)\right) \longleftrightarrow \langle \mathbf{GS}|\boldsymbol{U_1}|\mathbf{GS}\rangle = \mathbf{exp}(\mathbf{2}\boldsymbol{\pi i P_x}) = \exp\left[i\oint \mathrm{Tr}\,A_k\,dk\right]$$

$|GS\rangle =$ **Generic many-body ground state in**  **of size *L***

[Periodic Boundary Condition]

Warm-up: Dipole/Polarization

If band Insulator

$$\boldsymbol{U_1 = \exp\left(\frac{2\pi i}{L}\sum x\,\hat{N}(x)\right)} \longleftrightarrow \boldsymbol{\langle \mathrm{GS}|U_1|\mathrm{GS}\rangle = \exp(2\pi i P_x)} = \exp\left[i\oint \mathrm{Tr}\,A_k\,dk\right]$$

[1. Many-body invariant]

[2. Band Index]

…which agree each other.

[3. Edge Charge]

Vacuum

Insulator ($P_x \neq 0 \bmod 1$)

Vacuum

$\boldsymbol{Q_c \bmod 1 = -P_x}$

$\boldsymbol{Q_c \bmod 1 = P_x}$

Recent Progresses: Electric Multipoles

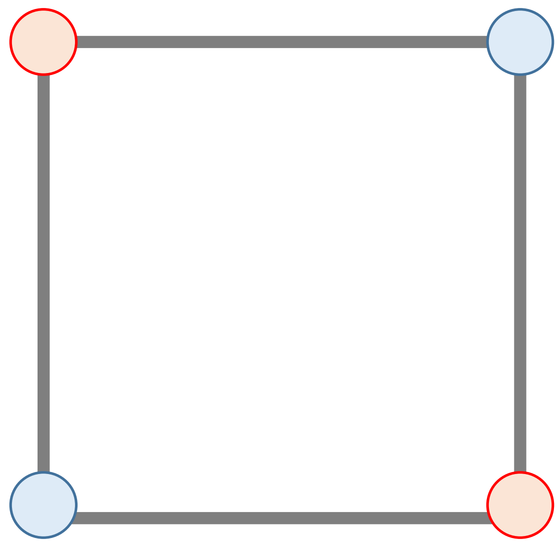
Quantized electric multipole insulators

Wladimir A. Benalcazar,[1] B. Andrei Bernevig,[2] Taylor L. Hughes[1*]

Dipolar Insulators

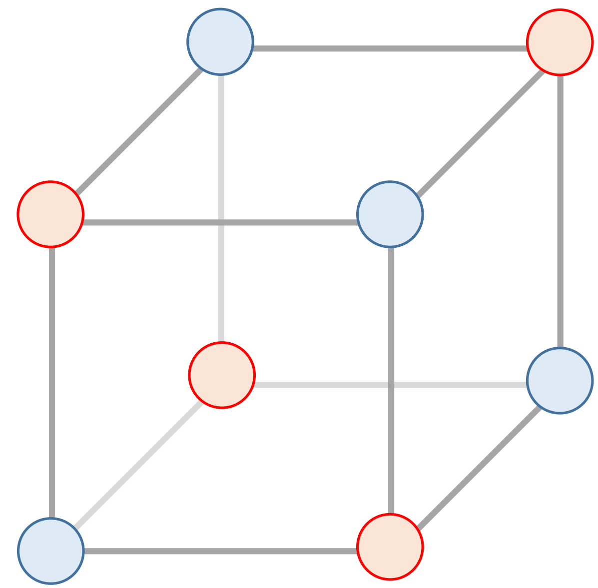
Boundary Charge

Quadrupolar Insulators



Corner Charge

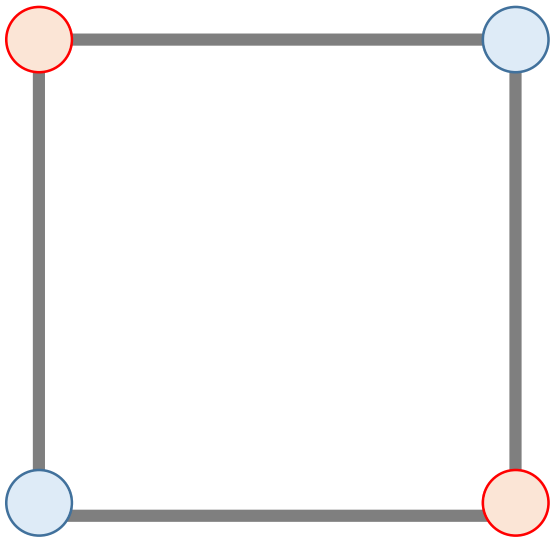
Octupolar Insulators



Corner Charge

Focus: Two Primary Higher-Order TIs:

Quadrupolar Insulators



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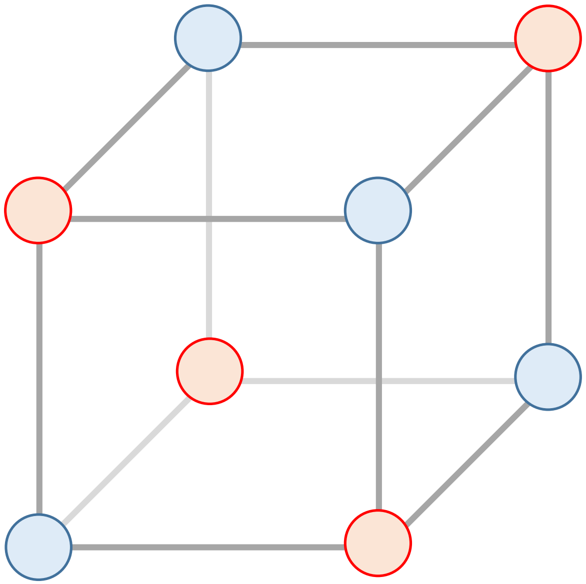

[Ref. Benalcazar-Bernevig-Hughes, Science, 2017] suggested…

$$P_x = P_y = 0$$

$$\boldsymbol{Q_{xy} \neq 0 \bmod 1}$$

$$P_x = P_y = 0$$

$$Q_{ab} = Q_{ba} = 0$$

$$\boldsymbol{O_{xyz} \neq 0 \bmod 1}$$

…which can be shown only when "discrete", e.g., $Q_{xy} = \frac{1}{2}, 0 \bmod 1$

Quantized electric multipole insulators

Wladimir A. Benalcazar,[1] B. Andrei Bernevig,[2] Taylor L. Hughes[1*]

[Science 2017]

...found insulators with quantized (discrete) electric multipoles

$$Q_{xy} = \frac{1}{2} \bmod 1 \quad \& \quad O_{xyz} = \frac{1}{2} \bmod 1$$

+ "Symmetry-Protected" Band Indices ["Nested Wilson Loops"]

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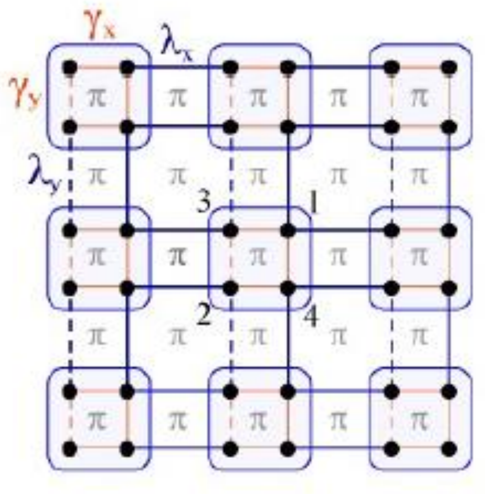
+ "Symmetry-Protected" Band Indices ["Nested Wilson Loops"]

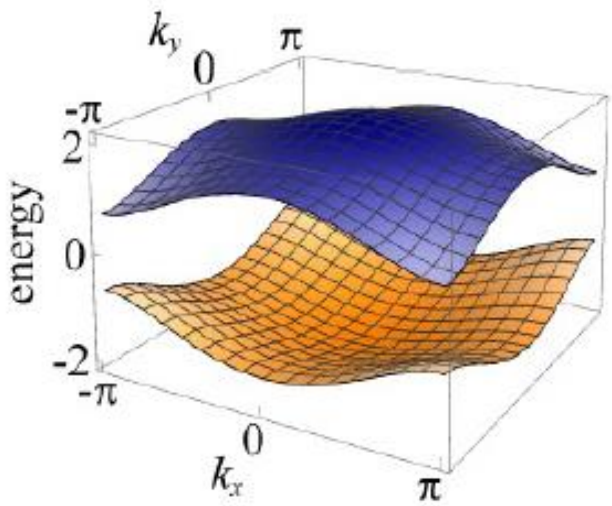
Not Generic Measure of Multipoles

What is done? [Ref. Benalcazar-Bernevig-Hughes, Science, 2017]

