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Energy flow for soliton ratchets

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Abstract. – We study the mechanism of directed energy transport for soliton ratchets. The energy flow appears due to the progressive motion of a soliton (kink) which is an energy carrier. However, the energy current formed by internal system deformations (the total field momentum) is zero. We show that the energy flow is realized via an *inhomogeneous* energy exchange between the system and the external ac driving. We also discuss effects of spatial discretization and combination of ac and dc external drivings.

A transport mechanism of potential relevance in various areas of physics, chemistry and biology is based on the ratchet effect [1], *i.e.* the generation of directed currents by zero-mean external perturbations. It is based on the breaking of relevant space-time symmetries of the underlying system evolution equations [2]. A paradigmatic model corresponds to a classical particle moving in a spatially periodic potential under the influence of zero-mean fluctuations [1], where the energy current is directly connected with the mean particle momentum and the corresponding kinetic energy of the particle. For spatially extended systems, *e.g.* an annular Josephson junction, which is described by a partial differential equation (PDE), the ratchet phenomenon manifests as a unidirectional motion of a collective kink excitation (soliton) [3–8]. Here the unambiguous *ab initio* definition of a current may become a much more complicated task, because the kink excites other modes in the system during its motion, which may contribute to an energy current as well.

A well-known model in the field of soliton ratchets is the driven-damped sine-Gordon equation [9], which is also used for modelling the abovementioned annular Josephson junction [8]:

$$\varphi_{tt} - \varphi_{xx} = -\alpha\varphi_t - \sin\varphi + E(t), \tag{1}$$

where $E(t)$ is a zero-mean time-periodic driving force, $E(t+T) = E(t)$, $\int_0^T E(t)\mathrm{d}t = 0$. We impose the kink-bearing periodic boundary condition:

$$\varphi(x+L,t) = \varphi(x,t) + Q, \qquad \varphi_t(x+L,t) = \varphi_t(x,t), \tag{2}$$

where $Q = 2\pi m$ is the topological charge with integer $m = 1, 2, \ldots$, and L is the system size.

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Let us consider the easiest case $m = 1$, *i.e.* the presence of one kink in the system $Q = 2\pi$. The kink velocity V is defined, *e.g.*, as [4,5]

$$V(t) = \frac{1}{Q}\int_0^L x\varphi_{tx}\mathrm{d}x. \tag{3}$$

In the case of a soliton ratchet the mean value of $V(t)$ will be nonzero. Since the kink carries some energy, one expects a mean nonzero energy current as well. Recently, it has been proposed to observe this directed energy transport using the definition of the *internal energy current* J^I and its density j^I [5]:

$$J^I(t) = \int_0^L j^I\mathrm{d}x, \qquad j^I(x,t) = -\varphi_x\varphi_t. \tag{4}$$

J^I is also known as the *total momentum* of the system [9–11].

Choosing either eq. (3) or eq. (4), the symmetry analysis provides identical necessary conditions for the appearance of a ratchet effect [5]. If the ac driving $E(t)$ possesses a shift symmetry,

$$E(t) = -E(t + T/2), \tag{5}$$

then the combined symmetry transformation

$$x \to -x, \qquad \varphi \to -\varphi + Q, \qquad t \to t + \frac{T}{2} \tag{6}$$

leaves eq. (1) invariant and changes the sign of V and J^I. Consequently, if eq. (1) allows for only one attractor solution, both quantities will have average value zero. Violating (5) we loose the symmetry (6) and may expect nonzero values for the mean values of V and J^I. This can be done, *e.g.*, by the choice [5,6,8]

$$E(t) = E_1\cos(\omega t) + E_2\cos(2\omega t + \Theta), \qquad \omega = \frac{2\pi}{T}. \tag{7}$$

The soliton ratchet effect has been observed in terms of the mean soliton velocity V both numerically [3–5] and experimentally in an annular Josephson junction [8]. However, by differentiating eq. (4) and using eq. (1) together with the boundary condition (2) it follows that [10,12]

$$J^I_t(t) = -\alpha J^I(t) - QE(t). \tag{8}$$

Thus, for any ac driving force $E(t)$ with zero mean the time-averaged value of the total momentum, $J^I = \lim_{t\to\infty}\frac{1}{t}\int_0^t J^I(\tau)\mathrm{d}\tau$, is zero. Suppose that a kink moves with some nonzero mean velocity V. The kink carries also some nonzero energy. Hence the progressive kink motion should lead to the appearance of an energy flux in the system.

