

## Anderson Localization or Nonlinear Waves: A Matter of Probability

M. V. Ivanchenko,<sup>1,2</sup> T. V. Laptyeva,<sup>2</sup> and S. Flach<sup>2</sup>

<sup>1</sup>*Theory of Oscillations Department, University of Nizhniy Novgorod, Nizhniy Novgorod, Russia*

<sup>2</sup>*Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Strasse 38, 01187 Dresden, Germany*

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In linear disordered systems Anderson localization makes any wave packet stay localized for all times. Its fate in nonlinear disordered systems (localization versus propagation) is under intense theoretical debate and experimental study. We resolve this dispute showing that, unlike in the common hypotheses, the answer is probabilistic rather than exclusive. At any small but finite nonlinearity (energy) value there is a finite probability for Anderson localization to break up and propagating nonlinear waves to take over. It increases with nonlinearity (energy) and reaches unity at a certain threshold, determined by the initial wave packet size. Moreover, the spreading probability stays finite also in the limit of infinite packet size at fixed total energy. These results generalize to higher dimensions as well.

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Anderson localization (AL) is a fundamental physical wave phenomenon with exponential localization of eigenstates in linear wave equations due to disorder [1,2]. It was originally used to explain metal-insulator transitions [3], and has been recently related to experimentally observed localization of weak intensity light [4] and noninteracting matter waves [5] on disordered lattice systems. Because of this, the fate of AL in the presence of many-body interactions or corresponding nonlinearity (a problem treated in nonlinear science previously [6]) has been brought to the frontier of theoretical and experimental physics [7,8].

The pioneering experiments demonstrate the principal possibility to study the interaction (nonlinearity) induced delocalization, also challenging the severe length and time scale limitations which restrict a more subtle quantitative analysis, e.g., the wave packet subdiffusion exponents [4,9]. Theoretical studies therefore remain central here. Two faces of Anderson localization are usually probed: the absence of the wave packet spreading [10–19] (the focus of this Letter) and vanishing conductivity [20,21]. Wave packets, for not too weak nonlinearities, have been found to spread subdiffusively, disobeying AL at least over many time scales [10–12], and remarkable progress in studying the regimes and characteristics of wave packet spreading has been achieved [15–19].

Still, the original questions are open. It remains debated whether the observed spreading will continue infinitely or slow down and even stop to restore AL once the wave packet densities become substantially small, as conjectured in Refs. [13,14,22]. It is unknown whether there exists a lower bound on the nonlinearity strength, beyond which wave packets do obey AL, and therefore do not spread at all; or are some time scales diverging in this limit? Unavoidable finite size, time, energy, and precision limitations will make the most advanced numerical results not entirely convincing. To achieve a breakthrough, a novel approach is needed.

The probabilistic description of AL in terms of the measure of localized regular trajectories in phase space (periodic orbits and tori) appears to be a promising solution for nonlinear disordered systems. Indeed, the mere assumption that a wave packet is launched on a chaotic trajectory leads to the conclusion that chaos remains forever: Arnol'd conjecture, unproved but widely accepted, states the uniqueness of the chaotic region in phase space [23]. If initial conditions belong to a chaotic trajectory, then it will be unbounded in phase space, and, characterized by mixing, by visiting all parts of the chaotic domain [24]. This corresponds to unlimited spreading of a wave packet. The spreading is absent only if the initial conditions belong to a periodic orbit or torus, and this regular trajectory is bounded in the phase space.

The progress has been quite limited up to now due to the difficulty of the task. Persistence of tori has been proved for a special class of infinite systems [25] and for finite tori dimensionality [26] only. The nonzero probability of existence of infinite-dimensional tori in the generic case (for small enough energies) has been conjectured by empirical and numerical arguments in Ref. [22]. Quantitative estimates for AL probabilities are lacking; qualitative conclusions advocate survival of localization.

In this Letter we show the probabilistic nature of AL in nonlinear disordered systems, disproving the common rival conjectures of exclusive localization or propagation. At any small but finite nonlinearity (energy) value there is a finite probability for AL to break up and for nonlinear waves to propagate. It increases with nonlinearity (energy) and reaches unity at a certain threshold, determined by the initial wave packet size. Moreover, the spreading probability stays finite also in the limit of infinite packet size at fixed total energy. This finite value is between zero and one for quartic anharmonicity (as realized for two-body interactions and in optical Kerr media), strictly one for the

lower degree of anharmonicity, and has zero as the lower bound for the higher.