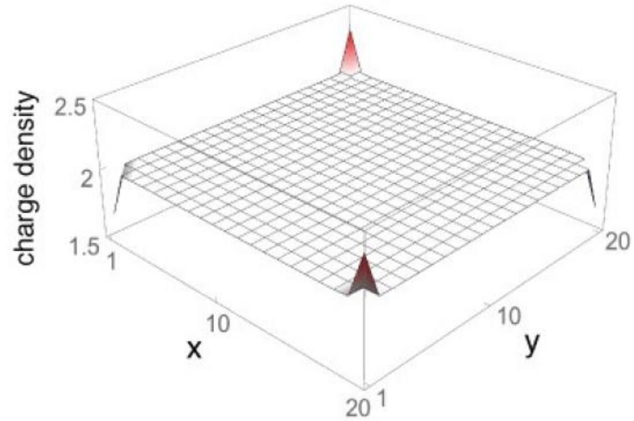
E.g., for the quadrupoles: symmetry **[1. C_4]** or **[2. $M_x \times M_y$ two mirrors]**



Tight-binding Model



(1) Gapped Spectrum



(2) Corner Charge $Q_c = \pm\frac{1}{2}$

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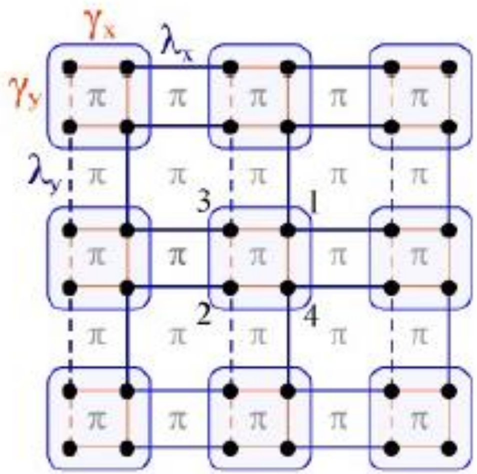

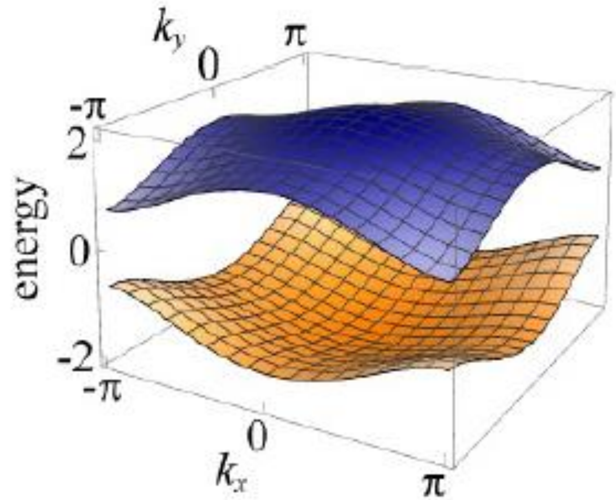

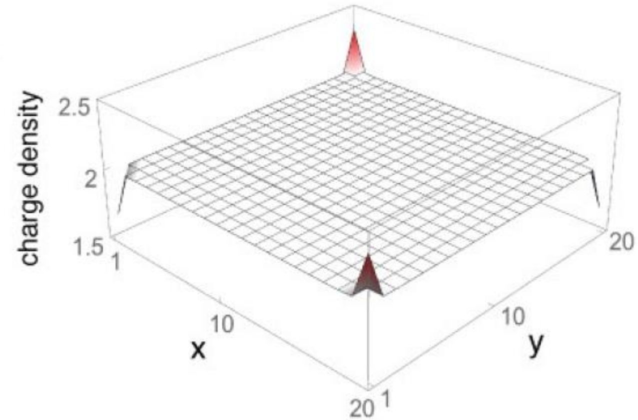

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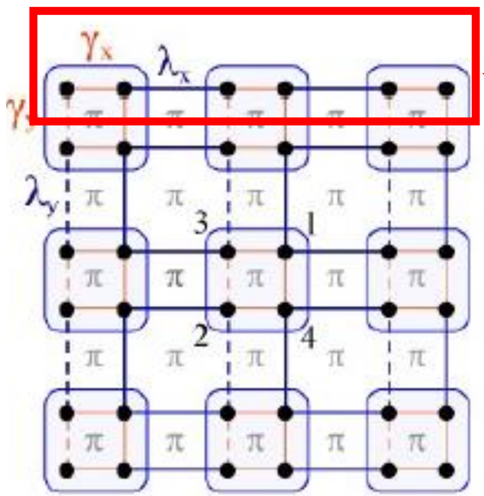


(3) "Su-Schrieffer-Heeger model" with $P_x = P_y = \frac{1}{2}$ mod 1

[Boundary Polarization]

Consistent with $Q_{xy} = \frac{1}{2}$ with boundary $\perp \hat{n}$:

$$P_a^{Bdry} = \hat{n}_a Q_{ab} \ \& \ Q_c = \hat{n}_1^a Q_{ab} \hat{n}_2^b = \frac{1}{2}$$

What is done? [Ref. Benalcazar-Bernevig-Hughes, Science, 2017]

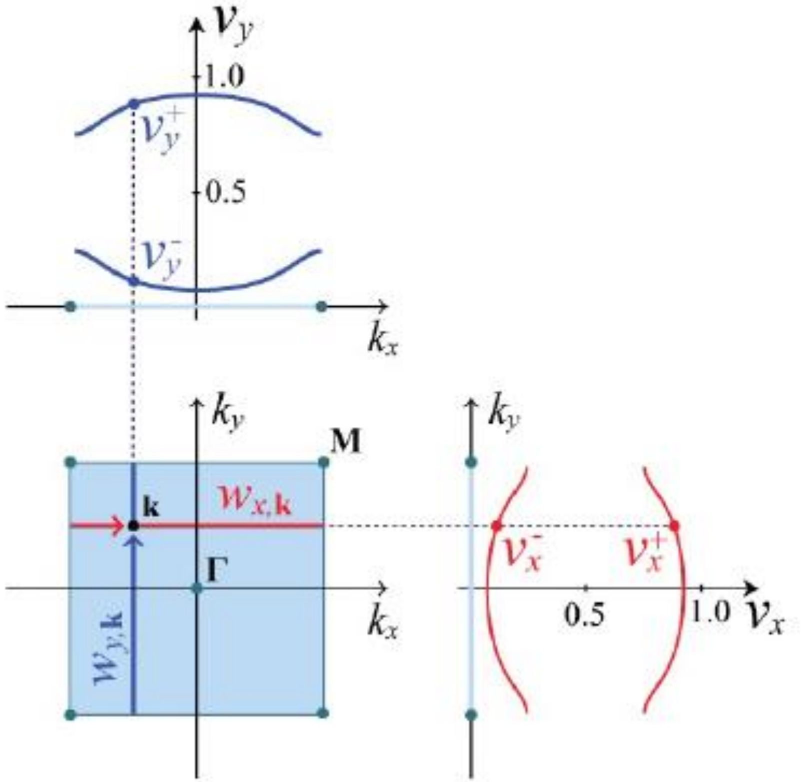
"Nested Wilson Loop Approach" = Capturing Boundary Polarization

Consider a Wilson line:

$$\mathcal{W}_{\mathcal{C},\mathbf{k}} \equiv e^{iH_{\mathcal{W}_{\mathcal{C}}}(\mathbf{k})}$$

...and investigate its polarization

[i.e., Wilson of Wilson = Nested Wilson]



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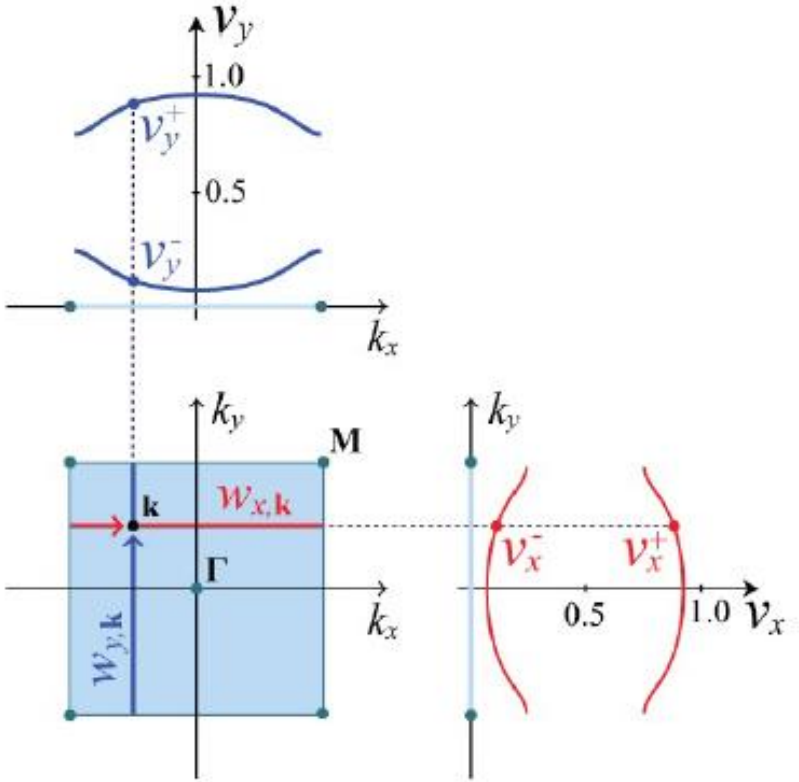


Reasoning:

