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Moving discrete breathers?

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Abstract

We give definitions for different types of moving spatially localized objects in discrete nonlinear lattices. We derive general analytical relations connecting frequency, velocity and localization length of moving discrete breathers and kinks in nonlinear one-dimensional lattices. Then we propose a new numerical algorithm to find these solutions. ©1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The search for moving radiationless spatially localized excitations is an interesting topic in the field of nonlinear dynamics of systems of many interacting degrees of freedom.

Many of the integrable one-dimensional models, both discrete and continuous in space, possess moving objects, either breathers or kinks [1]. The integrability of these models provides one with action–angle variables. Tuning the action as parameters one can continuously go from stationary solutions to moving ones (examples are the nonlinear Schrödinger equation [1], the Toda [2] and Ablowitz–Ladik lattices [3]). This fact allows us to think about the whole family of solutions as a particle-like entity. Still we are lacking a precise understanding of the connection between integrability and possible existence of moving localized objects. The reason is the hidden character of the symmetries which provide integrability.

Another reason for the existence of moving solutions can be some continuous symmetry of the Hamiltonian, e.g. the invariance under Lorentz transformations. The Lorentz transformation generates moving objects provided the corresponding stationary object exists. The system does not need to be integrable, so moving kinks exist for instance in Φ^4 theory in $1+1$ dimensions [1]. However, stationary breathers appear to be nongeneric for space-continuous models already in the case of one spatial dimension [4,5]. The reason for that lies in the fact that the phonon frequency spectrum Ω_q (q is a wave vector) of small-amplitude (linearized) vibrations of a continuous system is typically unbounded from above. As per the definition a stationary breather is a time-periodic spatially localized solution of the equations of motion. By expanding the solution in a Fourier series with respect to time one has to deal with the Fourier components associated to each multiple $k\Omega_b$ of the breather frequency Ω_b with integer k. The unboundeness of the phonon

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spectrum leads to an infinite number of high frequency resonances, which generally prevents the appearance of space-localized time-periodic solutions [6]. An exception is e.g. the sine-Gordon system and some isolated perturbations of it. Note that from these results, it follows that moving breathers are also nongeneric solutions for Lorentz-invariant field equations, since we can always perform a Lorentz transformation and obtain stationary breathers if moving breathers exist. Since stationary breathers are nongeneric for space-continuous models, moving breathers have to be nongeneric too (see results in Section 3.1).

Space-discrete models can generically allow for the existence of stationary discrete breathers [7,8] (see [9] for an extensive discussion and for a list of references). One reason for that is that the discreteness of space produces a lower cut-off in the wavelength of small amplitude plane waves, and thus a finite upper bound for the phonon spectrum. Then one has the possibility to choose frequencies Ω_{b} such that $k\Omega_{\mathrm{b}} \neq \Omega_{\mathrm{q}}$ for all integer k. However, discreteness in space implies the loss of say the continuous Lorentz symmetry. Thus in general there is no clear way how to generate moving breathers out of stationary ones on lattices.

There have been many attempts to solve for moving breathers. Moving kinks can be considered to some extent as moving breathers with zero frequency. Proofs of existence of moving kinks in FPU chains have been obtained in [10] by finding them as minimizers of a variational problem and in [11] by analytical continuation from the integrable Toda lattice. Numerical solutions for moving kinks have been obtained e.g. in [12,13]. Fourier transformations in space are used in the first one, while the second paper treats space-periodic solutions, but does not use Fourier transformations. There is a large amount of work reporting on moving breathers in FPU chains (e.g. [14,15]) and also for antiferromagnetic chains [16]. A study of their connection with stationary breathers was started in [17]. In [18] this connection was used to numerically obtain moving breathers by exciting pinning modes of stationary breathers. Finally, moving breathers have been obtained numerically for DNLS chains in [19]. Perturbation theories have been applied to the deformed Ablowitz–Ladik lattice [20] (see also [21]). So far, we are not aware of existence proofs for moving breathers.

To look for moving breather solutions we need to have some good definition of them. We define the simplest type of a moving solution as a solution that repeats itself after the time T_{s} shifted by one lattice site. Such a solution is a fixed point of the map $RG_{T_{\mathrm{s}}}$, where G_t is the evolution operator in the phase space of the system, R is the translation operator that shifts all indices by 1. More sophisticated solutions can be obtained by considering fixpoints of the map $R^n G$, i.e. solutions that repeat themselves after the time T_{s} shifted by n sites. We assume the lattice spacing to be 1, so the velocity V is then just n/T_{s}.

Depending on the boundary conditions at infinity we speak about moving breathers or kinks. A trivial example of a radiationless moving discrete kink can be obtained by considering identical billiard balls on the line separated by some distance l. Then kicking one ball we get an eternal motion where after the nth kick the nth ball transfers all its energy to the $(n+1)$th one. The interaction potential of balls can be more or less arbitrary the only restriction is that it is short-ranged enough, so only two balls interact at each moment of time. Below we will in general discuss boundary conditions corresponding to moving breathers, i.e. the lattice is asymptotically in the same ground state at large distances from the center of the excitation no matter what direction from the center is chosen.

In Section 2 we will outline a tail analysis of possible moving breathers. In Section 3 examples are given, and numerical calculations of solutions are presented in Section 4. In Section 5 we briefly discuss related work and summarize our results.