Evidently J^I does not account for this flow. We will derive the missing second energy flow channel, which appear due to the finite spatial extent of the kink and the external field and damping.

The PDE (1) corresponds to the energy density ρ,

$$\rho[\varphi(x,t)] \equiv \rho(x,t) = \frac{1}{2}(\varphi_t^2 + \varphi_x^2) + 1 - \cos(\varphi). \tag{9}$$

Using eq. (1) we obtain

$$\rho_t = -j_x^I - \alpha\varphi_t^2 + E(t)\varphi_t. \tag{10}$$

The last two terms describe energy losses through dissipation and the energy exchange between the system and external driving $E(t)$. A central result of this work is that these terms describe a new *exchange* energy current J^E. This exchange current corresponds to an additional energy transmission channel provided by a spatially *inhomogeneous* energy exchange between the system, the external ac driving and the dissipation. Thus, the complete current balance equation for the full current J reads:

$$J = J^I + J^E. \tag{11}$$

We consider a large system size, *i.e.* $L \gg L_k$, where L_k is the kink localization length. Far from the kink center the field is oscillating in time while being homogeneous in space. We also assume that the kink travels only over distances $L_p \ll L$ during one period of the ac driving. Because of the external ac force the field evolution is uniquely locked to the driver (as was the case for all previous considerations [3–6, 8], *i.e.* assuming that the system allows only for one attractor). Thus we observe a moving localized excitation (kink) which propagates on a background formed by a spatially homogenous ground state. It is convenient to separate $\varphi(x,t)$ into a localized kink part, $\varphi^k(x,t)$, where $\varphi^k(x \to 0,t) = 0$; $\varphi^k(x \to L,t) = 2\pi$, and a background (*vacuum*) part $\varphi^v(t)$ which depends only on time [10]:

$$\varphi(x,t) = \varphi^k(x,t) + \varphi^v(t). \tag{12}$$

The vacuum part alone must satisfy eq. (1). Because it is also a solution of the system in the absence of a kink when $Q = 0$, it cannot contribute to any energy transport [5].

On the attractor the dynamics of the system (1) is given by

$$\varphi^k(x,t+T) = \varphi^k(x-VT,t), \qquad \varphi^v(t+T) = \varphi^v(t), \tag{13}$$

where $T = 2\pi/\omega$ and V is the averaged kink velocity, $V = \langle V(t)\rangle_T = \frac{1}{T}\int_0^T V(t)\mathrm{d}t$. Note that all integral system characteristics, such as the total energy of the system, the kink velocity, and energy currents from eq. (11) are periodic functions of time with period T.

Let us compute the full energy current J produced by a moving kink. We use the periodicity of the total system energy $W(t) = \int_0^L \rho(x,t)\mathrm{d}x$ in time. The amount of energy carried through a point x between time t and $t+T$ is equal to

$$\Delta w(x,t) = \int_0^x [\rho(x',t) - \rho(x',t+T)]\mathrm{d}x' = -\int_0^x \mathrm{d}x' \int_t^{t+T} \rho_t(x',t')\mathrm{d}t'. \tag{14}$$