$H_{W_c}(k) \approx H_{edge}(k)$ **[adiabatic equivalence]**

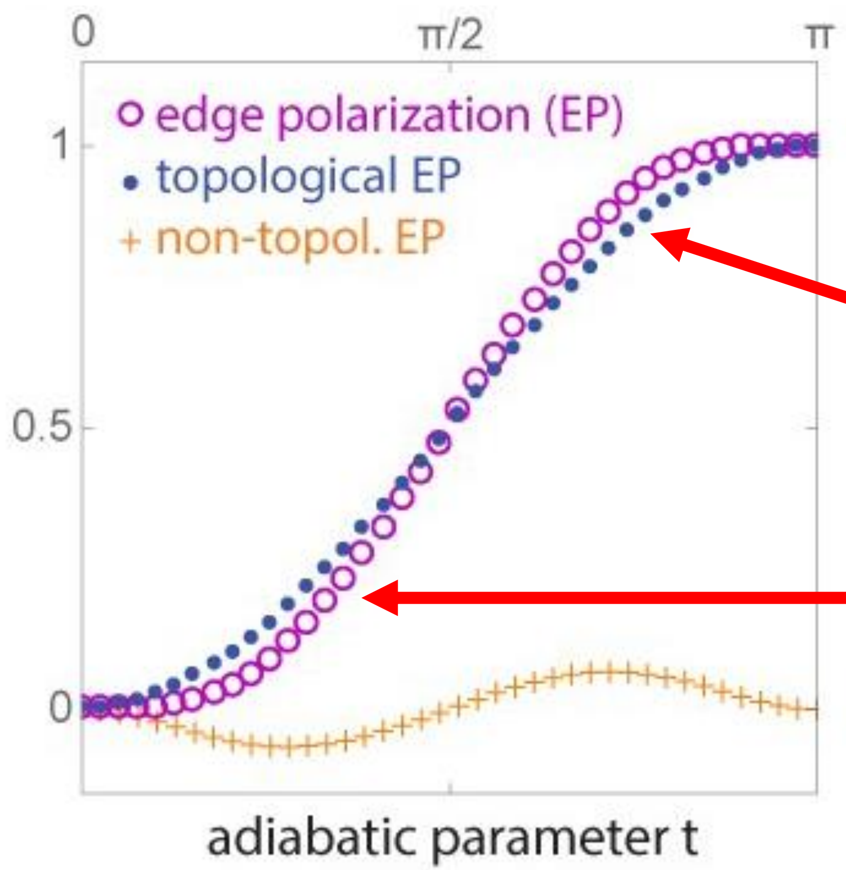
➡ **Polarization of** $H_{W_c}(k)$ = **Polarization of** $H_{edge}(k)$ **[when C_4 or mirrors present]**

By definition, nested Wilson loop is not a generic measure!

For example, when the symmetries are relaxed…

Disagreement with the physical quadrupole moment

[Ref. Benalcazar-Bernevig-Hughes, Science, 2017]



Nested Wilson Loop

Quadrupole moment

A (successful) "topological band index" but not a physical measure.

So, what is missing? [Ref. Benalcazar-Bernevig-Hughes, Science, 2017]

1. Generic momentum-space invariants (for free fermion!) for multipoles

[i.e., Nested Wilson loop seems fundamentally different from $P_x = \frac{1}{2\pi}\oint A_k$ in 1d]

2. Generic many-body & real-space invariant for multipoles

[i.e., No analogue of $\boldsymbol{U_1} = \mathbf{exp}\left(\frac{\boldsymbol{2\pi i}}{\boldsymbol{L}}\sum \boldsymbol{x}\,\widehat{\boldsymbol{N}}(\boldsymbol{x})\right)$ for multipoles]

3. Of course, no link is given for 1 & 2 (since they are absent)

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Our progress

In short,

We look for Generic Definitions of Electric Multipoles in Crystals

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

See also: William Wheeler, Lucas Wagner, and Taylor Hughes, arxiv:1812.06990 (2018)

2. Multipoles in Crystals

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

See also: William Wheeler, Lucas Wagner, and Taylor Hughes, arxiv:1812.06990 (2018)

We propose:

[1] **Quadrupole** in **a crystal** is defined by:

$$Q_{xy} = \frac{1}{2\pi} \mathrm{Im} \log \langle GS| U_2 |GS\rangle \quad \text{with} \quad U_2 = \exp\left(\frac{2\pi i}{L_x L_y} \sum xy\, \rho(x)\right)$$

[2] **Octupole** in **a crystal** is defined by:

$$O_{xyz} = \frac{1}{2\pi} \mathrm{Im} \log \langle GS| U_3 |GS\rangle \quad \text{with} \quad U_3 = \exp\left(\frac{2\pi i}{L_x L_y L_z} \sum xyz\, \rho(x)\right)$$

Here: **|GS⟩ = many-body states on Torus** (we will generalize later)

Essentially,

$$\langle U_2\rangle = |\langle U_2\rangle| \ \mathrm{Exp}\left(2\pi i Q_{xy}\right)$$

$$\langle U_3\rangle = |\langle U_3\rangle| \ \mathrm{Exp}\left(2\pi i O_{xyz}\right)$$

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

Data first and then Proof

Q. If I perform the explicit computation on my computer:

$$Q_{xy} = \frac{1}{2\pi} \operatorname{Im} \log \langle GS | U_2 | GS \rangle \quad \text{with} \quad U_2 = \exp\left(\frac{2\pi i}{L_x L_y} \sum xy\, \rho(x)\right)$$

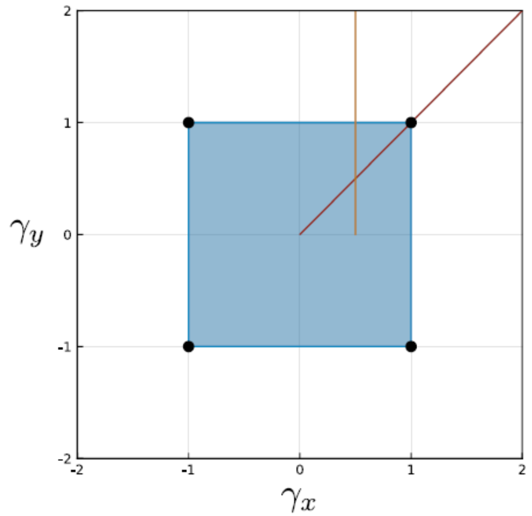
…on the models in literature, do I find:

Topological state: $Q_{xy} = \frac{1}{2} \bmod 1$

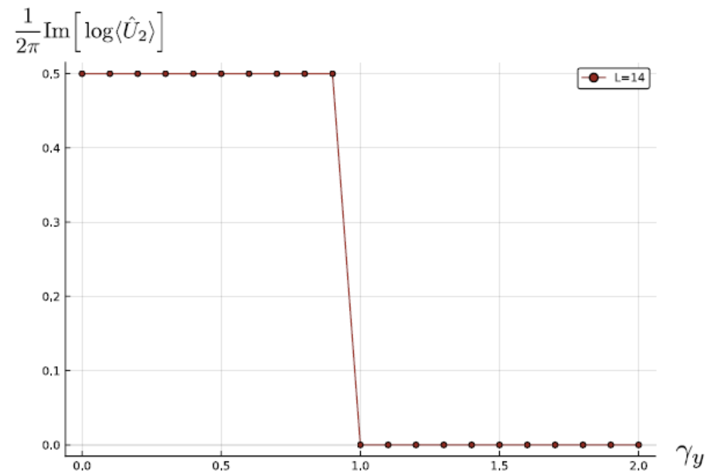
Trivial state: $Q_{xy} = 0 \bmod 1$

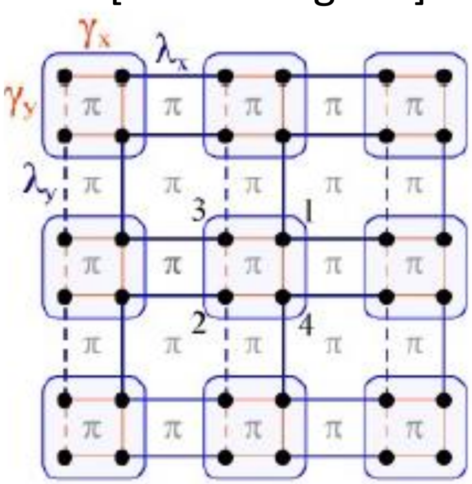
…from computer?

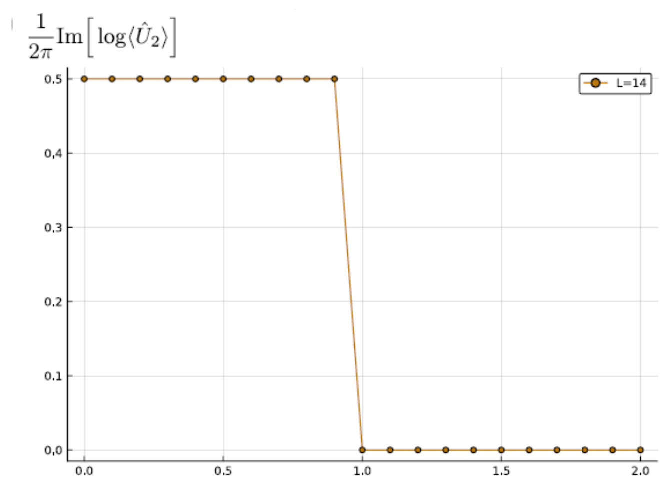
Symmetry-Protected Quadrupoles 1.