2. Tail analysis of possible moving breathers

The method of analyzing the decay properties in the tail of a breather is very fruitful. From such an analysis one precisely derives the nonresonance conditions a stationary nonmoving breather must fulfill in general to exist. This concept has been extremly useful to explain why stationary breathers are generic in discrete systems, while they are nongeneric in field theories. A necessary condition for existence of a breather is that the solution is localized in space. But if it is localized in space, we can analyze the tails, where all amplitudes are small, and check whether the solution really can decay to zero at infinite distance. This leads to the well-known nonresonance condition for stationary breathers [9].

Since a stationary breather is characterized by an internal frequency Ω_b, we have to incorporate this timescale into the definition of a moving breather. Consider a one-dimensional lattice, describing the interaction of degrees of freedom associated to each lattice site. Each degree of freedom is given by a pair of canonically conjugated variables (e.g. displacement and momentum) labeled with the site index. Call one of those variables $u_n(t)$. We define a one-frequency discrete moving breather solution as [1]

$$u_n(t) = F(\Omega_b t, n - Vt). \tag{1}$$

Here $F(x, y)$ is a function period 2π periodic with respect to x and localized with respect to y:

$$F(x + 2\pi, y) = F(x, y), \qquad F(x, y \to \pm\infty) \to 0. \tag{2}$$

If T_s and $2\pi/\Omega_b$ are commensurate so that $kT_s = l2\pi/\Omega_b$, where k and l are integers, then such a breather repeats itself after time kT_s shifted by k sites and belongs to the simplest moving breather defined above. In the general noncommensurate case the breather will never repeat itself although coming arbitrarily close to it.

In the same manner breathers having two or N internal frequencies can be defined. This hierarchy of objects incorporates everything that we intuitively percept as an object moving through the lattice.

Thinking of moving breathers in terms of fixed points allows to define other interesting objects on a discrete lattice. Consider a fixed point of some general map $G_{T_s}X$, where X is an element of the lattice symmetry group. If X is the identical transformation, we get stationary breathers. The translation operator gives us moving ones. For one-dimensional lattice the only symmetry group element left is the reflection, which gives us reflector-breathers, which mirror themselves after time T_s. Higher dimensional lattices provide more choices namely taking a rotation as X we get rotation-breathers, [2] and then taking as X a superposition of rotation and translation we get "walking" breathers.

[1] Of course one can choose more complicated forms, which will not be discussed here. Simplified ansatz versions have been frequently used for fitting numerical results, e.g. [15].

[2] Not to be confused with rotobreathers, see [22–25].

It is clear from the beginning that looking for discrete breathers in terms of exact analytical solutions is a hard task. Approximate methods like a rotating wave approximation even if justifiable would turn us away from the phase space of the system and thus do not help too much to understand what a moving discrete breather is. A productive way is to look at moving breathers from a general point of view and although not solving any particular problem to exactly find model independent relations that all the moving breathers should fulfill. We will do this here for the class of moving breathers described by (1).

We derive these relations below considering the tail of a moving breather where the motion can be considered linear because of the small amplitude of oscillations.

Let us consider a moving breather with one internal frequency as defined above in (1). We write the function $F(x, y)$ in a Fourier series with respect to x:

$$F(x, y) = \sum_k e^{ikx} f_k(y). \tag{3}$$

Inserting this ansatz into the Hamiltonian equations of motion we obtain coupled equations for the functions $f_k(y)$. In the spatial tails of the breather these equations decouple with respect to the label k. [3] Let us consider some particular value of k with frequency $\Omega_k = k\Omega_b$. The equations are linear so we seek for a solution in exponential form:

$$f_k(y) = e^{-iq_k y}. \tag{4}$$

Here q_k is a complex number. The $u_n(t)$ term corresponding to a given kth harmonic then takes the following form:

$$u_{nk}(t) = e^{i(\omega t - q_k n)}. \tag{5}$$

Here $\omega = \Omega + Vq_k$. Using the equations of motion we finally obtain

$$\omega = G(q_k), \tag{6}$$

where $G(q) = \Omega_q$ is the dispersion relation of the system analytically continued to the complex plane.

[3] The decoupling implies that in the tails, where all f_k become small, nonlinear terms in f_k can be dropped. This linearization can be violated if additional symmetries are present or if some of the f_k functions decay very slowly in space [26].

With the definition of ω we have

$$\Im z = V \Im q_k, \tag{7}$$

where $(\Im q_k)^{-1}$ is the localization length of a given harmonic, $\Re\omega = \Omega + V\Re q_k$, $\Omega = k\Omega_\mathrm{b}$.

This equation connects the frequency, velocity and the localization length of a given harmonic with each other. For the particular case of a breather without internal frequency, $\Omega_\mathrm{b} = 0$ (shock wave), we obtain the following relation between the velocity and the localization length:

$$V q_k = G(q_k). \tag{8}$$

This equation shows that for a shock wave with a given velocity, only a discrete set of complex wave vectors q_k is allowed. For the linear dispersion relation $G(q) = Cq$, only the velocity $V = C$ is allowed and then q can be arbitrary. For the parabolic dispersion relation $G(q) = Aq^2, q$ is always real namely $q = V/A$, so no localization is possible because the localization length is the inverse of the imaginary part of q. For the quadratic dispersion relation with a cut-off at finite frequency

$$Vq = G(q) = Aq^2 + D, \tag{9}$$

we have a complex root when $(V)^2 - 4AD < 0$. Therefore, a cut-off in the dispersion relation at finite frequencies in the dispersion relation appears to be relevant in order to have localization of a moving solution. Note that such a cut-off can be caused either by a gap in a space-continuous system or simply by a space-discrete system.