Here we assume that at $x = 0$ we have a homogeneous vacuum state during the full cycle of the ac driving, thus there is no energy current through this point. Due to the presence of ac driving the system is not time homogeneous and $\Delta w(x,t)$ in eq. (14) depends on t. With (2), (13) it follows: $\Delta w(x,t+T) = \Delta w(x-VT,t)$ and $\Delta w(x+L,t) = \Delta w(x,t)$ and

$$J = \frac{1}{T}\int_0^T \mathrm{d}t \int_0^L \mathrm{d}x j(x,t), \qquad j(x,t) = \frac{1}{T}\Delta w(x,t). \tag{15}$$

Using eqs. (2), (13) and the multiplicative integration rule [13] we obtain from (14), (15):

$$J = V\langle \int_0^L \rho[\varphi(x,t)]\mathrm{d}x - L\rho[\varphi^v(t)]\rangle_T \tag{16}$$

with $\langle \ldots \rangle = 1/T \int_0^T \ldots \mathrm{d}t$. The quantity being averaged in the r.h.s. of (16) is the difference between the energy of system with and without a kink, or simply the kink energy $W^k = W_Q(t) - W_0(t)$, so that the mean total current, eq. (16) generated by the moving kink reads

$$J = VW^k. \tag{17}$$

It follows that the total energy current has the same symmetry properties as the kink velocity (3). This result proves the initial intuitive guess that a moving kink indeed generates a nonzero energy current in the system.

Because for any ac driving $E(t)$ with zero mean the averaged internal current J^I is equal to zero, we arrive at the following energy current balance:

$$J = J^E, \qquad J^I = 0. \tag{18}$$

In order to obtain an expression for the exchange current density, we use (10) and (15) and arrive at

$$j^E(x,t) = -\frac{1}{T}\int_t^{t+T} \mathrm{d}t' \int_0^x \mathrm{d}x' \phi(x',t'), \tag{19}$$

$$\phi(x,t) = \phi[\varphi] = \alpha\varphi_t^2 - E(t)\varphi_t, \tag{20}$$

$$J^E = \frac{1}{T}\int_0^T \mathrm{d}t \int_0^L \mathrm{d}x j^E(x,t). \tag{21}$$

Evidently, the exchange current J^E has the same symmetry as V, J and J^I.

We solved eq. (1) numerically [14] in order to test the current balance (18). A crucial parameter is the mesh size h used for the spatial discretization of eq. (1). For any finite value of h the internal current J^I is nonzero, as also obtained in [5]. However it scales according to $J^I \sim h^2$ for $h \leq 0.1$ and vanishes in the continuum limit $h \to 0$ in full accord with (18). We chose $h = 0.1$ here, for which the values of J and J^E are determined with an error of less than 3%, while $J^I/J \sim 0.003$ (for $h = 0.32$ used in [5] the latter ratio increases to 0.02 for $\alpha = 0.2$ and 0.18 for $\alpha = 0.05$). The results of numerical calculations of J, eq. (16), and J^E, eq. (21), are shown in fig. 1. We obtain very good agreement with (18). Note that $L = 500$,

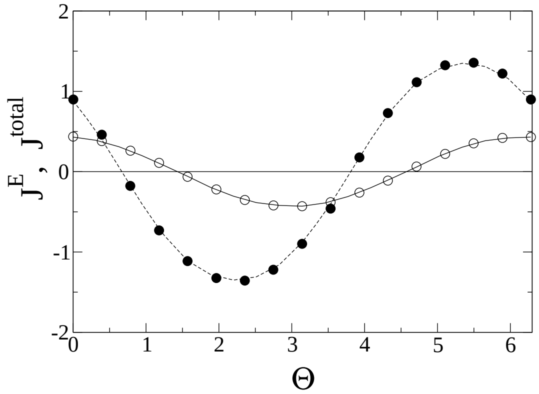

Fig. 1 – The dependence of the mean exchange current J^E on Θ for $\alpha = 0.2$ (solid line) and $\alpha = 0.05$ (dashed line). Circles correspond to the numerical results for J^{total}. Other parameters: $E_1 = E_2 = 0.2$, $\omega = 0.1$, $Q = 2\pi$, $L = 500$.