[Phase Diagram]







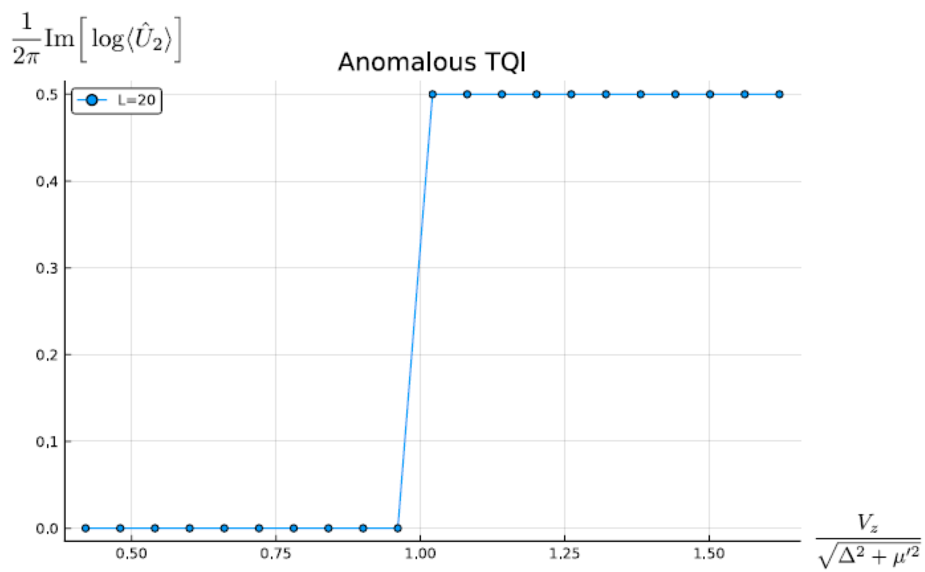
Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

Symmetry-Protected Quadrupoles 2.

An anomalous higher-order topological insulator

S. Franca,[1] J. van den Brink,[1,2] and I. C. Fulga[1]
[1]Institute for Theoretical Solid State Physics, IFW Dresden, 01171 Dresden, Germany
[2]Institute for Theoretical Physics, TU Dresden, 01069 Dresden, Germany
(Dated: November 30, 2018)

despite having a trivial topological invariant. We introduce a concrete example of an anomalous HOTI, which has a quantized bulk quadrupole moment and fractional corner charges, but a vanishing nested Wilson loop index. A new invariant able to capture the topology of this phase is then constructed. Our work shows that anomalous topological phases, previously thought to be unique to periodically driven systems, can occur and be used to understand purely time-independent HOTIs.



[Note: there is a modified index from nested Wilson loops]

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

So far, consistent with nested Wilson loop indices

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So far, consistent with nested Wilson loop indices

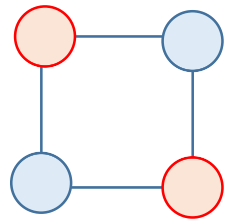
Can I go beyond nested Wilson approaches?

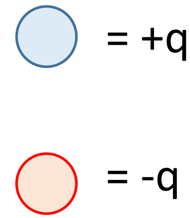
I.E., regime where quantizing symmetries are relaxed

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

Remind: Corner charge $Q_c = Q_{xy}^{ph}$ physical Quadrupole moments

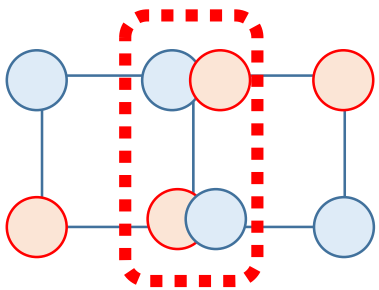
$$Q_{xy}^{ph} = q$$



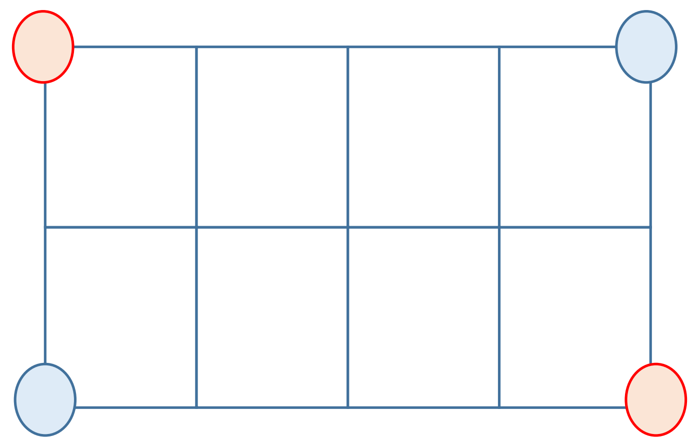


When this is uniformly stacked,

Ex:



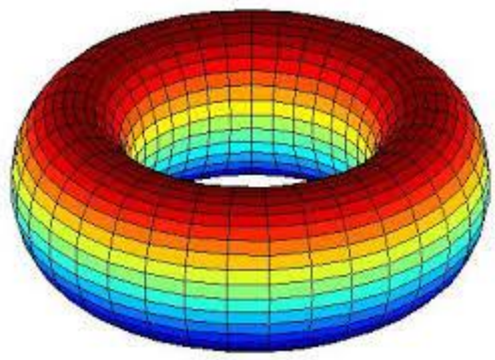
Inside is cancelled ($q - q = 0$) !



Only the boundary is left

So we compare the following two:

1. Periodic BC on Torus

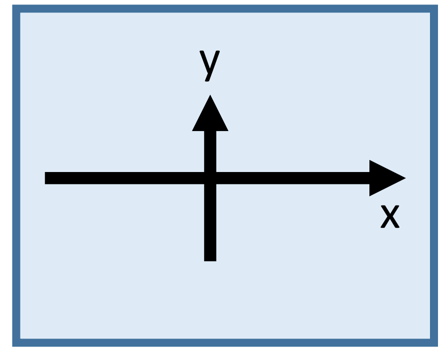


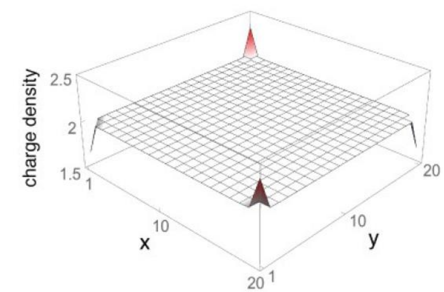
Many-body Calculation

$$Q_{xy} = \frac{1}{2\pi} \mathrm{Im} \log \langle \mathrm{GS} | U_2 | \mathrm{GS} \rangle$$

with $U_2 = \exp\left(\frac{2\pi i}{L_x L_y} \sum xy\, \hat{\rho}(x) \right)$

2. Open BC





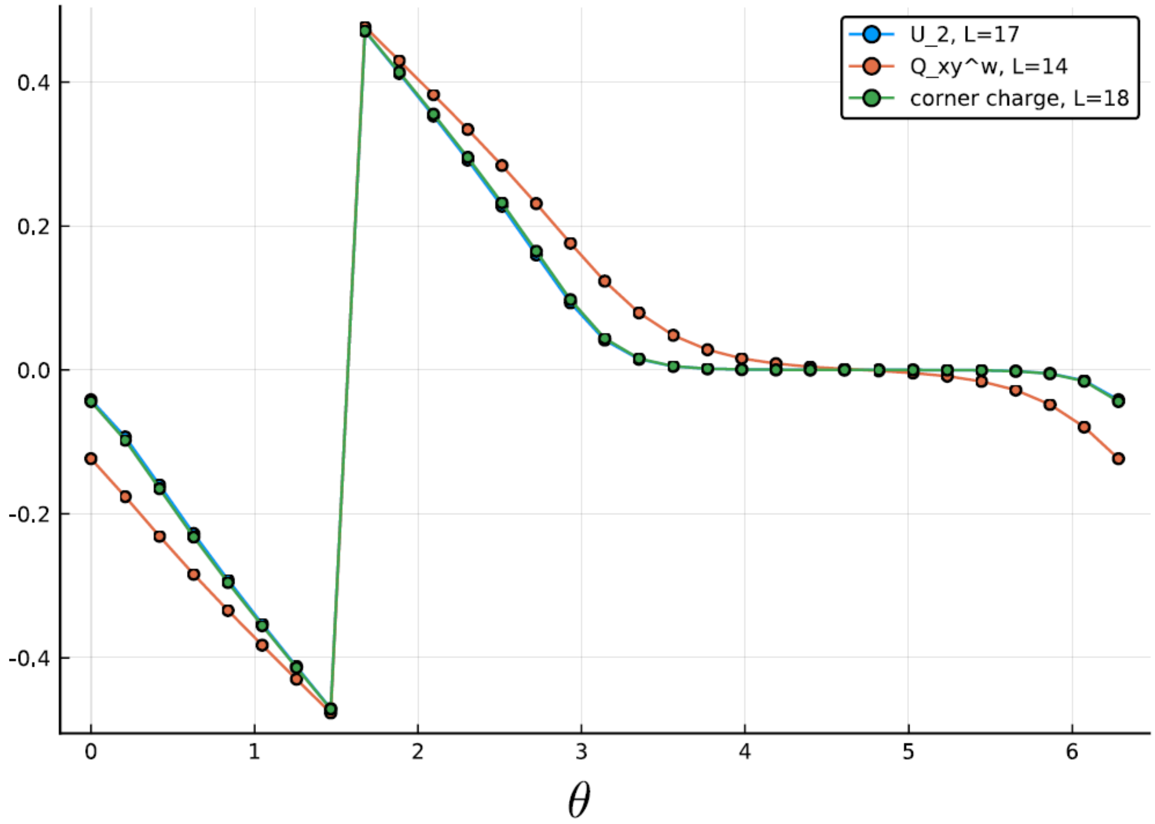
Single-particle Observable

$$Q_c = \int d^2x \, \langle \mathrm{GS} | \, \hat{\rho}(x) - \bar{\rho} \, | \mathrm{GS} \rangle$$

$$= \sum \langle \hat{\rho}(x) - \bar{\rho} \rangle_{\text{single particle}}$$

Do I find $Q_{xy} = Q_c \; \left(= Q_{xy}^{ph} \right)$?