For moving breathers with nonzero internal frequency a necessary condition of existence is obviously that all harmonics $f_k(y)$ are localized, namely for any $\Omega = k\Omega_\mathrm{b}$, there is a solution of Eq. (7) with $\Im q_k \neq 0$.

Therefore we obtain that for any nonlinear lattice the frequency of the breather, its velocity and the localization length of its harmonics are not independent but connected by Eq. (7).

3. Examples

In this chapter we will consider examples of systems to illustrate the general considerations from above.

3.1. Space-continuous models

3.1.1. Klein–Gordon models in 1+1 dimensions

Consider the partial differential equation for the field $U(x,t)$:

$$U_{,tt} = -U + CU_{,xx} - F_{nl}(U), \tag{10}$$

where the function $F_{nl}(x)$ if expanded in a Taylor series around $x = 0$ contains only nonlinear terms in x. Examples are e.g. the Φ^4 Klein–Gordon equation with $F_{nl}(x) = x^3$ or the sine-Gordon equation with $F_{nl}(x) = \sin(x) - x$.

Let us search for a moving solution in the form

$$U(x,t) = \sum_k u_k(x - Vt)\mathrm{e}^{\mathrm{i}k\Omega_\mathrm{b}t}. \tag{11}$$

Inserting (11) into (10), assuming that the amplitude of the solution is small at large distances from some center, skipping the nonlinear terms in (10) we obtain for each integer k:

$$(V^2 - C)\frac{\mathrm{d}^2 u_k}{\mathrm{d}z^2} - 2\mathrm{i}Vk\Omega_\mathrm{b}\frac{\mathrm{d}u_k}{\mathrm{d}z} + (1 - k^2\Omega_\mathrm{b}^2)u_k = 0. \tag{12}$$

For simplicity we skip the k index and define $u = u_k$, $\Omega = k\Omega_\mathrm{b}$, $z = x - Vt$. To solve (12) we make the ansatz $u(z) \sim \mathrm{e}^{\lambda z}$. Note that $\lambda = \mathrm{i}q_k$ (see (4)). Decomposing λ into real and imaginary parts $\lambda = R + \mathrm{i}I$ (R, I real), we find

$$(V^2 - C)(R^2 - I^2) + 2V\Omega I + 1 - \Omega^2 = 0, \tag{13}$$

$$(V^2 - C)2RI - 2V\Omega R = 0. \tag{14}$$

The physically relevant parameters of our solution are the velocity V and the exponent of the spatial decay R characterizing the localization length of the object. The physical frequency describing the true oscillations is given by $\omega = \Omega - VI$. Solving (13) we find

$$I^2 = \frac{V^2}{C}\left[\frac{1}{C - V^2} - R^2\right], \tag{15}$$

$$\omega = -\frac{C}{V}I. \tag{16}$$

It follows that $V^2 < C$ (Lorentz invariance of (10)) and $V^2 \geq C - 1/R^2$. These two curves define the allowed region in the $\{R^2, V^2\}$ plane of possible solutions, which is shown in Fig. 1. The velocity of any

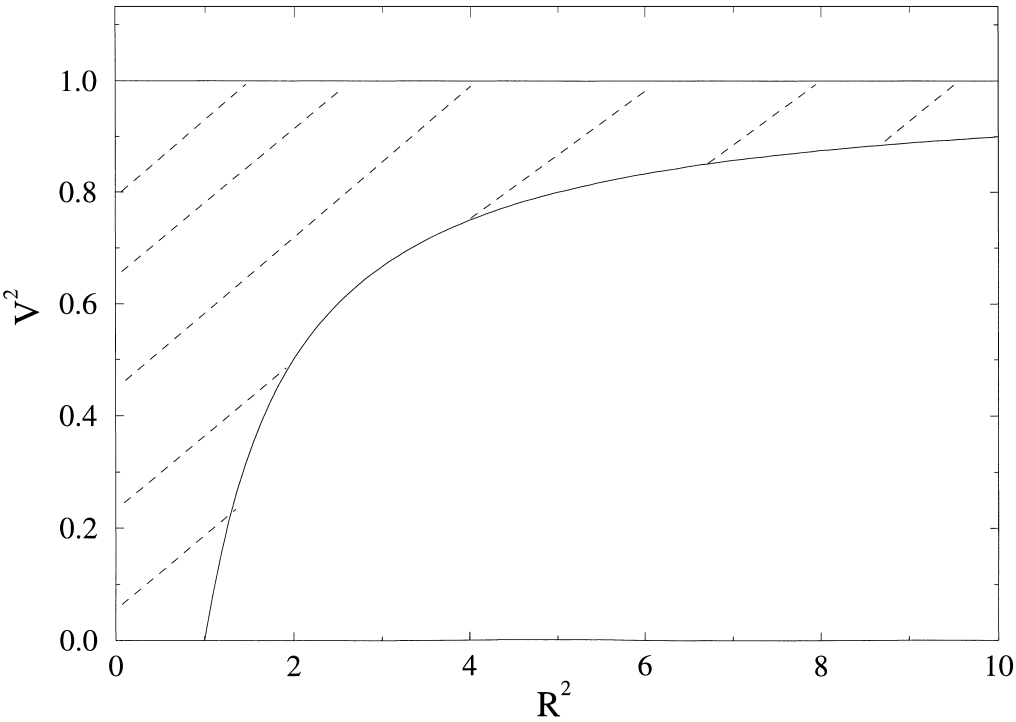

Fig. 1. Phase diagram for localization conditions on the existence of moving localized objects in Klein–Gordon models in 1+1 dimensions (see text).

moving object has to be below the speed of light. For large (weakly localized) objects with $R^2 < 1/C$ the solution will always have some nonzero frequency ω, i.e. there is a gap in the allowed frequency spectrum containing $\omega = 0$. For small (strongly localized) objects with $R^2 \geq 1/C$ the gap closes, and one can always design a tail solution with zero frequency $\omega = 0$ (this corresponds to the lower bound of the allowed region in Fig. 1). The case of zero frequency is nothing but the tail (or front) of a shock wave. Note that there are no restrictions with respect to the value of R^2, so the tail solutions can be infinitely strongly localized in space.