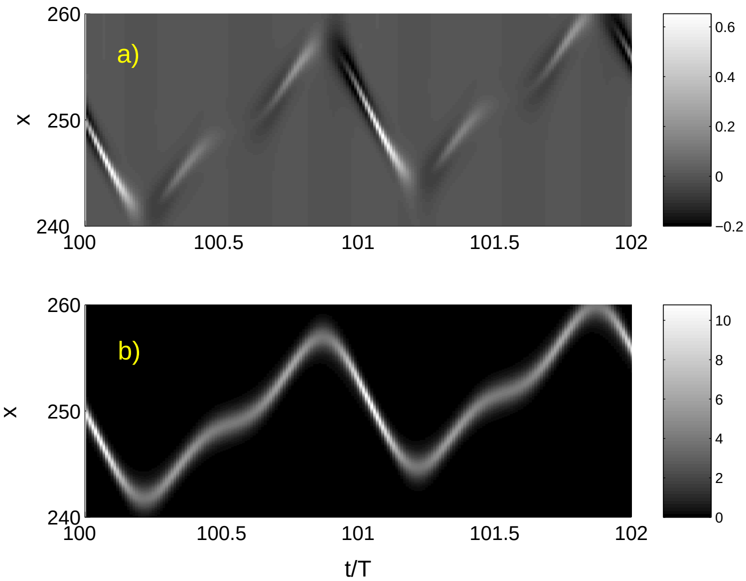

Fig. 2 – Space-time evolution of the soliton ratchet for $\alpha = 0.2$ and other parameters as in fig. 1. (a) Contour plot of the function $\phi(x,t)$, eq. (20). (b) Contour plot of the energy density (9).

$L_k \approx 10$ and $L_p \approx 30$. According to the collective coordinate theories [6,7] a nonzero average kink velocity implies the excitation of shape modes on the kink. Thus the appearance of a strong exchange current is related with the kink deformations.

In fig. 2(a) we plot the space-time evolution of the function $\phi(x,t)$ as defined in (20), which describes the energy exchange between the field φ, the external ac drive $E(t)$ and the friction term. The energy is exchanged and transported in a cyclic way: first the kink absorbs energy in its rare tail, then it releases energy in its front, then it absorbs energy in its front and finally releases energy in the rare tail. In fig. 2(b) we plot the space-time evolution of the energy density $\rho(x,t)$ (9). The excitation of internal shape modes on the kink is clearly observed —the kink is much more compressed when moving opposite to its average propagation direction as compared to the times when it moves in the same direction.

It is very instructive to apply our approach based on the generalized current balance equation (11) to the well-studied case of a constant force, $E = \text{const}$ [9]. Due to the time homogeneity of the system $\varphi(x,t) = \varphi(\xi)$, $\xi = x - Vt$. Still we deal with the motion of a spatially localized kink with a *finite extension*. The internal current J^I can be evaluated using eq. (8):

$$J^I = -\frac{EQ}{\alpha}. \tag{22}$$

The internal current density

$$j^I(\xi) = V\varphi_x^2|_{x\equiv\xi}. \tag{23}$$

Similarly the exchange current density can be obtained as

$$j^E(\xi) = \int_0^{\xi} \mathrm{d}x(\alpha V^2\varphi_x^2 + VE\varphi_x). \tag{24}$$

Due to the time homogeneity the kink energy W_k is now independent of time, and the total current can be computed using (17). We plot the current densities in fig. 3. The internal

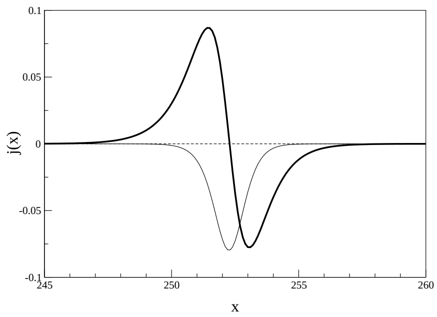

Fig. 3 – The densities of the scaled internal current, $0.02j^I(x)$ (thin line), and the exchange current, $j^E(x)$ (thick line), for the constant force case $E = 0.2$. The other parameters are $Q = 2\pi$, $L = 500$ and $\alpha = 0.2$.

