Classical and quantum radiation of perturbed discrete breathers

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We show that the linearized phase space flow around a discrete breather solution is not capable of generating persistent energy flow away from the breather even in the case of instabilities of extended states. This holds both for the classical and quantized description of the flow. The main reason for that is the parametric driving the breather provides to the flow. Corresponding scaling arguments are derived for both classical and quantum cases. Numerical simulations of the classical flow support our findings.

I. INTRODUCTION

Discrete breathers (DBs) have been intensively studied in the past decade. They are known to be generic solutions of the dynamics of nonlinear spatially discrete translationally invariant Hamiltonian systems. DBs are time-periodic and spatially localized excitations and belong to one-parameter families of solutions of the underlying equations of motion.[1]

DBs have been observed in various experimental situations ranging from Josephson junction ladders,[2] coupled nonlinear optical waveguides[3] and driven micromechanical cantilever arrays[4] to layered antiferromagnets[5] and high-T_c superconductors,[8] surface and bulk lattice vibrations of solids,[6] and Bose-Einstein condensates loaded on optical lattices.[7] DBs are also predicted to exist in the dynamics of dusty plasma crystals.[9] The characteristic spatial scales range from micrometers down to Å. Especially in Josephson junction ladders DB excitations have been studied very extensively, including their interaction with the modes of the lattice part which is not excited by the DB.[10] Thus the issue of stability of DB states, of their interaction with lattice modes, and possible mechanisms of radiation of energy by DBs due to this interaction becomes an important and timely issue.

The fact that DB solutions are generic for nonlinear Hamiltonian lattices implies that in the majority of cases the underlying Hamiltonian equations of motion are not integrable. This puts limitations and complications on the study of perturbed DB states. If the perturbation is considered to be of small amplitude, its evolution can be described using a linearization of the phase space flow around the DB. The resulting linear coupled ordinary differential equations (ODEs) with time-periodic coefficients can be studied within the framework of Floquet theory.[11]

Taking into account higher order terms in the phase space flow will ultimately lead to more complicated nonintegrable equations, which can be studied only approximatively.[12] Several issues are at stake when discussing the evolution of perturbed breathers. One can consider localized or extended perturbations on one hand. On the other hand there are differences in the way breathers react to such perturbations depending on the amplitude of the latter.

Let us first discuss the case of a linearized phase space flow around a DB. Formally the obtained Floquet equations

decouple the dynamics of the DBs (which is assumed to be given) from the evolution of the perturbation. Perturbations which grow in time will then invalidate the abovementioned linearization. Perturbations which decay in time do not contradict the linearization, but in fact the energy stored in the initial perturbation cannot simply disappear if we consider Hamiltonian dynamics. Consequently there is a subtle way this energy will have to be transferred to the DBs, which again is beyond the linearization frame. Marginally stable perturbations, which neither grow nor decay seemingly do not violate the assumed linearization. Nevertheless it has been shown that for extended perturbations such a case may be accompanied by a nonzero energy flux emitted out of or into the breather core.[13] Again it would violate the linearization frame. Even though it may do so, such predicted radiation scenaria are confirmed in numerical simulations, underpinning the use of the linearization picture.

Additional sources of radiation can appear when taking into account nonlinear corrections to the phase space flow of a DB perturbation, even if the linearized case did not provide with such sources. For instance a marginally stable localized linearized perturbation will yield an energy radiation due to the appearance of new frequency combinations and resonances with the spectrum of small amplitude plane waves.[14]

Here we will be concerned with a particular case within the linearized phase space flow frame, which corresponds to the abovementioned extended perturbations yielding a nonzero energy flux out of the breather core. The question we want to pose is whether this energy flux can be sustained if the perturbation we choose is local in space. The perturbation will have some overlap with the extended ones. So there will be some radiation, but at the same time the initial localized perturbation will simply disperse away from the breather lowering its amplitude. The question then is whether these two counteracting processes balance each other or not. The question is of relevance also in connection with recently discussed radiation mechanisms of strongly excited quantum breathers.[15] We will provide with answers for both cases.