So far we discussed the solutions without checking whether the initial ansatz (11) can be completely fulfilled in the tails, i.e. for all integers k. Note that in ansatz (11) we parametrize the solution using Ω_b and V. To answer that question, we can argue in the following way. Suppose we choose a point in the $\{R^2, V^2\}$ diagram in the allowed region. This point gives us a set of values for ω and I. Suppose that $k = 1$. Then we obtain some unique value for Ω_b. Now we can go the inverse way and say that for that value of Ω_b and V we find the corresponding values for R and I. But what happens for other values of k? Since by increasing k we increase ω, we have to check whether at a fixed value of V we can realize any value for ω in Fig. 1 by changing R throughout the allowed region. The answer is no. Indeed fixing V we always obtain a finite line segment in Fig. 1. On the right end of this segment $\omega = 0$, and on the left $\omega \leq \omega_{\max}(V) < \infty$, with no singularities in between. Thus we can never realize ansatz (11). So we conclude that in general moving breathers do not exist in Klein–Gordon field problems. The only possiblity is to have a shock wave (or kink), i.e. to set $\Omega_b = I = \omega = 0$, which is possible.

Note that we are in complete agreement with the results on stationary breathers in space-continuous models (see Section 1). Indeed, having taken the mathematical proofs that stationary breathers are nongeneric [4,5] for nonlinear Klein-Gordon equations, it immediately follows that moving breathers will be nongeneric too, since in the opposite case one could form a stationary breather out of a moving one by Lorentz

transforming, and be in conflict with the results on the nongeneric occurrence of stationary breathers.

3.1.2. (Non)linear Schrödinger equation

Consider the partial differential equation for the complex field $\Psi(x,t)$:

$$\dot{\Psi} = \mathrm{i}(C\Psi_{,xx} + F(\Psi)). \tag{17}$$

Again we search for a solution in the form $\Psi(x,t) = \phi(x - Vt)\mathrm{e}^{\mathrm{i}\Omega t}$. Repeating the same procedures as in the previous case we arrive at the equations

$$I = \frac{V}{2C}, \tag{18}$$

$$\omega = C\left(R^2 - I^2\right). \tag{19}$$

Since (17) is not Lorentz-invariant, we do not find restrictions on the choice of the velocity V. In fact we find no restrictions at all, the whole parameter space $\{R, V\}$ is allowed.

3.2. Space-discrete models

3.2.1. Discrete (non)linear Schrödinger equation

The equations of motion are given by

$$\dot{\Psi}_n = \mathrm{i}[|\Psi_n|^{\mu}\Psi_n + C(\Psi_{n-1} + \Psi_{n+1})]. \tag{20}$$

Here the nonlinear term is characterized by some $\mu > 0$. Again we search for a moving solution in the form $\Psi_n = \phi(n - Vt)\mathrm{e}^{\mathrm{i}\Omega t}$. Due to the nonlocality of the difference operator (as compared to the differential one) the differential equation for $\phi(z)$ contains now retarded and advanced terms (the tail is again only considered, nonlinear terms are neglected):

$$-V\phi'(z) = \mathrm{i}[-\Omega\phi(z) + C(\phi(z+1) + \phi(z-1))]. \tag{21}$$

Skipping the intermediate calculations we arrive at the result (without loss of generality we can choose $C = 1$)

$$I = \arcsin\left[\frac{VR}{2\sinh R}\right], \tag{22}$$

$$\omega = 2\cosh R\cos I. \tag{23}$$

A necessary condition is thus $V \le 2\sinh(R)/R$ and is shown in Fig. 2. In the region of possible solutions for each pair of $\{R, V\}$ we now have two solutions due to the periodicity of (22) and (23) in I–to each solution with a given value of $I_1 = I$ we can construct a second solution with $I_2 = \pi - I$. For any given value of R there is a finite upper bound on the value of V. However, with increasing R, the threshold value of the upper bound of V also increases. Again we can find infinitely strongly localized and infinitely fast moving tails.

3.2.2. Klein–Gordon chains

These models describe the dynamics of atoms on a substrate and interacting with each other:

$$\ddot{u}_n = -\alpha u_n - C(2u_n - u_{n-1} - u_{n+1}) + F_{nl}(u_n). \tag{24}$$

As in the field case we search for a moving solution in the form

$$u_n(t) = \sum_k A_k(n - Vt)\mathrm{e}^{\mathrm{i}k\Omega_{\mathrm{b}}t}. \tag{25}$$

In the tails of the assumed existing solution we obtain

$$V^2\frac{\mathrm{d}^2A_k(z)}{\mathrm{d}z^2} - 2\mathrm{i}k\Omega_{\mathrm{b}}V\frac{\mathrm{d}A_k(z)}{\mathrm{d}z} = (\Omega^2 - \alpha - 2C)A_k(z) + C(A_k(z+1) + A_k(z-1)). \tag{26}$$

