Decohering localized waves

Kristian Rayanov,[1,2,3] Günter Radons,[2] and Sergej Flach[1,3]
[1]*Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, D-01187 Dresden, Germany*
[2]*Institute of Physics, Chemnitz University of Technology, D-09107 Chemnitz, Germany*
[3]*Centre for Theoretical Chemistry and Physics, New Zealand Institute for Advanced Study, Massey University, Auckland 0745, New Zealand*

(Received 5 February 2013; revised manuscript received 12 April 2013; published 8 July 2013)

In the absence of confinement, localization of waves takes place due to randomness or nonlinearity and relies on their phase coherence. We quantitatively probe the sensitivity of localized wave packets to random phase fluctuations and confirm the necessity of phase coherence for localization. Decoherence resulting from a dynamical random environment leads to diffusive spreading and destroys linear and nonlinear localization. We find that maximal spreading is achieved for optimal phase fluctuation characteristics, which is a consequence of the competition between diffusion due to decoherence and ballistic transport within the mean free path distance.

DOI: 10.1103/PhysRevE.88.012901 PACS number(s): 05.45.−a, 03.75.Gg, 63.20.Pw

I. INTRODUCTION

Ever since the absence of diffusion of a quantum particle due to randomness on a lattice was predicted by P. W. Anderson [1], wave localization has become an intensively studied phenomenon. Of particular interest is the wave character of the quantum objects that localize. In contrast to classical particles, waves are fields characterized by amplitude and phase and can tunnel under a potential barrier but also be back-scattered above a given barrier. In the absence of confinement, wave localization implies that the wave does not escape to infinity even if it was energetically allowed. Consequently, Anderson localization is an interference phenomenon that relies on phase coherence.

Localization on a lattice also occurs as result of an applied DC field where it leads to Bloch oscillations [2,3]. It can even show up in translationally invariant lattices in the form of discrete breathers as a result of nonlinear interactions [4,5]. Another source for wave localization can be a quasiperiodic potential interpolating between uncorrelated disorder and perfect periodicity. In this case, localization has been predicted by Aubry and André [6] only when the strength of the quasiperiodic potential exceeds a certain critical value. The latter case is largely debated. Initially, the additional importance of high enough incommensurability was stressed [7–9]. More recently, the occurrence of a metal-insulator transition has even led to the claim that localization in the Aubry-André model has particle character [10], and therefore the phase coherence should not matter.

In this work, we will directly probe the sensitivity of linear and nonlinear localization to a loss of phase coherence. In experiments, decoherence generally arises due to random temporary fluctuations as a result of inevitable coupling of the ideal system to its environment [11,12]. Therefore, our considerations are also related to the conceptual understanding of the effects of decoherence, which is assumed to play a key role in the translation of the quantum world to the "classical" picture obtained through generic measurement processes [13].

For Anderson localization, the destructive consequences of noise and decoherence have been well-studied during the past decades. In disordered electronic systems, where it was initially predicted, localization is hindered by phonon and electron assisted variable range hopping [14]. For coupled quantum dots it has been conjectured that Anderson localization is destroyed for asymptotically large times even if the only disturbance of the ideal system comes from a local measurement at one quantum dot [15] (note that these times could diverge according to Ref. [16]). Anderson localization has also been predicted as localization in momentum space [17]. Here, the delocalizing impact of decoherence has been discussed as well, both theoretically [18] and experimentally [19,20]. However, direct observation of Anderson localization of matter waves has been achieved only recently in one [21] and three [22,23] dimensions with the creation of Bose-Einstein condensates of ultracold atoms (see also [24] for a recent review). Apart from quantum systems, Anderson localization has also been demonstrated with light waves in coupled optical wave guides [25].

In the following, we will investigate wave packets that are localized not only as a consequence of disorder but also of quasiperiodic potentials and DC fields (linear localization) and of interactions (nonlinear localization). Our aim is to stress that, like for Anderson localization, phase coherence is essential for all of these localization phenomena and, hence, they are interference effects. This especially means that localization in the Aubry-André model has pure wave character and is of quantum origin in the case of matter waves, where it has also been experimentally demonstrated recently [26].