We first address the paradigmatic Fröhlich-Spencer-Wayne (FSW) type  $d$ -dimensional classical lattice [25], for which the linear eigenmodes of a disordered system are compact single-site excitations with random frequencies, and where infinite-dimensional Kolmogorov-Arnold-Moser tori [25] exist. This model can be considered as the strong disorder limit of generic classical Klein-Gordon (KG) and semiclassical discrete nonlinear Schrödinger arrays, extremely successful in reproducing and explaining experimental results [4,9], where the strong disorder regime is commonly realized. The FSW Hamiltonian reads

$$\mathcal{H} = \sum_n \left[ \frac{p_n^2}{2} + \frac{\varepsilon_n x_n^2}{2} + \sum_{m \in D(n)} \frac{(x_m - x_n)^\gamma}{2\gamma} \right], \quad (1)$$

where  $x_n$  is the displacement of the  $\mathbf{n} = (n_1, \dots, n_d)$ th particle from its original position,  $p_n$  its momentum,  $D(\mathbf{n})$  is the set of its nearest neighbors, and the random uncorrelated  $\varepsilon_n \in [1/2, 3/2]$  are uniformly distributed. Unless explicitly specified, a chain  $d = 1$  with quartic anharmonicity  $\gamma = 4$  is considered. Without the loss of generality we assume  $\varepsilon_0 = 1$ .

As argued above, AL can be observed under two simultaneous conditions: existence of a localized regular trajectory and the initial conditions belonging to it. Our general strategy is to study the probability for such trajectory to exist at a given energy, with realizations of disorder varied. We consider the wave packets of different initial width  $L$  and derive the probability of  $L$ -site localized periodic and quasiperiodic solutions of (1).

We start with periodic orbits, which are single-site localized solutions, and derive conditions of their destruction (when regular trajectories delocalize the corresponding wave packets are assumed to spread). Let us construct an exact time-periodic orbit of (1) localized at  $n = 0$ . We apply perturbation theory  $x_n(t) = \sum_{k=0}^{\infty} x_n^{(k)}(t)$  in the small-amplitude limit, taking  $x_n^{(0)}(t) = A_n \delta_{0,n} \cos t$  as the zero-order approximation (note that  $\varepsilon_0 = 1$ ). In first order we find

$$x_{\pm 1}(t) = A_{\pm 1} \cos t, \quad A_{\pm 1} = \frac{3A_0^3}{4(\varepsilon_{\pm 1} - \varepsilon_0)}. \quad (2)$$

In higher orders it follows

$$x_{\pm n}(t) = A_{\pm n} \cos t, \quad A_{\pm n} = \frac{3A_{\pm(n-1)}^3}{4(\varepsilon_{\pm n} - \varepsilon_0)}. \quad (3)$$

For the perturbative solution (3) to converge, the amplitudes must decay:

$$\left| \frac{A_{\pm n}}{A_{\pm(n-1)}} \right| = \frac{3A_{\pm(n-1)}^2}{4|\varepsilon_{\pm n} - \varepsilon_0|} < \frac{1}{\kappa}, \quad \kappa > 1. \quad (4)$$

The probability for this condition to hold at site  $\pm n$  is determined by its random on-site potential  $\varepsilon_{\pm n}$  and the oscillation amplitude of its neighbor:

$$\mathcal{P}^{(\pm n)} = 1 - \frac{3}{2} \kappa A_{\pm(n-1)}^2 \geq 1 - \frac{3}{2} \kappa^{3-2n} A_0^2. \quad (5)$$

The probability to obtain a localized time-periodic solution in the infinite chain  $\mathcal{P} = \prod_{n=1}^{\infty} [\mathcal{P}^{(n)}]^2$  is bounded from above by the probability to have decreasing amplitudes at least in the first neighbors

$$\mathcal{P} \leq \mathcal{P}^{(1)} \mathcal{P}^{(-1)} = (1 - 3\kappa E)^2 \equiv \mathcal{P}^+, \quad (6)$$

and from below by the probability to get at least an exponentially decaying profile in the infinite system

$$\mathcal{P} \geq \prod_{n=1}^{\infty} \left( 1 - \frac{3}{2} \kappa^{3-2n} A_0^2 \right)^2 \geq \left( 1 - \frac{3\kappa E}{1 - \kappa^{-2}} \right)^2 \equiv \mathcal{P}^-, \quad (7)$$

assuming  $3\kappa E \ll 1$ , where  $E = A_0^2/2$  is the energy of the central site.