Repeating the intermediate calculations as above we arrive at the following equations (note that we skip the index k, so below $\omega = k\Omega_{\mathrm{b}} - VI$):

$$\omega = -\frac{C}{VR}\sinh R\sin I, \tag{27}$$

$$\left[\frac{C}{VR}\right]^2\sinh^2 R\sin^2 I = V^2R^2 + \alpha + 2C(1 - \cosh R\cos I). \tag{28}$$

The allowed region in the parameter space $\{R, V^2\}$ is not only similar to the one of the discrete nonlinear Schrödinger case (Fig. 2) but also more complicated. Solutions exist below a certain line $V^2(R)$. This line consists of two parts. For $\cosh R > 2C/(\alpha + 2C - \sqrt{\alpha^2 + 4\alpha C})$, the line is given by

$$\text{(a)} \qquad V^2 = \frac{1}{R^2}(2C\cosh R - \alpha - 2C). \tag{29}$$

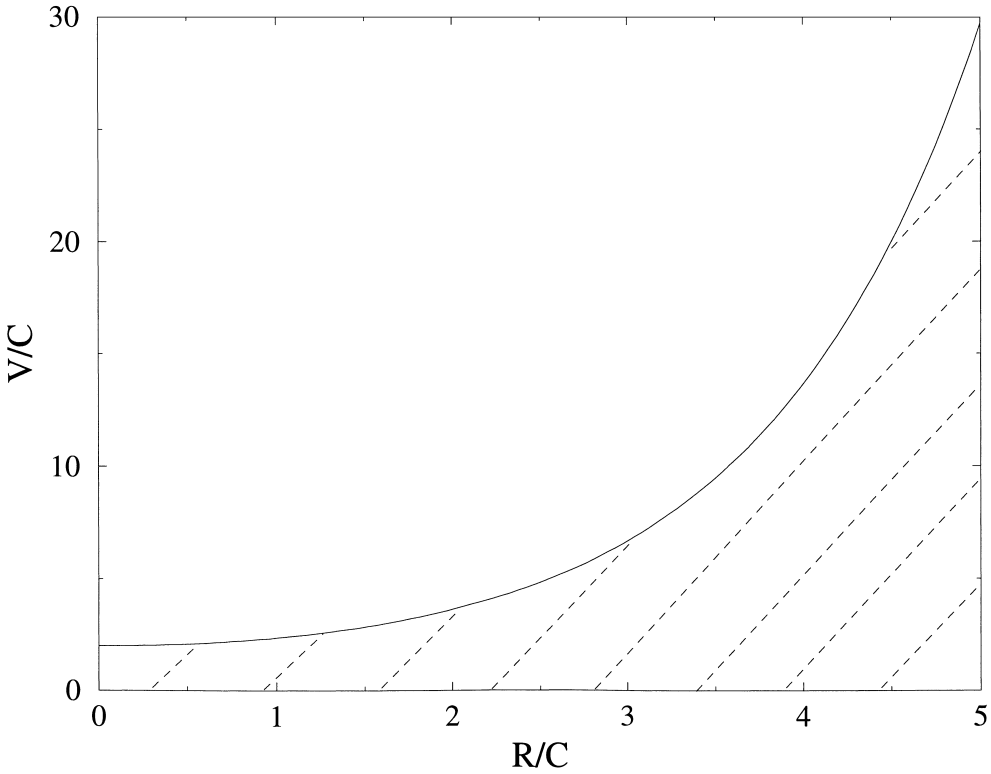

Fig. 2. Same as Fig. 1, but for the DNLS chain.

For smaller values of R we have line (b):

$$\text{(b)} \qquad V^2 = \frac{1}{2}\frac{\sinh^2 R}{R^2}(\alpha + 2C - \sqrt{\alpha^2 + 4\alpha C}). \qquad (30)$$

In this second case the continuation of line (a) (29) separates solutions with multiplicity 4 (to the left) from solutions with multiplicity 2 (to the right) as shown in Fig. 3.

Let us fix a value of V. This value defines some half-infinite line segment of allowed solutions in $\{R, V^2\}$. For ω we find $0 \le \omega < \infty$. Thus for any $V \neq 0$ we can generate a breather solution in the tails, for any value of Ω_b! Surprisingly the problem of resonances, as in the case of a stationary breather, does not appear on that stage.

We can also find a solution to $\Omega_b = \omega = I = 0$, i.e. we can again generate shock waves (or kinks) with any velocity (in the tails). They correspond to solutions on line (a).

3.2.3. Acoustic chains

This case is obtained by performing the limit $\alpha \to 0$. The phonon spectrum is acoustic. Relations similar to (27) and (28) were also obtained in [14]. In that case solutions exist below line (a) (29), which now extends down to $R = 0$ (Fig. 4) . All solutions are of multiplicity 2. In contrast to the previous case for velocities $V^2 < C$ we can generate moving breather tails only for frequencies above some threshold value. This gap value shrinks to zero and remains zero as the velocity is increased above $V^2 > C$. Also shock waves (kinks) which correspond to all points on line (a) can again be generated. Again in contrast to the previous case, these shock waves or kinks must have velocities $V^2 > C$ to be generated.