Beyond nested Wilson loop



Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

Seems working.

But why do they work?

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

Path-integral Interpretation of the overlap:

$$\langle \mathbf{GS}|\mathrm{U}_2|\mathbf{GS}\rangle = \frac{1}{Z}\,\mathbf{Tr}\,\mathbf{e}^{-\beta H}\mathrm{U}_2 \propto \;\mathbf{exp}\left(i\mathbf{S}_{\mathbf{eff}}\left[A_\mu\right]\right)$$

Applying Dyson's formula:

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

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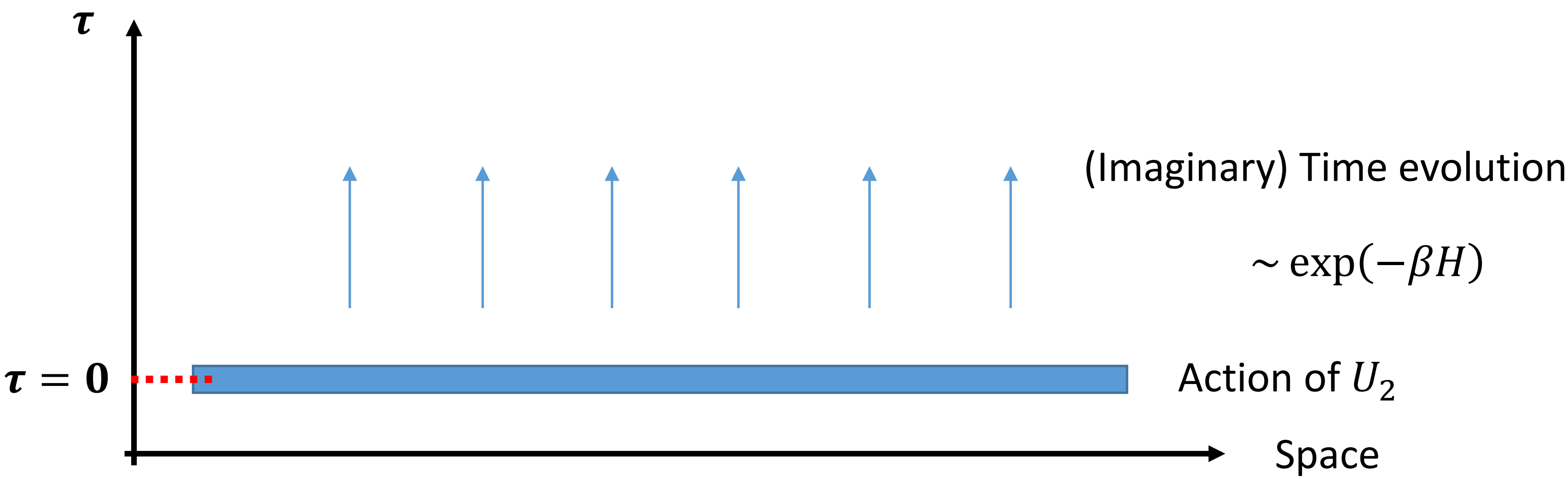


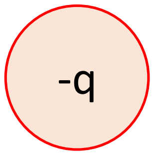
Here A_μ is generated by U_2, i.e., $A_\mu = \delta_{\mu 0}\delta(\tau)\frac{2\pi}{L_x L_y}xy$

So, what is $S_{eff}[A_\mu]$?

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

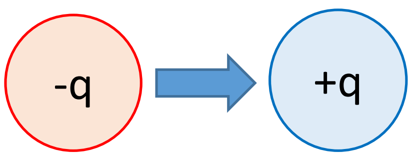
Effective Responses of Multipoles:

1. Charge (monopole)



$$\mathbf{S_{eff}} = \iiint \mathbf{dtd^2x \;\; q\, V(x, y)}$$

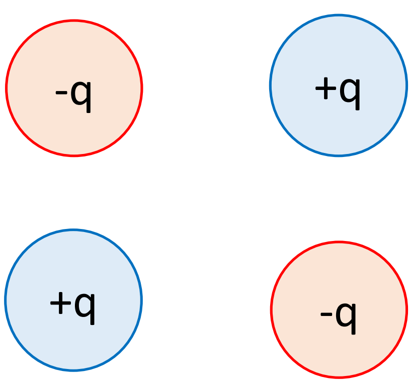
2. Dipole (1st multipole)



$$\mathbf{S_{eff}} = \iiint \mathbf{dtd^2x \;\; q\, V(x, y) - q\, V(x + d, y)}$$

$$\approx \iiint \mathbf{dtd^2x \;\; qd \;\; \partial_x V(x, y)} = \iiint \mathbf{dtd^2x \;\; P \cdot E_x}$$

3. Quadrupole (2nd multipole)

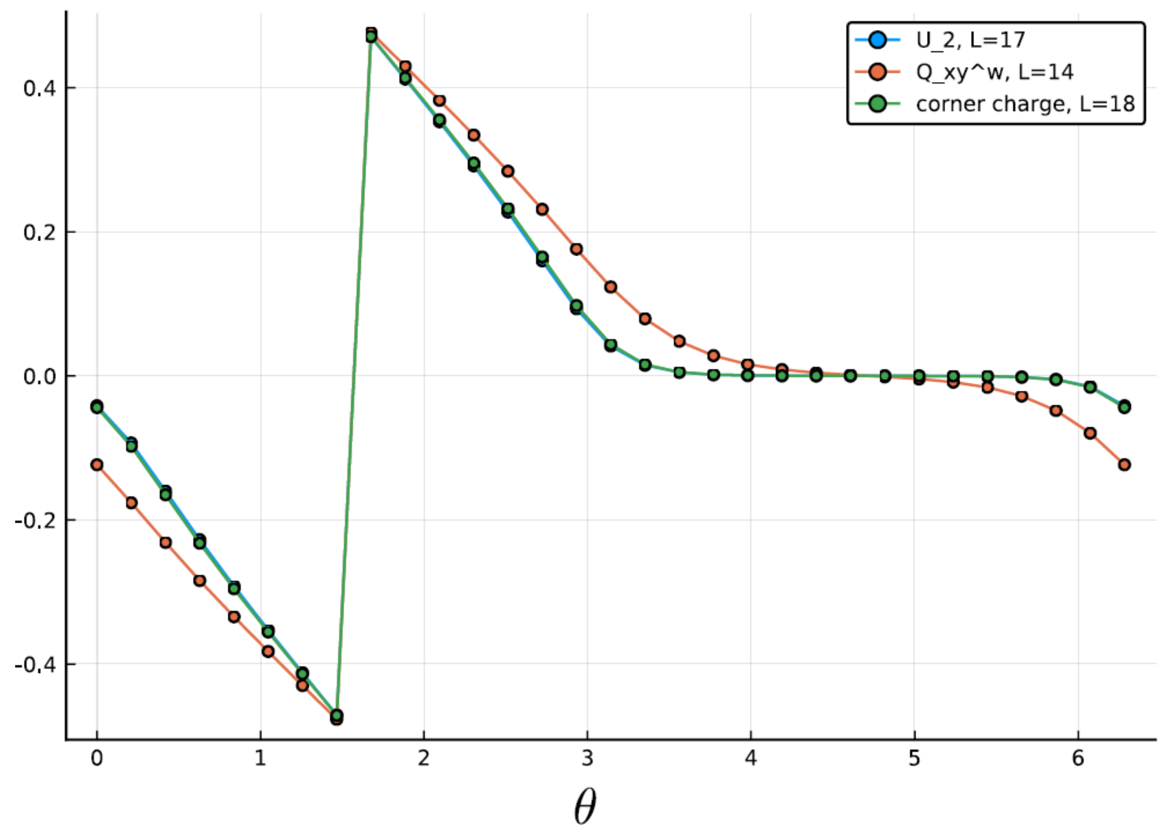


$$\mathbf{S_{eff}} = \iiint \mathbf{dtd^2x \;\; [q\, V(x, y) - q\, V(x + d, y) + \cdots]}$$

$$= \iiint \mathbf{dtd^2x \;\; Q_{xy}^{ph}} \frac{[\partial_x E_y + \partial_y E_x]}{\mathbf{2}}$$