II. THE CASE OF CLASSICAL BREATHERS

For the sake of simplicity we consider first a one-dimensional lattice with the equations of motion

$$\ddot{x}_l + V'(x_l) + W'(x_l - x_{l-1}) - W'(x_{l+1} - x_l) = 0, \qquad (1)$$

which corresponds to the Hamiltonian

$$H = \sum_l \left[\frac{1}{2}\dot{x}_l^2 + V(X_l) + W(x_l - x_{l-1}) \right]. \qquad (2)$$

The index l denotes the lattice site, and can run over a finite or infinite lattice. Extensions to higher lattice dimensions are straightforward. A local minimum energy state $x_l = \dot{x}_l = 0$ is provided by $V(0) = W(0) = V'(0) = W'(0) = 0$ and $V''(0), W''(0) > 0$. Small amplitude excitations can be obtained by linearizing Eq. (1) with the ansatz

$$x_l(t) \sim e^{i(\omega_q t - ql)} \qquad (3)$$

which results in the dispersion relation for plane waves

$$\omega_q^2 = V''(0) + 4W''(0)\sin^2\frac{q}{2}. \qquad (4)$$

For varieties of anharmonic potentials V, W it is well known[1] that the Eq. (1) allow for families of discrete breather solutions of the type

$$\bar{x}_l(t) = \bar{x}_l(t + T_b), \quad x_{|l| \to \infty} \to 0. \qquad (5)$$

Here the breather frequency $\Omega_b = 2\pi/T_b$ is a tunable parameter which satisfies the nonresonance condition

$$m\Omega_b \neq \omega_q. \qquad (6)$$

In addition given a breather family we can generate new families by discrete translations $l \to l + l_0$.

In the next step we consider small perturbations ϵ_l around a given breather solution for a given value of its frequency ω_b. We insert the ansatz $x_l(t) = \bar{x}_l(t) + \epsilon_l(t)$ into the equations of motion (1) and linearize it with respect to the perturbation

$$\ddot{\epsilon}_l + V''[\bar{x}_l(t)]\epsilon_l + W''[\bar{x}_l(t) - \bar{x}_{l-1}(t)](\epsilon_l - \epsilon_{l-1})$$
$$- W''[\bar{x}_{l+1}(t) - \bar{x}_l(t)](\epsilon_{l+1} - \epsilon_l) = 0. \qquad (7)$$

Integration of these Floquet equations over one breather period T_b maps the phase space $(\epsilon_l, \dot{\epsilon}_l)$ onto itself and is equivalently described by a Floquet matrix F. This matrix is symplectic and can be obtained, e.g., numerically. Its eigenvalues λ and eigenvectors $\vec{y}_\lambda$ describe the linear stability properties of the DB and the scattering of plane waves by the DB as well. If $|\lambda| = 1$ the corresponding eigenstate y_λ is marginally stable—it neither grows nor decays in time. Eigenvectors may be localized or extended. Since the breather is localized, the eigenvalue spectrum of the localized eigenstates is discrete, while the eigenvalue spectrum of delocalized eigenstates is continuous for an infinite lattice. Indeed, in that case delocalized eigenvectors far from the breather asymptotically take the form (3). Consequently their eigenvalues are given by

$$\lambda_q = e^{i\omega_q T_b}. \qquad (8)$$

Note that this is true only for an infinite lattice. In the following we will remind the reader about finite size corrections to this picture.

Let us now consider a lattice with a particular breather solution such that

$$\omega_q \pm \omega_{q'} = m\Omega_b, \quad q \neq q', \qquad (9)$$

so that $\lambda_q = \lambda_{q'}$. This twofold degeneracy is the origin of inelastic multichannel scattering[13] performed at these wave-numbers q and q'. Note, that such a situation corresponds to the parametric resonance in the system (7), where the breather acts as a parametric driver (exponentially localized in space). Thus an important question arises: can one pump energy into the system in the regime of a parametric resonance, provided that parametric driving is local in space? The answer is no, as long as we deal with an infinite system size. Here, however, we note that for a finite lattice the outcome will be close to the above statements, but not identical. Indeed, as shown by Marín and Aubry,[16] the degeneracy of the corresponding Floquet eigenvalues is lifted in a finite lattice, leading to their departure from the unit circle, so that $|\lambda| \neq 1$. However, these departures are the smaller the larger the lattice size is. Marín and Aubry have shown using a band analysis that $||\lambda| - 1| \sim 1/N$, where N is the total number of lattices sites. This result is obtained first by noting that the twofold degeneracy is lifted due to the time-periodic localized DB perturbation and can be accounted for by estimating the corresponding matrix element. Because both initially degenerate eigenstates are delocalized, whereas the DB is localized, the matrix element will be of the order of $1/N$. Perturbation theory of two degenerate states tells immediately that the degeneracy will be lifted to an amount of $1/N$ as well.