In our numerical simulations, we specifically consider fluctuations in the phases, resulting from coupling to a random environment. The resulting delocalization is directly related to the loss of coherence. In the long time limit, the loss of localization is generally observed as a diffusive spreading regime, which is in agreement with theoretical predictions for dynamical disorder on otherwise translational invariant lattices [27] and continuous systems [28]. Note that here we have in addition nonlinearity or a lack of translational invariance. We discuss the decay of initially localized wave packets also on intermediate time scales and find that maximal spreading is not achieved for the strongest or most frequent dephasing. Therefore, there exists a nontrivial optimal rate and strength of dephasing, which maximizes the wave packet's extent at a given time. As this maximal delocalization does not necessarily correspond to the asymptotic diffusion and results from the interplay between a localization mechanism and time-dependent perturbations, these results are not directly explained by the analytical predictions for the asymptotes.

II. LOCALIZED WAVE MODELS

We consider the model of the one-dimensional discrete nonlinear Schrödinger equation,

$$i\frac{\partial}{\partial t}\psi_l = \epsilon_l\psi_l + \psi_{l+1} + \psi_{l-1} + \beta|\psi_l|^2\psi_l, \qquad (1)$$

where ψ_l is a complex field at site l with onsite energy ϵ_l and nonlinearity strength β. It can be derived from the Hamiltonian

$$\mathcal{H} = \sum_l \epsilon_l|\psi_l|^2 + (\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l) + \frac{\beta}{2}|\psi_l|^4 \qquad (2)$$

by $\dot{\psi}_l = \partial\mathcal{H}/\partial(i\psi_l^*)$. Varying the total norm $S := \sum_l |\psi_l|^2$ is equivalent to varying β; therefore, it is always possible to fix $S = 1$. Then $|\psi_l|^2$ can be identified as the norm density on site l. Both the energy $\mathcal{H}$ and the norm S are integrals of motion. The discrete nonlinear Schrödinger equation allows for the investigation of all localization phenomena described above.

Linear localization of wave packets is obtained for $\beta = 0$ and suitable potentials $\epsilon_l \neq 0$. The resulting set of linear differential equations may be decoupled in terms of normal modes (NMs). Separation of variables [$\psi_l(t) = A_l e^{-i\lambda t}$] leads to the eigenvalue problem

$$\lambda A_l = \epsilon_l A_l + A_{l+1} + A_{l-1} \qquad (3)$$

for the only site-dependent (in general complex) amplitudes A_l. The normalized eigenvectors $A_{\nu,\{l\}}$ are the NMs having eigenfrequencies λ_ν. For localized NMs, any initial excitation containing only a finite number of them will stay localized in time, leaving the NMs oscillating with their eigenfrequency, respectively, but independently from each other.

We consider the following three cases of onsite energies that yield localized NMs. (I) Disorder (Anderson model): the onsite energies are random and chosen uniformly from the interval $[-\frac{W}{2},\frac{W}{2}]$, where W denotes the disorder strength. The eigenfrequencies λ_ν lie in the interval $[-2-\frac{W}{2}, 2+\frac{W}{2}]$. The NMs decay exponentially as $A_{\nu,l} \sim e^{-l/\xi}$, with a localization length $\xi(\lambda_\nu)$. (II) DC Field (Wannier-Stark ladder): the field E determines the linear growth of the onsite energies, $\epsilon_l = El$. The eigenfrequencies are $\lambda_\nu = E\nu$, and the NMs are given by Bessel functions of first kind $J_l(x)$ as $A_{\nu,l} = J_{l-\nu}(2/E)$ [3]. (III) Quasiperiodic potential (Aubry-André model): the onsite energies satisfy $\epsilon_l = \zeta\cos(2\pi\alpha l)$. The commensurability is characterized by the irrational parameter α, while the parameter ζ describes a relative strength, similar to W in the Anderson model. A condition for localized NMs is that α be as far as possible from a rational number [9], where a standard choice is to take the inverse golden mean $\alpha = \frac{\sqrt{5}-1}{2}$ [29]. Then NMs are exponentially localized for $\zeta > 2$ in real space and for $\zeta < 2$ in Fourier space (self-duality). Consequently, there exists a metal-insulator transition at $\zeta = 2$ in either space.

Adding a nonlinearity, in general, destroys the integrability of the linear system by inducing frequency shifts, which result in excitations of overlapping NMs [30–32]. For localized NMs, small enough nonlinearities have been shown to cause delocalization with subdiffusive spreading [31,33,34]. In contrast, a large enough nonlinearity can lead to localization in form of discrete breathers when the nonlinear frequency shifts exceed the linear eigenfrequency spectrum (self-trapping) [5,35]. Discrete breathers are exponentially localized and their time-dependence is characterized by a single frequency Ω_b as $\psi_l(t) = A_l e^{i\Omega_b t}$.