It follows from (6) that it is not possible to construct a single-site localized ( $|A_n/A_0| \ll 1$ ,  $\forall n \neq 0$ ) time-periodic orbit for  $E > 1/3$ . For smaller energies, the lower bound of spreading probability always remains nonzero and scales linearly with the total wave packet energy  $E$ , backing the arguments in [22].

Next we consider  $L$ -site localized solutions to (1) loosely corresponding to tori. Their existence probability is maximized for sparse packets, when the most excited sites are separated by intervals of weakly excited ones (due to reduced perturbative corrections or, equivalently, resonance probabilities, see also [22]). We assume such a sparse excitation with  $L$  sites with the energy  $E/L$  per site, separated by at least two nonexcited sites, as a zero-order approximation for the perturbation theory. In the first order the problem is reduced to  $L$  independent single-site problems and, with (6), the upper bound for the localization probability reads

$$\mathcal{P}_L = \left( 1 - \frac{3\kappa E}{L} \right)^{2L}. \quad (8)$$

Note that one can proceed to arbitrary high orders assuming sparser zero-order approximations, ultimately obtaining the lower bound analogue to (8) by exponentiating (7).  $\mathcal{P}_L$  is a monotonically increasing function of  $L$  with the limiting localization probability for the infinite-size packet

$$\mathcal{P}_{\infty} = e^{-6\kappa E}. \quad (9)$$

Equations (8) and (9) are the central results of this work. Note that a wave packet of size  $L$  has  $2L$  phase space variables to be defined. The probability of obtaining regular AL states (or not) is defined through the ratio of the volume  $v_l$  of all points in this  $2L$ -dimensional phase space, which yield AL localization, to the full available volume  $v_l + v_s$  where  $v_s$  is the volume of all points which yield spreading:  $\mathcal{P}_L = v_l/(v_l + v_s)$ . We can now conclude that for a wave packet of size  $L$  no regular AL states are expected if the energy density  $h \equiv E/L > 1/3$ . But even for  $h < 1/3$  there is always a finite probability to observe spreading trajectories.

It is most interesting that, given a fixed total energy  $E$ , the probability for AL is approaching a finite value below one in the limit of infinite packet size  $L$  (9). Therefore, even in this limit (of zero energy density and infinite packet size) there always remains a nonzero probability to spread, i.e.,  $v_s/(v_l + v_s) \neq 0$  in this limit.

Remarkably, the derivative  $\partial \mathcal{P}_L / \partial L \sim L^{-2}$  is also vanishing as a power law, and not as an exponential for large  $L$ . Therefore, at variance to the case of AL, there is no new length scale emerging. In particular, already the first moment  $\langle L \rangle$  obtained with such a probability distribution function diverges.

It is instructive to generalize our results to the case of arbitrary degree of nonlinearity  $\gamma$  and lattice dimensionality  $d$ . We consider localized solutions of equally excited sites that occupy volume  $V$  of characteristic size  $L$ . As before, sparse solutions maximize their existence probability  $\mathcal{P}_V$ . As the volume scales as  $V \propto L^d$ , we straightforwardly arrive at the probability to find an AL wave packet:

$$\mathcal{P}_V = \left(1 - \frac{\kappa_\gamma E^{\gamma/2-1}}{\sqrt{\gamma/2-1}}\right)^{2Vd}. \quad (10)$$

We find that for any  $\gamma$  no regular AL wave packets are expected if the energy density  $h$  exceeds a  $\gamma$ -dependent threshold. For smaller  $h$  there is always a finite probability to launch a spreading wave packet. It is most interesting that the fraction of localized AL wave packets  $v_l/(v_l + v_s)$  tends to zero in the limit  $V \rightarrow \infty$  at fixed  $E$  for  $\gamma < 4$ , and tends to unity for  $\gamma > 4$  (note that this is only the upper bound for the AL probability), as shown schematically in Fig. 1.