It is instructive to revisit here the consequencies. Assume we have a large but finite lattice and such an extended slightly unstable eigenstate. Taking the perturbation along this eigenstate, and integrating over one period of the breather T_b, this perturbation will grow everywhere in the system, though not very strongly. How can that happen, if the breather itself is localized, say exponentially? The influx of energy is provided by the breather, and is confined to a finite part of the lattice. The only possibility is that there is a nonzero energy flux in the outer regions of the lattice due to some slightly inhomogeneous profile of the Floquet eigenstate. Then the breather simply feeds energy in (or out) of a confined small part of the lattice, and this energy then travels along the rest of the lattice. If so, the legitimate question arises whether this can be used as a possible source of radiation of waves by a breather, if a corresponding generic localized perturbation of the breather is excited.

To answer this question let us assume that all the elements of a given (extended) Floquet eigenvector are of the same order of magnitude $\vec{y}_\lambda \sim A$. Then taking an initial perturbation being equal to such an eigenvector the stored energy is given by $E_\lambda \sim NA^2$. If we choose a growing unstable eigenvector, after one period T_b the amplitude of the perturbation will become $(1 + 1/N)A$. Consequently the energy after that time increases to $N(1 + 1/N)^2 A^2$. Thus the energy grows during that time by an amount of $\Delta E \sim 2A^2$ for large N. Assume now that we make a local perturbation of the breather given by some vector $\vec{B}$ with its elements again being at most of

order A. Then it can be represented as a superposition of the Floquet eigenvectors:

$$\vec{B} \sim \frac{1}{N}\sum_\lambda \vec{y}_\lambda. \tag{10}$$

Consequently the growth of energy in each unstable eigenstate during one period T_b of the breather amounts to $\Delta E_\lambda \sim A^2/N^2$. The total number of unstable eigenstates will be always less than N, so that the total growth of energy in the direction of all unstable eigenstates is limited by A^2/N. Recalling that A is fixed, this growth rate will scale to zero for an infinite lattice. Consequently we predict that the above discussed finite lattice instabilities do not result in an energy radiation of the breather when perturbed locally. In other words, a local perturbation will start to disperse away from the breather, lowering its amplitude in the breather core. At the same time it has some overlap with unstable eigenstates which would result in a growth of the same amplitude. However the dispersion acts more efficiently, and for large times the amplitude of the perturbation in the breather core will ultimately decay down to zero.

The above argumentation was provided for a one-dimensional system. It is straightforward to generalize it to higher lattice dimensions d as well. Assuming now that N represents the linear size of the system, the stored energy in an unstable eigenvector becomes $E_\lambda \sim N^d A^2$. After one period T_b the energy growth will be given by $\Delta E \sim 2N^{d-1}A^2$. For a local perturbation the growth of energy in each unstable eigenstate per T_b is then given by $\Delta E_\lambda \sim 2N^{-d-1}A^2$. Since there are at most N^d unstable eigenvectors, we obtain again an upper limit for the total energy growth A^2/N, independent of the dimension of the lattice.

In order to verify this prediction, we performed high precision numerical simulations. We consider a system with $V(x)=x^2/2-x^3/3$ and $W(x)=0.5Cx^2$. First we consider a lattice of size $N=40$. We identify a breather with frequency $\Omega_b=0.75$ for which the abovementioned finite lattice instabilities take place in the coupling constant region $C\gtrsim 0.093$. We compute the eigenvalues and eigenvectors of the corresponding Floquet matrix for two different values of the coupling constant: $C=0.092$ (breather is linearly stable) and $C=0.093$ (breather has finite size instabilities). We then perturb the breather in the direction of a stable (at $C=0.092$) and a slightly unstable (at $C=0.093$) extended eigenstates and monitor the energy flux at some distance from the breather

$$j_l(t) = \dot{\epsilon}_l(\epsilon_{l-1} - \epsilon_{l+1}) \tag{11}$$

as well as its integral over the observed time of simulation τ

$$J_l(\tau) = \frac{1}{\tau}\int_0^\tau j_l(t)dt. \tag{12}$$

We nicely reproduce the expected two different scenarios: oscillations of energy flux with zero average of its integral value in the case of a stable breather; and slow but still exponential growth of the energy flux out of the breather region in the case of an unstable breather, see Fig. 1.