3.3. Some discussions

Let us summarize the results obtained so far. We assumed the existence of some moving localized ob-

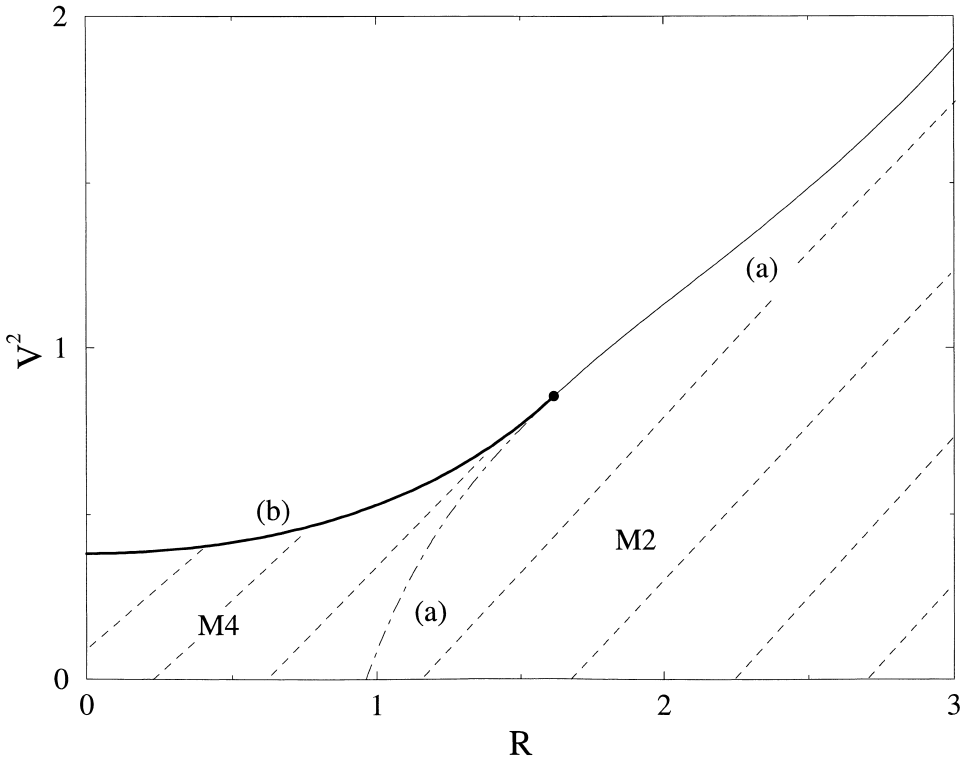

Fig. 3. Same as Fig. 1 but for a Klein–Gordon chain. The labels M2, M4 indicate the multiplicity of solutions in the given part of the parameter space. Here $C = \alpha = 1$.

ject (breather) which is parametrized in a proper way. Then we considered the equations of motion in the tails of the object, and checked under what conditions the linearized equations in the tails can be satisfied. We found, that Klein–Gordon PDEs do not allow [4] in general for moving breathers (as they do not for stationary ones), only shock waves or kinks are allowed. Discretizing these equations, we found that all restrictions are gone, and moving breather tails can be generated for any parameters. It is surprising that even the nonresonance conditions known to exist for stationary breathers do not appear here. If we consider a chain with an acoustic spectrum, then the nonresonance condition reappears in some sense, but the forbidden frequency gap shrinks to zero as the velocity is increased above the speed of sound. One of the reasons for the disappearance of the nonresonance condition in the tails of a possible moving breather is the presence of the second term on the l.h.s. of (26). This term causes a mixing of the real and imaginary parts of the eigenvalues. However, this term is also present for space-continuous systems. Since (26) also contains advanced and retarded terms, these terms cause a periodicity of the equations for the eigenvalues in the imaginary part of the eigenvalue. Thus one can find eigenvalues with infinitely large real parts at fixed velocities, in contrast to the space-continuous case.

Another result is that we can generate shock wave tails in all cases, with restrictions in some cases (acoustic chains) on the velocity.

4. Numerical methods

4.1. The general idea

Let us consider a chain with the full equations of motion [5] given by $\ddot{u}_n = -\partial H/\partial u_n$. Then the ansatz

[4] Note that "allow" implies only that the tail of the expected solution can be constructed, not that the whole solution exists.

[5] The reader should note that in this section we consider the full nonlinear equations of motion.

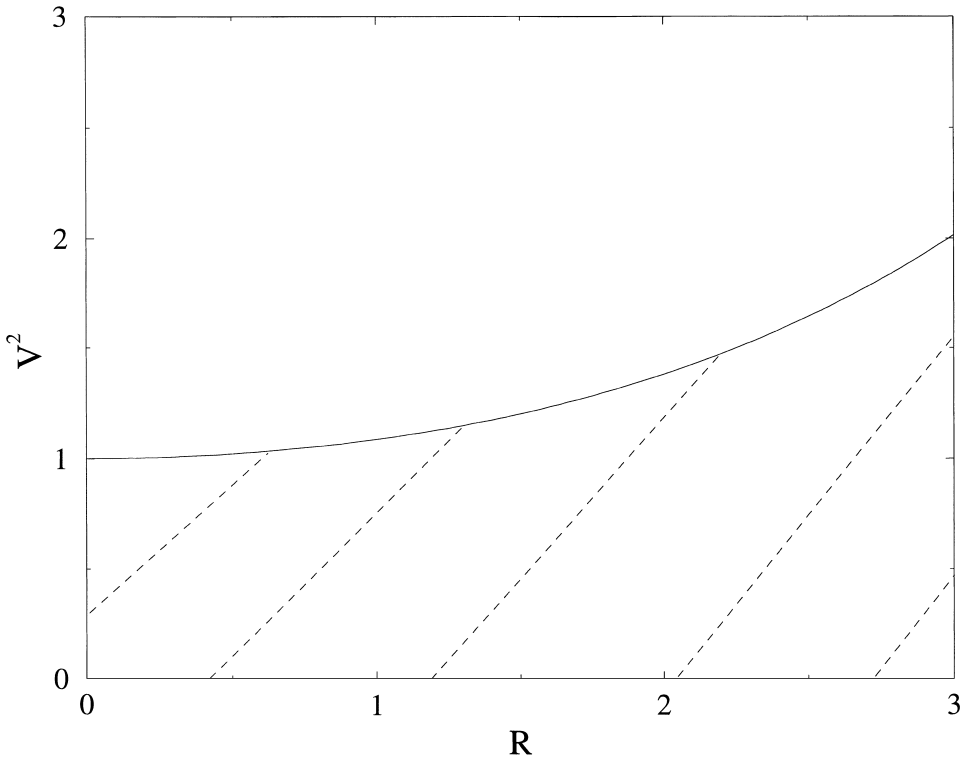

Fig. 4. Same as Fig. 1 but for FPU chains.