To study nonlinear localization independent of linear localization, we consider Eq. (1) with $\epsilon_l = 0, \forall l, \beta > 0$. Then the NMs of the underlying linear system [Eq. (3) with $\epsilon_l = 0$] are extended and localization can only be attributed to discrete breathers. Any not self-trapped part of a wave packet will spread ballistically over the entire chain. Note that overlap of several discrete breathers, in general, does not lead to new localized wave packets with a quasiperiodic time-dependence [5].

To quantify the spatial extent (localization volume) of an arbitrary wave packet we compute the participation number $P = 1/\sum_l |A_l|^4$, which measures the number of considerably excited sites [36], and the second moment (variance) $m_2 = \sum_l (m_1 - l)^2 |A_l|^2$, which measures the average norm spread around its center $m_1 = \sum_l l|A_l|^2$. Since m_2 has the units of a square distance, a localization volume in one dimension is defined proportional to $\sqrt{m_2}$ [37].

III. LOCALIZATION AS A COHERENT EFFECT

In this paper we want to establish a direct relationship between coherence and localization. The main goal is to show how temporal phase fluctuations always destroy localization. Consequently, phase coherence is necessary for linear and nonlinear localization, which is a manifestation of its crucial wave character. Considering localization in the form of localized NMs and discrete breathers, we will first identify maximally localized wave packets with maximal coherence. Then we will investigate the effects of phase fluctuations and decoherence.

To characterize a wave packets' degree of coherence, we will utilize the basic property of the second-order complex degree of coherence γ [38], which is closely linked to the visibility of interference fringes in related interference experiments. The essence of second-order coherence lies in the investigation of correlations between two space-time points. It is furthermore a prerequisite for coherence of higher orders that contain correlations of more space-time points. The second-order complex degree of coherence is defined as [38]

$$\gamma(l_1,l_2,\tau) := \frac{\Gamma(l_1,l_2,\tau)}{\sqrt{\Gamma(l_1,l_1,0)}\sqrt{\Gamma(l_2,l_2,0)}}, \qquad (4)$$

which is a normalized version of the mutual coherence function

$$\Gamma(l_1,l_2,\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \psi_1^*(t)\psi_2(t+\tau)dt. \qquad (5)$$

Complete coherence is achieved for $|\gamma(l_1,l_2,\tau)| = 1$, while partial coherence corresponds to $0 < |\gamma(l_1,l_2,\tau)| < 1$ and complete incoherence to $|\gamma(l_1,l_2,\tau)| = 0$. Note that the averaging in Eq. (5) assumes statistically stationary evolving fields ψ_1, ψ_2.

Inserting a generic solution of the form $A_l e^{i\omega t}$ with site-independent ω and time-independent amplitudes A_l into Eq. (4), it is straightforward to obtain $|\gamma| = 1$. Thus, a single NM or discrete breather always possesses complete coherence, when ω is identified with either λ_ν or Ω_b, respectively. The phases at different sites of these coherent solutions remain locked together in time. It can also be shown easily that a superposition of more than one NM will only be partially coherent. Since nonlinear localization is only expected in the form of discrete breathers, it is, in general, always fully coherent. Note that here we have not made any restrictions to the complex amplitudes A_l except for time-independence. Especially, we have not requested that the solutions (e.g., the NMs) be localized. Therefore, coherence is not a sufficient condition for localization.

Considering that the most localized wave packets consist of either a single NM or discrete breather in a linear or nonlinear model, respectively, it is clear that complete coherence is necessary for maximal localization. One can even specify a stricter condition for the phase relations of neighboring sites, apart from being only locked together in time. Norm conservation of Eq. (1) and time-independent amplitudes $|\psi_l|$ ensure that the norm density current $j_l = -2|\psi_{l+1}||\psi_l|\sin(\varphi_{l+1} - \varphi_l)$ is constant over the whole chain. Localization implies $|\psi_l| \to 0$ for $l \to \pm\infty$ and, therefore, $j_l = 0\ \forall l$. Hence, when $|\psi_l| \neq 0$, the phases must satisfy $\varphi_{l+1} - \varphi_l = m\pi$, $m = 0,\pm1,\pm2,\ldots$ This is equivalent to stating that localized NMs must have real amplitudes A_l, however, we want to stress here that this condition is generic and also applies to nonlinear localization.