Next we take a large lattice with $N=4000$ and identify breather states with $\Omega_b=0.75$ for two different values of the coupling constant $C=0.02$ [the breather is linearly stable, see Fig. 2(a)] and $C=0.1$ [there exists a considerable number of unstable extended eigenstates, see Fig. 2(b)]. We excite a local perturbation on top of the breather core in a random way on 10 adjacent sites with amplitude being of order one at each site. We then monitor the time dependence of the energy flux averaged over 20 lattice sites $j_{sum}(t)=\sum_n^{n+20}j_l(t)$ outside the breather core. In both cases we observe a decay of the measured radiation with growing time, see Fig. 3. For large times it follows the power law $j\sim t^{-4}$ [see insets in Fig. 3, to smoothen the oscillations and make the picture more clear in insets we plot the flux value averaged over four breather periods: $\hat{j}_{sum}(t)=1/(4T_b)\int_{t-4T_b}^t j_{sum}(\tau)d\tau$], which is explained in the Appendix. This clearly demonstrates that the finite lattice instabilities are not capable of sustaining a nonzero energy radiation of the breather induced by a local perturbation.

III. THE QUANTUM CASE

The general problem of quantizing Eq. (2) and understanding the correspondence between quantum eigenstates and classical discrete breathers is an issue of current research. Since the underlying models are nonintegrable in general, no analytic solutions can be obtained. Approximations have to be complemented by numerical studies. However, this is, in general, very hard to achieve, since one has to deal with the diagonalization of huge matrices. Indeed, even when studying one single oscillator, strictly speaking one has to consider an infinite dimensional Hilbert space. So already at the level of a single oscillator justified cutoffs in its Hilbert space have to be introduced. Taking into account many coupled oscillators leads to the necessity to reduce the number of states per oscillator. Reliable results in the high-energy domain of the quantum problem have been obtained so far only for small systems of two or three coupled oscillators.[17–20] These studies together with computations of larger systems at lower energies have confirmed that quantum breather states are many phonon bound states.[21] They correspond to some extent to classical breather excitations being able to tunnel along the lattice.[22] The tunneling rate is expected (and confirmed for small systems) to be exponentially small for large numbers of participating phonons. Then it is legitimate to assume that a localized excitation of the lattice with an energy corresponding to a large number of phonons will evolve for exponentially long times according to its classical DB analog. During that time, however, the evolution of small amplitude perturbations around the breather can be considered in its full quantum version. In order to proceed we observe that Eq. (7) corresponds to a time-dependent Hamiltonian

$$H_\epsilon = \sum_l \left[\frac{1}{2}\pi_l^2 + \frac{1}{2}V''[\bar{x}_l(t)]\epsilon_l^2 \right.$$
$$\left. + \frac{1}{2}W''[\bar{x}_l(t) - \bar{x}_{l-1}(t)](\epsilon_l - \epsilon_{l-1})^2 \right]. \tag{13}$$

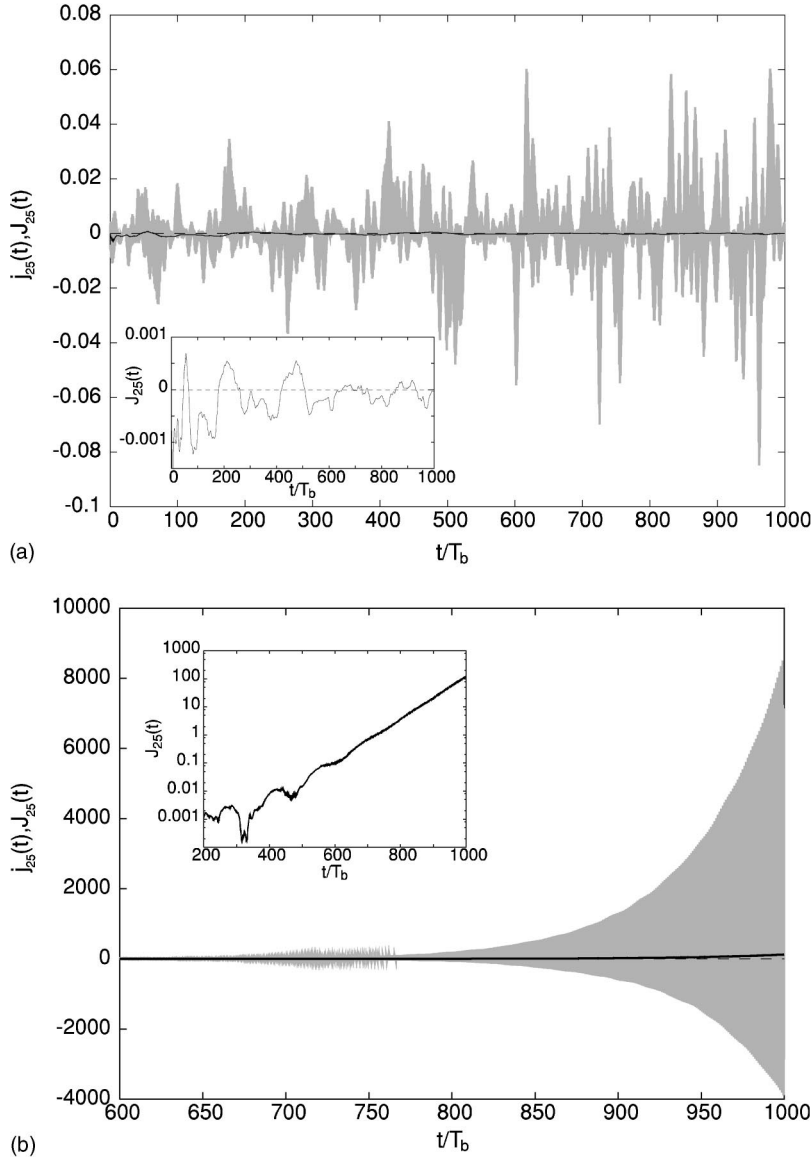Here π_l is the canonically conjugated momentum to ϵ_l. Now