For the gauge fields: $A_\mu = \delta_{\mu 0}\delta(\tau)\frac{2\pi}{L_x L_y}xy$

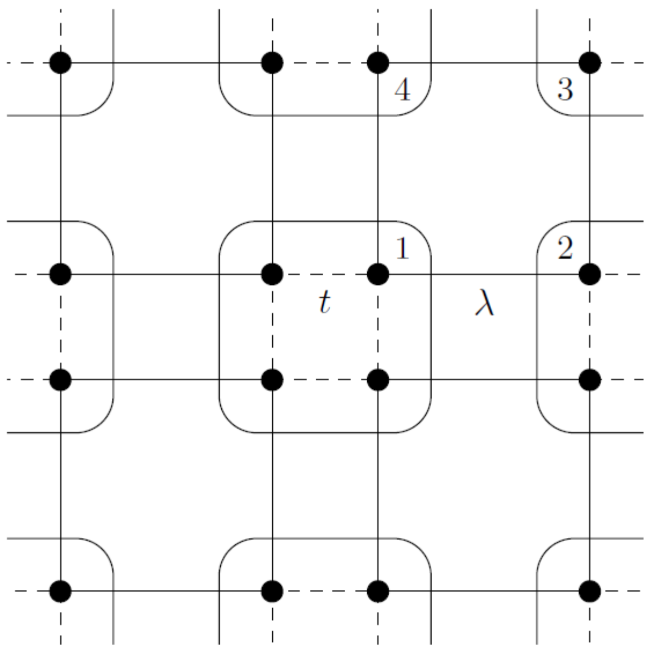
$$\langle \mathrm{GS}|\mathrm{U}_2|\mathrm{GS}\rangle \propto \exp\left(iS_{\mathrm{eff}}\left[A_\mu\right]\right) \propto \exp\left(2\pi i Q_{xy}^{ph}\right) = \exp\left(2\pi i Q_c\right)$$



Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

Spins?

[1] B Kang, K Shiozaki, and GYC (2018)



[Each dot is spin-1/2]

$$H_p = \lambda \sum_{a=x,y} (\sigma_1^a \sigma_2^a + \sigma_2^a \sigma_3^a + \sigma_3^a \sigma_4^a + \sigma_4^a \sigma_1^a)$$

[Ref. Dubinkin-Hughes (2018)]

At the exactly-soluble limits:

(1) $\lambda \neq 0$ and t $= 0$: Topological

- Dangling spin-$\frac{1}{2}$'s at the corners

$$\langle U_2 \rangle = -1$$

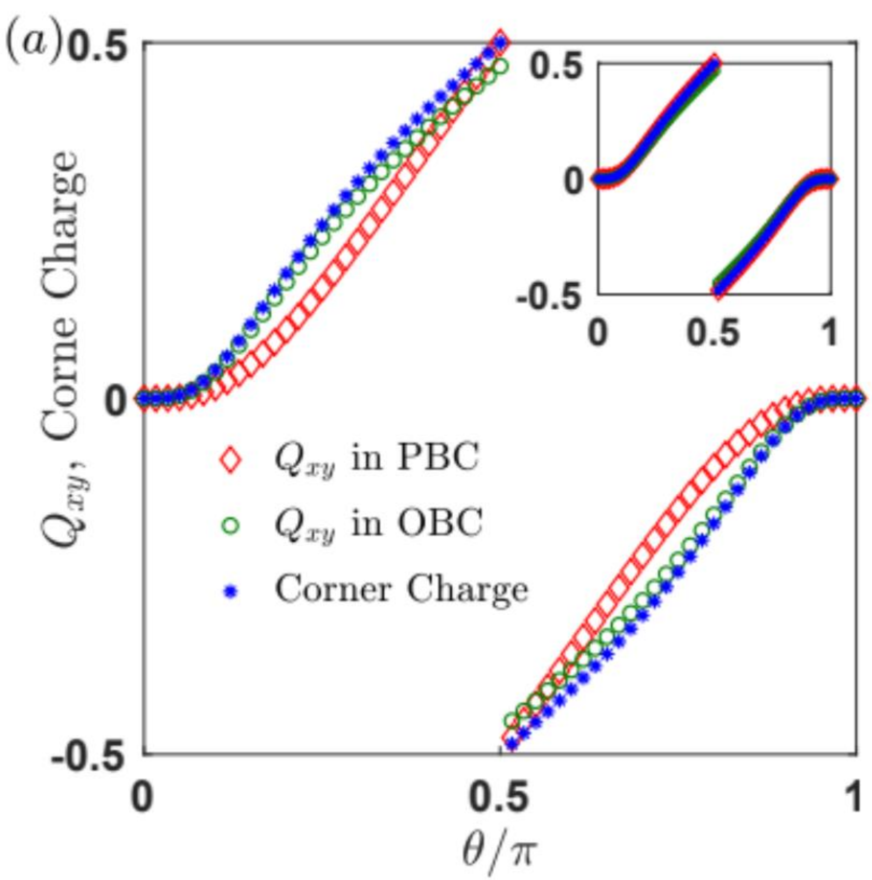
(2) $\lambda = 0$ and t $\neq 0$: Trivial

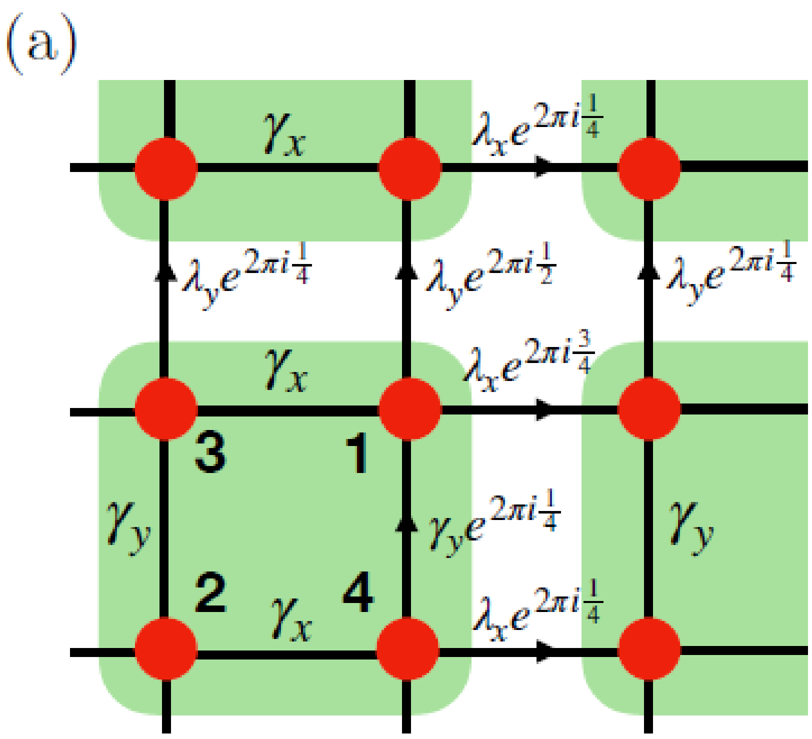
$$\langle U_2 \rangle = +1$$

Other models?

Reviewer 1's model:

To verify that the value of the order parameter proposed in this manuscript indeed describes the multipole moments in a more general setting, I calculated the value of the order parameter for a model which is a modification of the model in Eq 4 in that, instead of threading \pi flux per plaquette, only half of that flux is

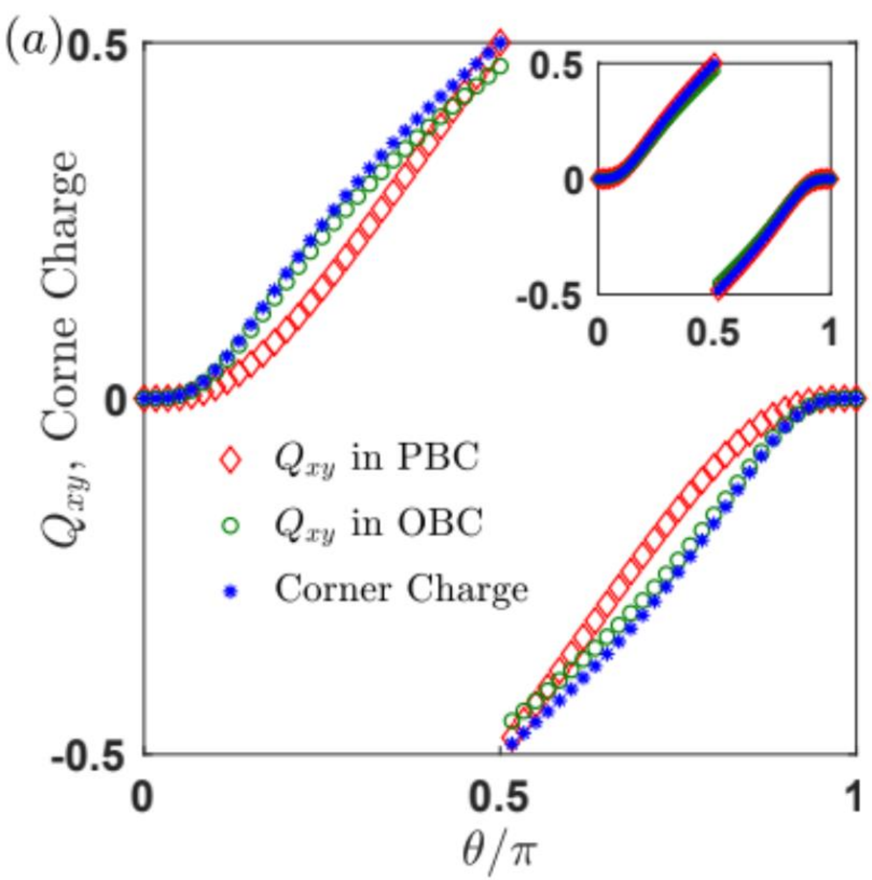




Other models?