(25) yields equations of the type (cf. (26))

$$V^2 \frac{\mathrm{d}^2 A_k(z)}{\mathrm{d}z^2} - 2\mathrm{i}k\Omega_\mathrm{b} V \frac{\mathrm{d}A_k(z)}{\mathrm{d}z} = \Omega^2 A_k(z) + \sum_n f_n(\{A_{k'}(z)\}, \{A_{k'}(z+n)\}, \{A_{k'}(z-n)\}). \tag{31}$$

The essential feature is that these coupled differential equations contain advanced and retarded terms. These terms arise due to the interaction on the lattice. Instead of directly trying to solve these equations, we consider a lattice governed by the equations

$$V^2 \ddot{A}_{kn}(t) - 2\mathrm{i}k\Omega_\mathrm{b} V \dot{A}_{kn}(t) = \Omega^2 A_{kn}(t) + \sum_{n'} f_{n'}(\{A_{k',n}(t)\}, \{A_{k',n+n'}(t)\}, \{A_{k',n-n'}(t)\}). \tag{32}$$

Here n is again the lattice site label, and with each lattice site n we have an associated infinite set of variables $\{A_{kn}\}, k = 0, \pm 1, \pm 2, \ldots$ Eq. (32) defines a phase space flow in the phase space of all variables $A_{kn}, \dot{A}_{kn}$. In general, trajectories generated by that dynamics are not related to solutions of (31). However, all fixed points of the map $RG_{t=1}$ (G_t is the evolution operator defined by (32) and R is the translation operator that shifts all lattice indices by 1) are solutions of (31). The main reason for this is that all delay and advance intervals are integers.

Once a fixed point (solution) is found, it can be continued using generalized Newton methods or steepest descent methods.

4.2. Example: DNLS

Let us investigate numerically moving breathers in the discrete nonlinear Schrödinger model (DNLS) with $\mu = 2$ (20) which is a nonintegrable model.

The DNLS system has an integrable counterpart which is the Ablowitz–Ladik model (ALM) having the following Hamiltonian:

$$\dot{\Psi}_n = \mathrm{i}(\Psi_{n-1} + \Psi_{n+1})(1 + |\Psi_n|^2). \tag{33}$$

The ALM has moving breathers of the form:

$$\Psi_n = \frac{\sinh(\mu)}{\cosh(\mu(n - Vt))} \mathrm{e}^{-\mathrm{i}kn} \mathrm{e}^{\mathrm{i}\Omega_\mathrm{b} t}. \tag{34}$$

Here k and μ are free parameters. Note that these solutions possess exact self-dual reflectional symmetry [21].

All other parameters are expressed through k and μ:

$$\Omega_b = 2\cosh(\mu)\cos(k), \tag{35}$$

$$V = \frac{2}{\mu}\sinh(\mu)\sin(k). \tag{36}$$

Here V is the velocity of a breather.

Now we can test the relations obtained in the previous section. Indeed, using the solution (34) in the tails it can be checked that the relation between the frequency, velocity and localization length (22) and (23) is fulfilled. Note that in this case we have only one Fourier component which is a specific property of models of the nonlinear Schrödinger type which do not generate higher Fourier harmonics with respect to time if a single Fourier component is excited.

Now let us consider the following model, which allows for a continuous tuning between ALM and DNLS (this is a realization of the ideas in [20,21]):

$$\dot{\Psi}_n = \mathrm{i}[\Psi_{n-1} + \Psi_{n+1} + |\Psi_n|^2[(1-\alpha)(\Psi_{n-1} + \Psi_{n+1}) + \alpha\Psi_n]]. \tag{37}$$

Here $0 \le \alpha \le 1$.

Let us look for a solution in the form

$$\Psi_n(t) = \mathrm{e}^{\mathrm{i}\Omega_\mathrm{b} t} g_n(t). \tag{38}$$

Then we have

$$\dot{g}_n = \mathrm{i}[g_{n-1} + g_{n+1} - 2\Omega_\mathrm{b} g_n + |g_n|^2[(1-\alpha)(g_{n-1} + g_{n+1}) + \alpha g_n]]. \tag{39}$$

For a moving breather we have

$$g_n(t) = g(n - Vt). \tag{40}$$

The moving breather is a fixed point of the map RG_T, $T = 1/V$. We look for a zero minimum of the functional $F = |RG_T X - X|$, where X is a point in the phase space of (39).

We proceed in the following way. First, the parameter α is put to zero. The initial point in the phase space is chosen to be the ALM moving breather (34). Then α is incremented by a small value $\Delta\alpha$. A minimization of the functional F is performed. Again α is incremented by a small value $\Delta\alpha$, etc. The algorithm stops when α reaches 1.

We were able to generate moving DNLS breathers with the value of the minimized functional less then 10^{-6} and $\Delta\alpha = 10^{-5}$. An example is shown in Fig. 5 ($\mu = 0.5, k = 1, V = 0.0364$).