FIG. 1. Energy flux $j_l(t)$ (solid gray lines) and integral energy flux $J_l(t)$ (solid black lines, see also insets) in breather dynamics with small perturbation for the cases (a) $C=0.092$, perturbation along stable eigenstate, (b) $C=0.093$, perturbation along unstable eigenstate. The system size is $N=40$ (breather is centered at 20th site), energy flux is monitored at the site $l=25$.

we may consider the corresponding quantum Hamiltonian operator

$$\hat{H}_\epsilon = \sum_l \left[\frac{1}{2}\hat{\pi}_l^2 + \frac{1}{2}V''[\bar{x}_l(t)]\hat{\epsilon}_l^2 \right.$$
$$\left. + \frac{1}{2}W''[\bar{x}_l(t) - \bar{x}_{l-1}(t)](\hat{\epsilon}_l - \hat{\epsilon}_{l-1})^2 \right]. \quad (14)$$

The operators satisfy the standard commutation relation $[\hat{\epsilon}_l, \hat{\pi}_m] = i\delta_{l,m}$.

The fact that the Hamiltonian (14) is a quadratic form of operators $\hat{\epsilon}_l$ and $\hat{\pi}_l$ is crucial: we may appeal to the Ehrenfest theorem[23] and conclude that the dynamics of the quantum system (14) is in full agreement with the dynamics of the corresponding classical system (13) and therefore no additional quantum effects which may lead to breather radiation should appear. To show this we switch to time-dependent Heisenberg operators $\hat{\epsilon}_l^H(t), \hat{\pi}_l^H(t)$ using standard relations

$$\hat{A}^H(t) = (\tilde{T}e^{i\int_0^t d\tau \hat{H}_\epsilon})\hat{A}(Te^{-i\int_0^t d\tau \hat{H}_\epsilon}), \quad (15)$$

where $\hat{A}^H(t)$ is the time-dependent Heisenberg operator corresponding to a time-independent operator $\hat{A}$, T and $\tilde{T}$ are time ordering and antitime ordering operators, respectively. The equation of motion for a Heisenberg operator $\hat{A}^H(t)$ reads

$$-i\frac{\partial \hat{A}^H(t)}{\partial t} = [\hat{H}_\epsilon(\hat{\pi}, \hat{\epsilon}, t), \hat{A}^H(t)]. \quad (16)$$