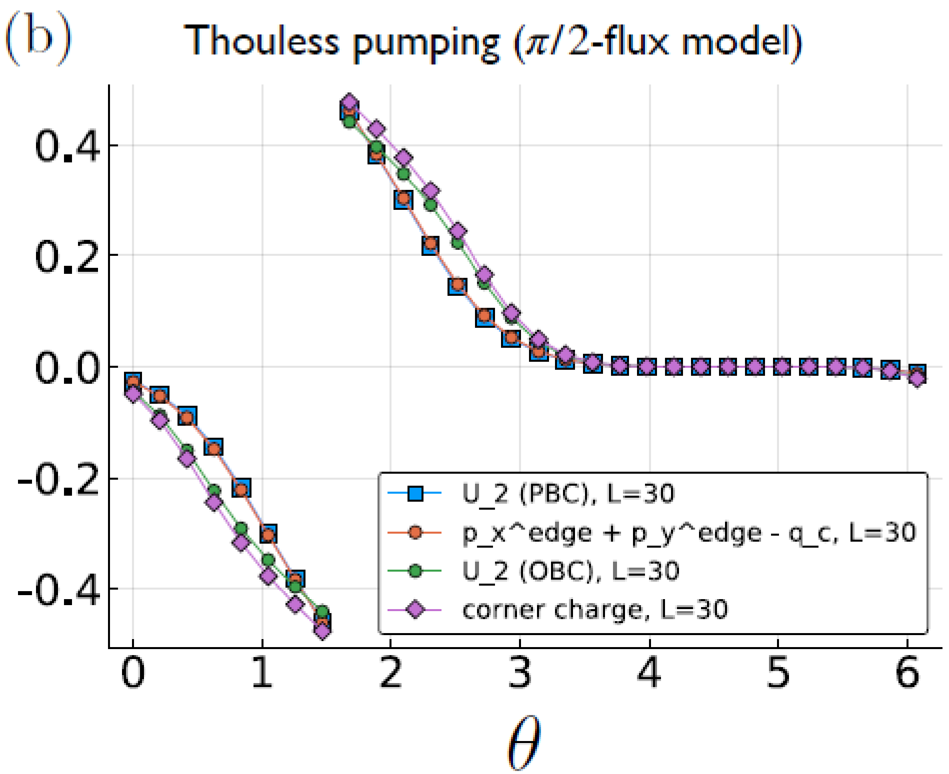
Review 1

To verify that the value of the order parameter proposed in this manuscript indeed describes the multipole moments in a more general setting, I calculated the value of the order parameter for a model which is a modification of the model in Eq 4 in that, instead of threading \pi flux per plaquette, only half of that flux is



With Boundary Polarizations

$$Q_c = P_x^{bdry} + P_y^{bdry} - Q_{xy}$$



Good agreement within numeric errors

Other models?

Higher Order Topological Insulators in Amorphous Solids

Adhip Agarwala,[1,2,*] Vladimir Juričić,[3,†] and Bitan Roy[2,‡]

[Amorphous, Disordered Fermionic (2019 Feb)]

Nonsymmorphic Topological Quadrupole Insulator in Sonic Crystals

Zhi-Kang Lin,[1] Hai-Xiao Wang,[2,1] Ming-Hui Lu,[3] and Jian-Hua Jiang[1,*]

[Nonsymmorphic, Bosonic (2019 Mar)]

Higher-order topological insulator out of equilibrium: Floquet engineering and quench dynamics

Tanay Nag,[1,2,*] Vladimir Juričić,[3,†] and Bitan Roy[2,‡]

[Nonequilibrium, Floquet-driven (2019 April)]

So far so good if there is a Wannier gap.

Cf. S Ono, L Trifunovic, and H Watanabe (2019)

In short,

We found working definitions of electric multipoles in Crystals

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

See also: William Wheeler, Lucas Wagner, and Taylor Hughes, arxiv:1812.06990 (2018)

Beyond Multipoles:

Byungmin Kang, and GYC, in preparation

For U(1) symmetric states:

$$S_{top} = \mathrm{Im}\left[S_{eff}\right]$$ **can be gradient expanded by** A_μ

We find a unitary U with spatial geometry M:

Byungmin Kang (KIAS)

Beyond Multipoles:

Byungmin Kang, and GYC, in preparation

For U(1) symmetric states:

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We find a unitary U with spatial geometry M:

1: Phase of $\langle GS(M)|U|GS(M)\rangle$ detects **the topology**

Ex: bulk dipole, bulk quadrupole, bulk octupole etc

Resta (1999); Kang, Shiozaki, and GYC (2018)

Ex: boundary polarizations, Chern numbers

Byungmin Kang, and GYC, in preparation

Byungmin Kang (KIAS)

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Ex: boundary polarizations, Chern numbers

Byungmin Kang, and GYC, in preparation

Byungmin Kang (KIAS)

2: $|\langle GS(M)|U|GS(M)\rangle|$ detects "**metallicity/gap**" of **the excitation**

Ex: Resta's conjecture, Wannier gap for multipoles

Resta (1999), Kang, Shiozaki, and GYC (2018); Dubinkin, May-mann, and Hughes (2019)

3. Conclusions & Outlooks

Conclusions

1. Proposed (definition of) many-body invariants for multipoles
2. Numerically confirmed the invariants
3. Analytic Supports from Effective QFT
4. Generalization to other topological states

Outlooks

1. Momentum-Space Indices of Our Many-Body Invariants
2. Cases without Wannier gap

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)
Byungmin Kang, and GYC, in preparation

Thanks for your attention!

Remarks on the modulus of Unitaries 1.

1. Resta's conjecture on $U_1 = \exp\left(\frac{2\pi i}{L_x} \sum x\, \rho(x)\right)$

$$|\langle U_1 \rangle| \to 0 \text{ as } \Delta_{gap} \to 0 \text{ ("metal")}$$

…so that P_x as the phase of $\langle U_1 \rangle$ ill-defined.

[Ref. Resta (1998); more precise statement in Kobayashi-Nakagawa-Fukusumi-Oshikawa (2018)]

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

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[Ref. Resta (1998); more precise statement in Kobayashi-Nakagawa-Fukusumi-Oshikawa (2018)]

2. Our conjecture on $U_2 = \exp\left(\frac{2\pi i}{L_x L_y} \sum xy\,\rho(x)\right)$

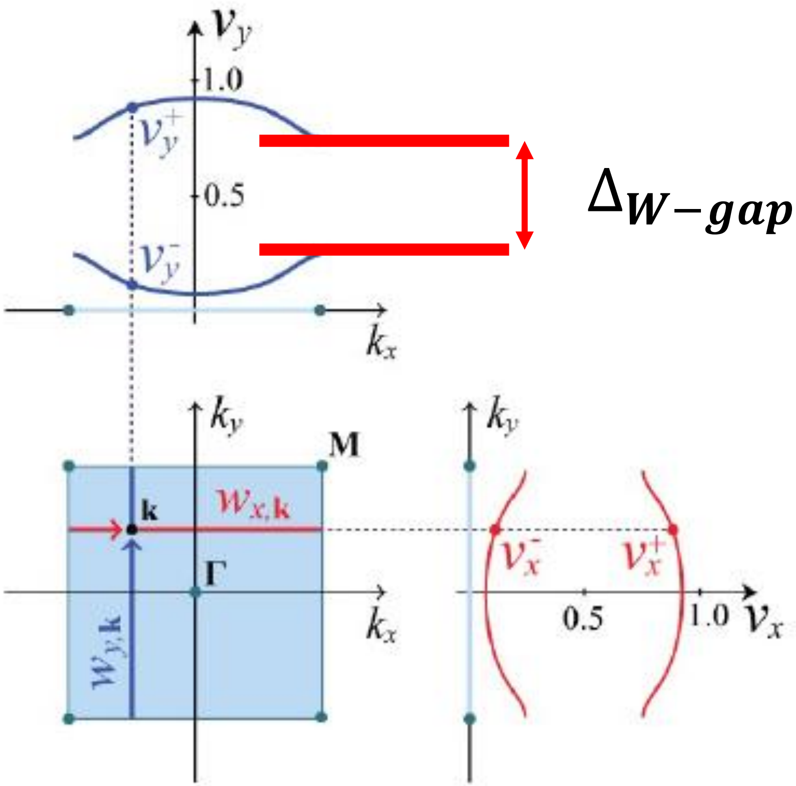
$$|\langle U_2 \rangle| \to 0 \text{ as } \Delta_{W-gap} \to 0$$ ["dipolar metal" but "charge insulator"?]

Note: $\Delta_{W-gap} \neq 0$ is necessary to define quadrupoles in nested Wilson loops

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

Remarks on the modulus of Unitaries 2.