To probe our analytical results we simulate the evolution of single-site initial excitations with energy  $E$  in (1) and (11) for  $d = 1$ ,  $\gamma = 4$ . We use a symplectic SABA-type numerical integration scheme [15,19] up to final times  $t_{\text{end}} = 10^9$ . To characterize the wave packet we calculate its second moment  $m_2 = \sum_n (n - \bar{n})^2 E_n / E$ , with  $\bar{n} = \sum_n n E_n / E$ , which measures the squared distance between both packet tails, and the participation number

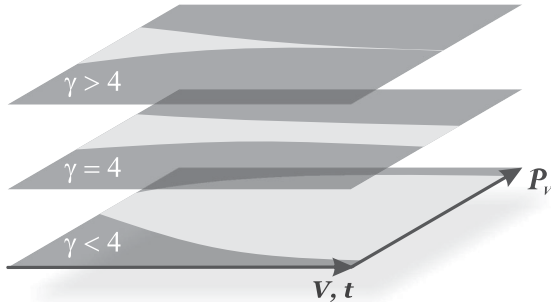


FIG. 1. Schematic dependence of the probability  $\mathcal{P}_V$  for wave packets to stay localized (dark area) together with the complementary light area of spreading wave packets versus the wave packet volume  $V$  (either initial or attained at some time  $t$ ) for three different orders of nonlinearity  $\gamma < 4$ ,  $\gamma = 4$ , and  $\gamma > 4$ .

$P = E^2 / \sum_n E_n^2$ , which tells the number of most strongly excited lattice sites.

We first choose an energy  $E = 1$  which according to our above results should yield spreading wave packets with probability one. Indeed in Fig. 2 we plot the growth of  $m_2$  for two disorder realizations, which survives a fit  $m_2 \propto t^{1/3 \dots 1/2}$  in accord with theoretical predictions [17,18]. For small energy  $E = 0.05$  we expect to observe both spreading and nonspreading wave packets. In the inset in Fig. 2 we show two typical cases—in one case the second moment does grow, in the second case it does not, therefore indicating AL.

To mount statistics we perform simulations of single-site excitations at a given energy  $E$  for  $N_r = 1000$  different disorder realizations. We measure the participation number  $P(t)$  at three large time values  $t = 10^8, 5 \times 10^8, 10^9$ , sort them by value, and plot in Fig. 3 versus the sorting index  $i$ . For very small energy  $E = 0.002$  all three curves for different times overlap and have small values of  $P$ , indicating AL. For larger energies, e.g.,  $E = 0.05$ , we find that the curves overlap up to some largest index  $i_m(E)$  and differ for larger indices.

To quantify the results we introduce the following numerical criterion for a single-site (or a single-mode) localized trajectory: the one with participation number  $P(t) < 1 + \alpha$ , choosing a small enough  $\alpha = 0.2$ . As  $P \approx 1 + 2(E_{-1} + E_1)/E_0$  for strongly localized trajectories, (5) yields a theoretical estimate for the nonspreading fraction:  $\mathcal{P} \approx 1 - 6\sqrt{2/\alpha}E$ . The numerically obtained dependence of the nonspreading fraction on the energy shows a linear decay and agrees well with this analytical estimate (Fig. 3, inset).