After substitution of the Hamiltonian (14) into Eq. (16) it follows that operators $\hat{\epsilon}_l^H(t)$ satisfy the discussed above classical equations (7), in which coordinates ϵ_l are substituted by the corresponding operators $\hat{\epsilon}_l^H$. Since these equations are linear we may average them with the time-independent wave function ψ_0, corresponding to the initial state of the system, and get an equation for the expectation value of the coordinate operator
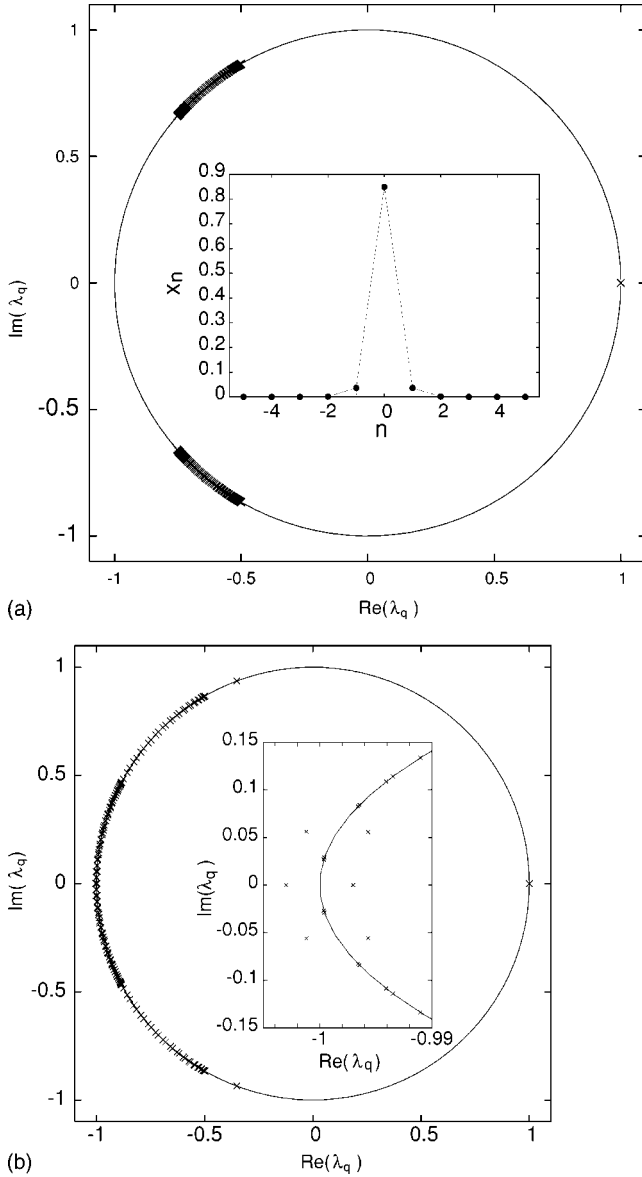
(a)

(b)

FIG. 2. Eigenvalues λ_q of the Floquet matrix F for the cases (a) $C=0.02, \Omega_b=0.75$, inset shows the breather profile (central part); (b) $C=0.1, \Omega_b=0.75$, inset zooms the region of instabilities. The system size in both cases is 80 sites.

$$\widetilde{\varepsilon}_l(t) = \langle \psi_0 | \hat{\varepsilon}_l^H(t) | \psi_0 \rangle \qquad (17)$$

which is identical to Eq. (7) with expectation value $\widetilde{\varepsilon}$ standing instead of the classical variable ϵ. Therefore all the conclusions made in the previous chapter as for time evolution of the classical coordinate ϵ_l hold for the expectation value $\widetilde{\varepsilon}_l$ of the quantum operator $\hat{\varepsilon}_l$ as well.

Alternatively, the question of possible radiation of phonons in the quantum case may be studied by considering a quantum Floquet problem. Let us split the Hamiltonian (14) into a time-averaged and an ac part

$$\hat{H}(t) = \hat{H}_{dc} + \hat{H}_{ac}(t). \qquad (18)$$

The ac part $\hat{H}_{ac}$ is local in space because of the locality of the discrete breather which is the origin of the ac drive. We can
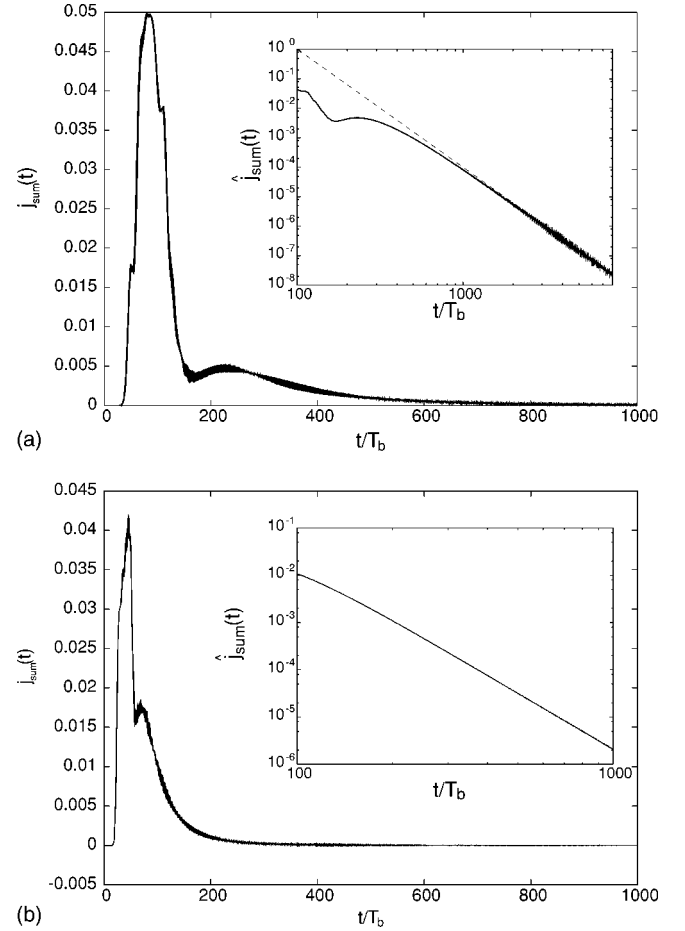