We plot…

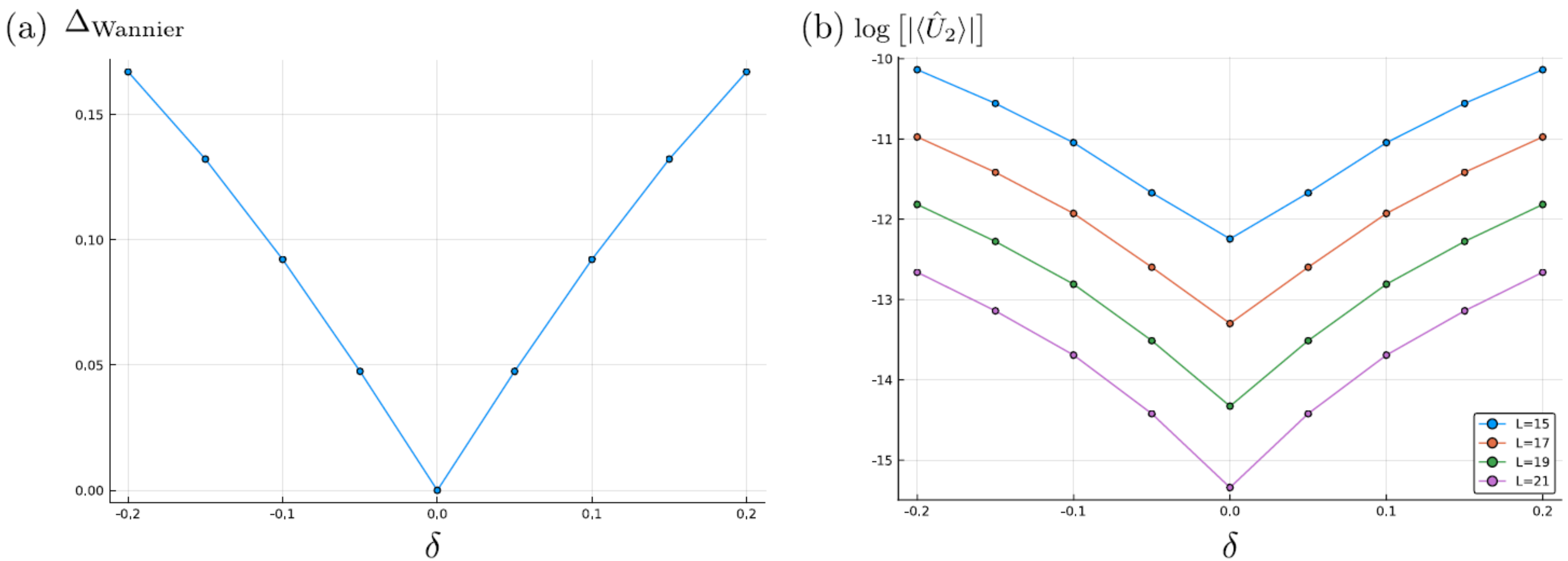


Δ_{W-gap}

v.s.

$$|\langle \boldsymbol{U_2} \rangle|$$

Remarks on the modulus of Unitaries 3.



$$|\langle U_2\rangle| \to 0 \text{ as } \Delta_{W-gap} \to 0 \text{ ("Dipole Metal")}$$

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

In short,

We found generic definitions of electric multipoles in Crystals

[phase part of unitary]

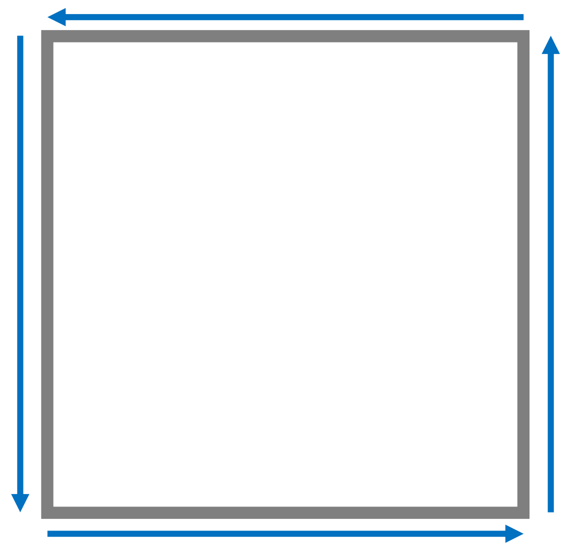
+ many-body measure of Wannier gap closing

[modulus of unitary]

Ref. Byungmin Kang, Ken Shiozaki, and GYC, arxiv:1812.06999 (2018)

Higher-Order Topology = Non-Trivial at "Boundary of Boundary"

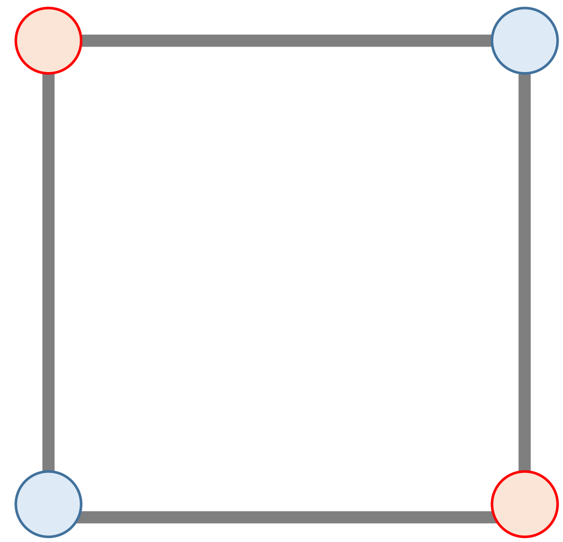
Ex: In **2D**,



2D Topological Insulators

E.g. QHEs or HgTe

Quadrupolar Insulators



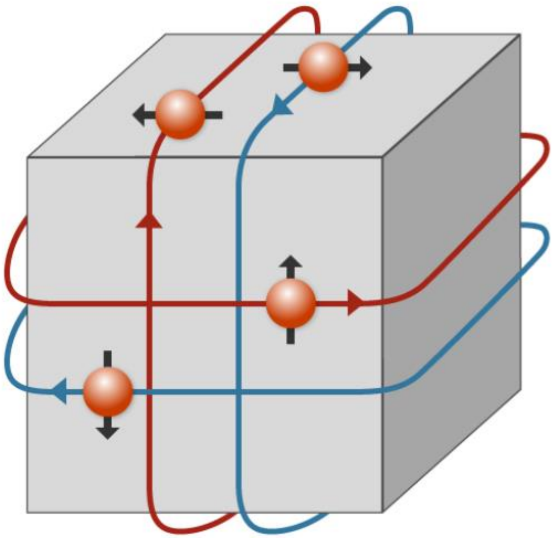
Corner States

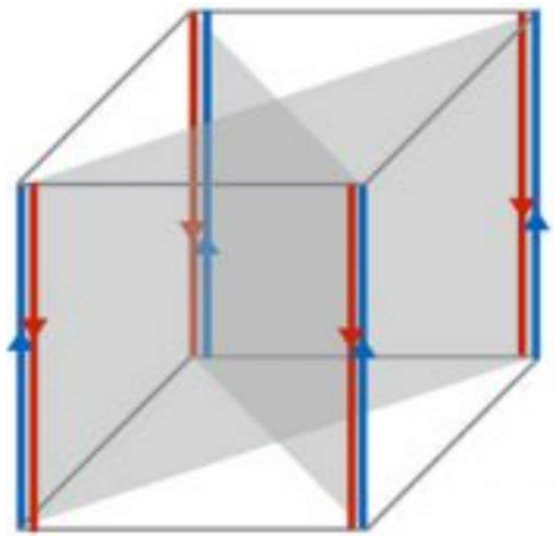
E.g., ?

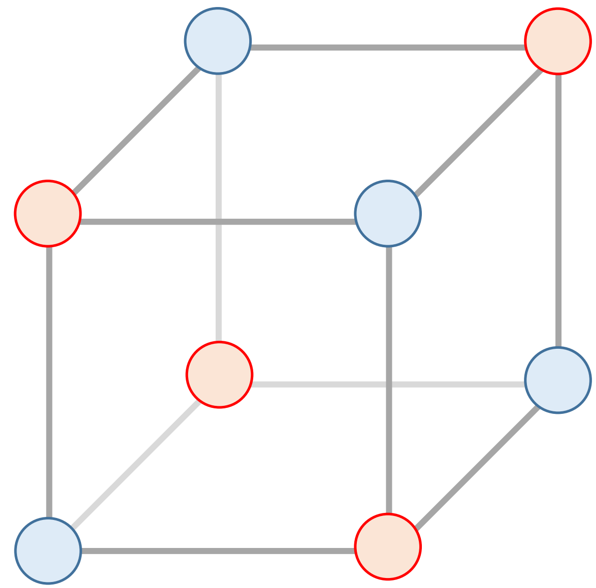
Higher-Order Topology = Non-Trivial at "Boundary of Boundary"

Ex: In **3D**,

Octupolar Insulators







3D Topological Insulators | **Hinge States** | **Corner States**

E.g. Bi_2Se_3 | E.g. Bismuth | **E.g., ?**