current density is single peaked, with the peak position corresponding to the position of the kink center. The exchange current density $j^E \neq 0$, which may come as a surprise, since in this case the kink moves with constant shape. Yet the kink is a spatially extended object, and it is this property which leads to a nonzero exchange current density. We also computed the kink velocity, the kink energy, the total current and the total internal and exchange currents for the case $E = 0.2$ and $\alpha = 0.2$, $L = 670$ and $h = 0.045$. The kink velocity $V = -0.6201$, the kink energy $W_k = 10.0813$ and thus the total current $J = -6.251$. The internal current $J^I = -6.282$, which is off the exact result $J^I = -2\pi$ (22) by an error $\delta = 0.001$. The total exchange current is obtained as $J^E = 0.0321$. The current balance equation (11) is satisfied within the same small error δ, which is an order of magnitude smaller than the computed value for J^E [15]. The appearance of an exchange current (or its density) is simply linked to the finite spatial extent of the kink, and the corresponding spatially (and thus temporally as well) inhomogeneous energy exchange between the field φ and the external field E and the friction term, similar to the soliton ratchet.

Let us combine both cases from above, *i.e.* both a constant and an ac components of the driving E. A careful tuning of the parameters leads to an exact cancellation of both force components and $V = 0$. At that point $J = 0$ (17). At the same time, according to eq. (22), the internal current is $J^I = -QE^{stop}/\alpha$, where E^{stop} is the dc component of E. This implies an exact balance between the two currents,

$$J^{total} = 0, \qquad J^E = -J^I. \tag{25}$$

It is also possible to have a situation where the sign of the total momentum of the system J^I, eq. (4), is *opposite* to the sign of the kink velocity V, and the internal current is pumping energy against the kink motion.

We resolved the soliton ratchet energy flow for nonvanishing kink velocities and vanishing total momentum. Our results are important for the general case of spatially extended systems coupled to external driving fields or simply other degrees of freedom (see also [11] and [17]). We identified a new energy pathway which is entirely mediated by the spatial and temporal inhomogeneity of the system assisted by the external driving and the finite spatial extension of the kink. Even for the case of a constant external field, the exchange current is found to be small but nonzero. And the role of this new pathway becomes dominant in the case of the ac driving, when the exchange energy flux is strongly enhanced by the kink shape oscillations.

The relative contributions of the two currents, internal and external, might change when considering spatially discrete systems. For the ac driving case of a soliton ratchet, corrections to the internal current invalidate (8) so that J^I also contributes to the total energy flow [5] even for a zero mean drive. We note that in the nonadiabatic case, $\omega \sim 1$, the separation ansatz eq. (12) may not be valid anymore. The kink may emitt phonon waves, which would also contribute to the energy transport [18].

Finally, we mention that the driven-damped sine-Gordon system in eq. (1) is the relevant physical model of the Josephson junction ring oscillator, where the soliton ratchet effect has been experimentally realized recently [8]. An intriguing question arises about possibility of detection of the new exchange current mechanism on the base of available experimental data (current-voltage characteristics, spectra of emitted radiation, etc.).

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[14] We deal with a discrete version of the system in eq. (1), $x \to hn$, $n = 0, 1, 2, \ldots, N$, $\varphi(x,t) \to \varphi(hn,t) = \varphi_n(t)$, and $\varphi_{xx} \to h^{-2}(\varphi_{n+1} - 2\varphi_n + \varphi_n)$. For the integration of the corresponding set of coupled ordinary differential equations we used a standard fourth-order Runge-Kutta method. The deviation of any computed number from its continuum limit value was monitored as a function of h.

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