FIG. 3. Energy flux $j_{sum}(t)$ for a local initial perturbation for the cases (a) $C=0.02$; (b) $C=0.1$. The system size is $N=4000$ (breather is centered at 2000th site), energy flux is monitored at sites $l \in [2020, 2040]$, perturbation is made on top of the 10 adjacent sites in the breather core in a random way. Insets show the averaged over four breather periods integral energy flux $\hat{j}_{sum}(t)$ (see the main body text for details). Dashed line in inset in (a) shows the asymptote $\hat{j}_{as}(t) = 10^8/(t/T_b)^4$.

consider the full orthonormal basis of eigenfunctions ϕ_ν of the dc part

$$\hat{H}_{dc}\phi_\nu = \epsilon_\nu \phi_\nu \qquad (19)$$

and expand the full wave function Ψ which satisfies the time-dependent Schrödinger equation

$$i\dot{\Psi} = \hat{H}(t)\Psi \qquad (20)$$

in that basis:

$$\Psi(t) = \sum_\nu C_\nu(t)\phi_\nu. \qquad (21)$$

This will lead to a set of coupled first order differential equations for the coefficients C_ν:

$$i\dot{C}_\nu = \epsilon_\nu C_\nu + \sum_\mu h_{\nu\mu}(t)C_\mu, \qquad (22)$$

where the matrix elements

$$h_{\nu\mu}(t)\langle\phi_\nu\hat{H}_{\mathrm{ac}}(t)\phi_\mu\rangle \tag{23}$$

have been introduced and $\langle\cdots\rangle$ denotes the scalar product in the space of ϕ. We also note that because $\hat{H}$ is Hermitian (in fact real symmetric)

$$\frac{d}{dt}|\Psi(t)|^2 = \frac{d}{dt}\sum_\nu|C_\nu(t)|^2 = 0. \tag{24}$$

In order to answer the question of radiation, let us start with noticing that Eq. (22) constitute a Floquet problem similar to the classical case. The difference is that the rank of the Floquet matrix is formally speaking infinite even for a finite lattice, since the Hilbert space dimension of a single site oscillator is infinite. The extended states in Eq. (19) can be characterized by the amount of excited one-phonon energies and classified accordingly as many-phonon excitations with a given number of phonons participating. This constitutes the main difference to the classical Floquet problem (7), where only pairs of one-phonon excitations appear (because in the classical case we compute frequencies instead of energies, and time reversal symmetry provides with two possible signs of the phonon frequency). For the quantum case we have an infinite number of one-, two-, three-phonon excitations, etc. To estimate the magnitude of the departure of quantum Floquet eigenvalues from the unit circle, we need again, as in the classical case, to first estimate the matrix elements. Since the DB solution has a main frequency contribution and higher harmonics with amplitudes exponentially decaying with increasing order, the main contribution will originate from the main frequency component of the DB, which couples the space of many-phonon states locally, e.g., the ground state with the two-phonon states, the two-phonon states with the four-phonon states, the four-phonon states with the six-phonon states, etc. In other words, we have to consider an infinite matrix with degenerate diagonal elements and nearest-neighbor interaction elements of the order of $1/N$ in analogy to the classical case. The eigenvalue spectrum of such a matrix will spread around the value of the diagonal elements to the same order $1/N$. Thus we conclude that the quantum Floquet eigenvalues will depart from the unit circle not farther than $1/N$ exactly as in the classical case.

What remains then is to repeat the final argument applied in the classical case. This can be done in full analogy to the previous chapter, noting that the observable of the flux $j_l(t)$ will be defined through the corresponding operator $\hat{j}$ and the product $j(t)=\langle\Psi|\hat{j}|\Psi\rangle$. This product will be a quadratic form of the time-dependent coefficients $C_\nu(t)$, which completes the above analogy. The conclusion is thus that despite the fact that the quantum Floquet problem involves an infinite number of bands, and despite the fact that the norm is conserved in the quantum case, a local perturbation around the breather will not lead to a persistent radiation of phonons. The argument that radiation must take place because the ground state of the unperturbed system (without DB) is not anymore the ground state of the system with a DB must then be misleading. In fact the computation of the exact quantum Floquet eigenvalue problem will show that there is always a locally deformed ground state (as well as the excited states). That deformation is clearly not the cause of radiation. The only possible cause—an instability of extended Floquet states—has been excluded by the above reasoning.