Let us mention an important point found during the numerical calculations which is the existence of a large (infinite?) number of very (infinitesimally?) close local nonzero minima of the functional F near the true fixed point. When a step $\Delta\alpha$ is made, the algorithm minimizes the functional F, and we hope that the value of the functional in the minimum will be zero. Surprisingly (and in contrast to the numerical calculations on stationary breathers), we found that the value of the functional at the minimum for a given α is not zero *no matter how small* $\Delta\alpha$ *is*.

On the other hand when $\Delta\alpha$ is decreased (thereby increasing the computation time) the minimum value of the functional at a given α tends to zero, although not being zero for any finite $\Delta\alpha$. From this we conclude that the true fixed point trajectory is surrounded by a dense set of other nonzero minima of F. The structure of the phase space near the moving breather is therefore highly nontrivial in contrast to stationary breathers where none of the above effects were found.

These findings are maybe connected to the fact that the spectrum of Floquet multipliers of a moving breather has unusual properties as compared to the Floquet spectrum of stationary breathers. Namely, the Floquet multipliers of the linearized map around a moving breather fill the unit circle densely. Especially, there exist Floquet multipliers with value $+1$, which would in general make continuation impossible for stationary breathers. The existence of these multipliers can be simply explained. Linearizing the map around a moving breather fixed point, we obtain an infinite set of eigenvalues with spatially extended eigenvectors. At large distances from the breather center

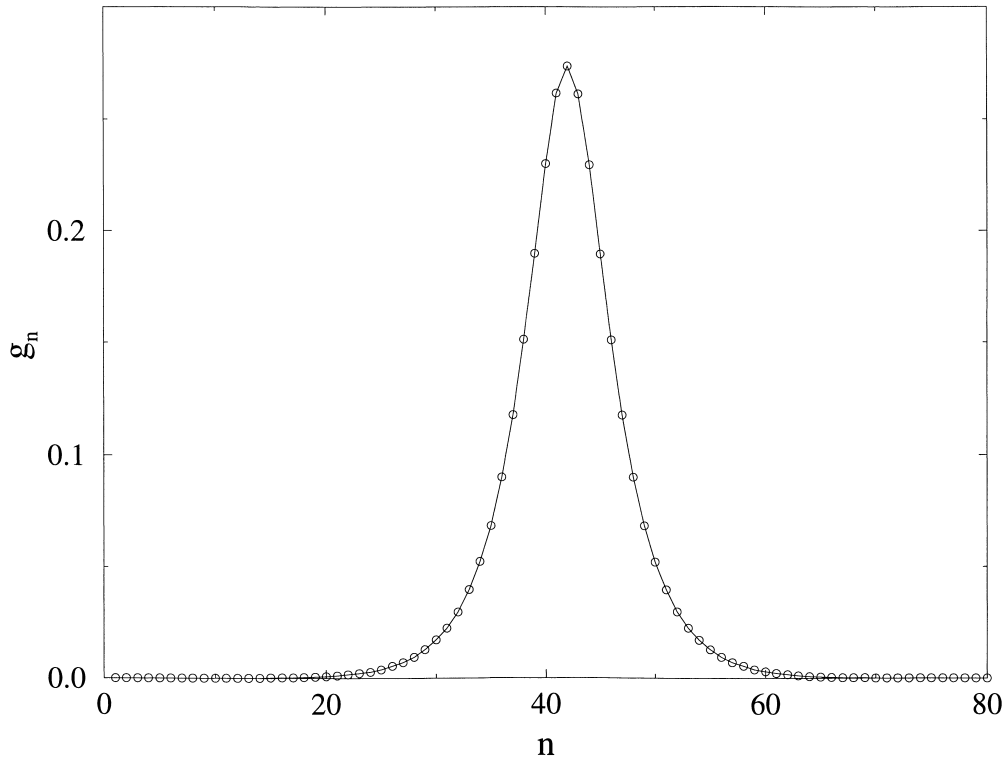

Fig. 5. A moving breather solution for DNLS (see text).

these eigenvectors will correspond to normal phonons. A phonon is given by

$$e^{i(\Omega_q t - qn)}. \tag{41}$$

It is always possible to cast it into the form

$$e^{i\Omega_b t} e^{-iq(n-Vt)} \tag{42}$$

with arbitrary numbers Ω_b, V by solving

$$\Omega_q = \Omega_b + Vq. \tag{43}$$

Indeed, we can always find q-values which will do the job. For the case of a stationary breather $V = 0$, we essentially recover the nonresonance condition, which can be fulfilled by choosing Ω_b to be outside the phonon band.

5. Conclusion

There are results reporting on a transition from moving to trapped breathers upon changing the parameters of lattice Hamiltonians [16,20]. We have not been able to trace these transitions in the tail analysis. Consequently, the origin of these transitions should be connected to the dynamics in the breather center. It is interesting to note that numerical observation of supersonic shock waves was reported for acoustic chains in [15] in accord with our results which allow the existence of the tails of such solutions.

Our main results can be summarized as follows. The nonresonance condition for the localization condition in the tails of stationary breathers is not present if one considers moving breathers. The problem of finding moving breathers can be transformed into the calculation of fixed points of a map which differs from the map of the original phase space onto itself, but is unambiguously related to that map. The numerical evaluation shows that continuation of fixed points of the map for moving breathers is not simple. We argue that this is related to the fact that the Floquet spectrum of the map for moving breathers contains a dense set surrounding $+1$ on the unit circle.

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