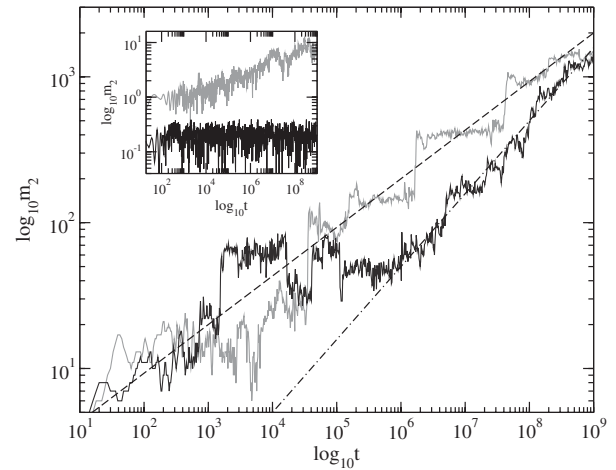


FIG. 2. Second moment (see text for definition) of a spreading single-site excitation versus time,  $d = 1$ ,  $\gamma = 4$ ,  $E = 1$ . Dashed and dash-dotted lines with the slopes  $1/3$  and  $1/2$ , respectively, guide the eye. Inset: Second moment of a spreading and a nonspreading single-site excitation for two different disorder realizations versus time,  $d = 1$ ,  $\gamma = 4$ ,  $E = 0.05$ .

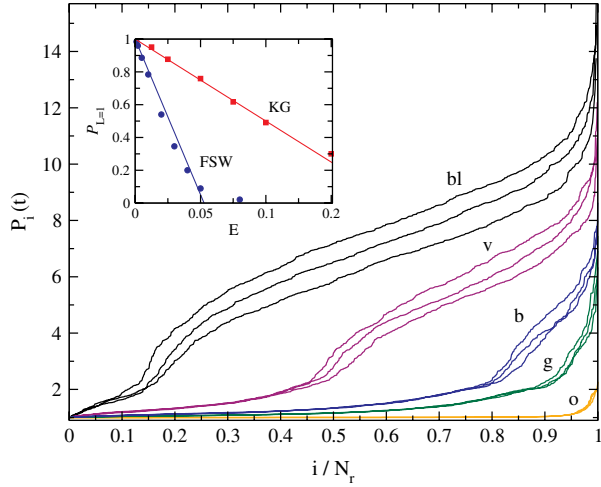


FIG. 3 (color online). Sorted participation numbers  $P_i$  (see text for definition) for  $N_r = 1000$  different disorder realizations  $1 \leq i \leq N_r$ , at  $t = 10^8, 5 \times 10^8, 10^9$  (bottom to top) for  $E = 0.08$  (bl),  $E = 0.05$  (v),  $E = 0.03$  (b),  $E = 0.02$  (g),  $E = 0.002$  (o). Inset: Single-site localized fractions ( $\mathcal{P}_{L=1}$ ): numerics by symbols, linear fits for FSW and for KG by solid lines.

A similar approach can be developed in the KG chain described by the Hamiltonian

$$\mathcal{H} = \sum_n \left[ \frac{p_n^2}{2} + \frac{\varepsilon_n x_n^2}{2} + \frac{u x_n^4}{4} + \sum_{m \in D(n)} \frac{(x_m - x_n)^2}{4W} \right]. \quad (11)$$

This is a generic model with a finite localization length of Anderson modes [12,15,16,18,19]. We performed a similar numerical analysis at  $u = 1$  as for the FSW model with launching a single eigenmode of the corresponding linear eigenvalue problem of  $u = 0$ . The parameter  $W = 6$ . The final outcome is plotted in the inset in Fig. 3. It shows again a linear decay of the AL probability  $\mathcal{P}_L$  with energy, similar to the FSW model.

In summary, we developed the theory of AL in nonlinear disordered systems in the zero-temperature limit. In contrast to linear systems, where AL halts wave propagation, the fate of AL becomes a matter of probability, and nonlinear waves may propagate. With increasing the strength of nonlinearity (energy) the probability of AL is reduced, and reaches zero at a certain level of nonlinearity depending on the initial wave packet size. At fixed total energy, the fraction of AL wave packets does not reach unity in the limit of an infinite packet size but stays finite yet less than one for quartic anharmonicity. For a weaker degree of nonlinearity the AL fraction in this limit vanishes completely, while for a stronger nonlinearity degree its upper bound tends to one. It follows that the previously conjectured slowing down or halt of wave propagation [13,14,22] is not realized at least for the quartic and weaker

anharmonicity. An analytic treatment of the generic KG-type system yields similar results and will be presented in more detail elsewhere. Our results not only resolve a fundamental theoretical problem in nonlinear wave physics but also have multifaceted experimental implications, as in design, as well as for interpretation.

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