IV. CONCLUSIONS

In this work we excluded a particular mechanism of radiation of perturbed breathers driven via weak finite size instabilities of extended states. While the statement is rigorous when treating the whole system classically, we arrive at a similar result also when treating the fluctuations quantum mechanically, leaving the breather solution to be a given classical one. One way to obtain nonzero radiation is to include higher order terms of the perturbation ϵ which together with possible localized Floquet eigenvectors of the linearized phase space flow will provide with a constant radiation rate of the breather into the plane wave continuum, both for a classical as well as a quantum treatment of the fluctuations. Another path is to quantize the breather itself in the quantum case. Then we can expect breather tunneling along the lattice, which will provide with some diffusion of the full breather energy out of the originally excited lattice part. Concluding we may say that discrete breathers are surprisingly robust objects. They can radiate energy into the continuum of a large lattice only via higher orders of perturbations around them.

APPENDIX

An estimation of a wave packet dynamics, resulted from a local perturbation on a lattice, can be made essentially in a similar way as it was done for continuous systems.[24] Let us start with an instructive case of a lattice without any breather. We excite a local initial perturbation, say, $\epsilon_l(0)=\delta_{l,0}$. Its representation in the reciprocal lattice space with wave number q is given by a q-independent constant $\epsilon_q=$const. Consequently the evolution of the perturbation after some time t will be given by

$$\epsilon_l(t) \sim \int_0^\pi \epsilon_q e^{i(ql-\omega_q t)}dq \tag{A1}$$

with ω_q given by the plane waves dispersion relation (4). Rewriting it as

$$\int_0^\pi e^{i(ql-\omega_q t)}dq = \int_0^\pi e^{iF(q)t}dq \tag{A2}$$

with $F(q)=q(l/t)-\omega_q$ we can estimate the integral by noting that for large values of t only q values contribute for which $\partial F/\partial q=l/t-v_q$ is small. Here v_q is the group velocity at wave number q. For large t only waves with small group velocities contribute, i.e., with wave numbers q close to the edges of the first Brilluen zone $q=\hat{q}=0,\pi$. It is straightforward to show that all extremum points $q=q^*$ of the function $F(q)$ give contribution to $\epsilon_l(t)$ of the same order in small parameter $1/t$ and do not cancel each other, hence it is enough to consider only a single extremum point. Expanding

F around its extremum $q=q^*$, the integral can be estimated to be

$$\int_0^\pi e^{iF(q)t}dq \sim e^{iF(q^*)t} \int e^{i0.5F''(q^*)(q-q^*)^2t}dq$$

$$\sim \frac{1}{\sqrt{t}}e^{i[\xi(l^2/t)+\hat{q}l-\omega_{q=q^*}t]}, \qquad (A3)$$

where $\xi = (dv_q/dq)^{-1}|_{q=\hat{q}}$. Using the definition (11) we then obtain

$$\dot{\epsilon}_l(t) \sim t^{-0.5}, \qquad (A4)$$

$$\epsilon_{l+1}(t) - \epsilon_{l-1}(t) \sim t^{-0.5}\sin(\hat{q}+2\xi l/t) \sim t^{-1.5}, \qquad (A5)$$

$$j_l(t) \sim t^{-2}. \qquad (A6)$$

We tested this prediction numerically and found complete agreement.

The observed $1/t^4$ dependence for a perturbation on top of the breather can be now explained by noting that the breather represents a local violation of the translational invariance. In such a case the abovementioned reciprocal lattice representation ϵ_q of a local perturbation becomes q dependent because plane waves are not the true eigenstates of the system anymore. It is easy to show that already for a single site time-averaged breather contribution the amplitude of the extended eigenstates at the breather site is proportional to $\sin q$. This implies that $\epsilon_q \sim \sin q$. Following then again the above reasoning, we obtain

$$\epsilon_l(t) \sim \sin(q^*)e^{iF(q^*)t} \int e^{i0.5F''(q^*)(q-q^*)^2t}dq$$

$$\sim t^{-1.5}e^{i[\xi(l^2/t)+\hat{q}l-\omega_{q=q^*}t]}, \qquad (A7)$$

$$j_l(t) \sim t^{-4} \qquad (A8)$$

confirming the numerical results in Fig. 3.

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