IV. DECOHERING AND DELOCALIZING

Now we will quantitatively analyze the impact of temporal phase fluctuations, which, in general, occur when coupling to a stochastic environment is considered. It is clear that for a maximally localized wave packet this will destroy the fixed phase relation of neighboring sites and lead to a local norm density current. Then complete coherence is lost and spreading takes place. However, the wave packet does not have to delocalize completely as long as some partial coherence remains. We will numerically show that an initial dephasing induced loss of coherence directly relates to the loss of localization and that persistent fluctuations finally delocalize the wave packet completely.

We integrate Eq. (1) using a symplectic SABA_1 integrator described in Ref. [39]. A dynamical random environment is included by considering dynamical onsite energies $\epsilon_l \to \epsilon_l + \varepsilon_l(t)$, where $\varepsilon_l(t)$ is a random process. It can be easily shown that $\varepsilon_l(t)$ defines a dephasing term in the equations of motion of the phases φ_l. Numerically, the dephasing is implemented by altering the phases between integration steps of the unperturbed equations of motion. As initial condition, we always choose a single NM or discrete breather as shown in Fig. 1.

We choose two different dephasing schemes. The first one is a complete random dephasing, which allows us to observe the effects of dephasing as clearly as possible. Therefore, the phases φ_l are replaced by completely new random phases chosen uniformly from $[0,2\pi]$ at certain times of the integration. Between these kicks on the phases, the dynamics of a wave packet is solely governed by the linear or nonlinear equations of motion. Each kick can be considered as defining a new initial configuration for the integration where all the information of the old phases is lost. The second scheme is a quasiperiodic dephasing, similar as in Ref. [12]. Its purpose is to probe the sensitivity of localization to only small phase fluctuations, which are even not completely uncorrelated. The new phases result from the old ones as $\varphi_l(t) = \varphi_l + b\sin(\mu_l t)$ with a time-independent frequency μ_l chosen on each site randomly and uniformly from $[0, \max\{\mu_l\}]$ and a constant strength of dephasing b. The frequencies μ_l are, in general, incommensurate but fixed during the integration. This leads to uncorrelated fluctuations between different sites and temporarily correlated fluctuations on each site. Here, the phase kicks are performed after each step of integration. To calculate the complex degree of coherence, we switch off dephasing after a certain number of kicks on the phases in order to obtain statistically stationary evolving fields. We use the time evolution at the center of norm and one neighboring site.

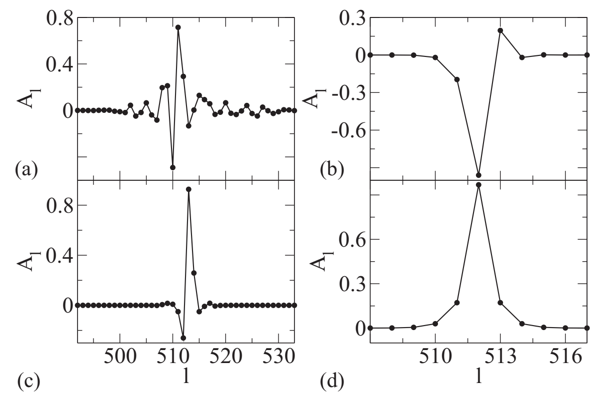

FIG. 1. Fully coherent initial conditions that remain localized unless dephasing is applied. (a), (b), (c) NM with eigenenergy close to zero for disordered ($W = 5$), linear growing ($E = 5$), and quasiperiodic ($\zeta = 5$) onsite potentials, respectively. (d) Discrete breather for ($\beta = 6$). The amplitudes are shown by the dots while the black lines guide the eye.

Dephasing of a NM (Fig. 2) shows that the decoherence is, in general, nonmonotonic with respect to the growing number of phase kicks, however, a decrease (increase) in coherence similarly leads to a decrease (increase) in localization. The reason is a selective excitation and damping of overlapping NMs. Therefore, the direct correspondence between decoherence and delocalization can be found in all linear models with localized NMs (Fig. 3). It holds for a small number of phase kicks as long as the excited NMs contain the two sites that are considered in the calculation of $|\gamma|$.

In contrast, after a few phase kicks on a discrete breather, either only a small part of the norm is radiated and the rest can again self-trap to form a new fully coherent wave packet, or the discrete breather delocalizes completely. However, the approach to a new time-periodic trajectory of a discrete breather in phase space can take up arbitrarily long times [5]. In this regime, we observed seemingly localized (over accessible integration times) wave packets with only small deviations from complete coherence. The deviations seem to

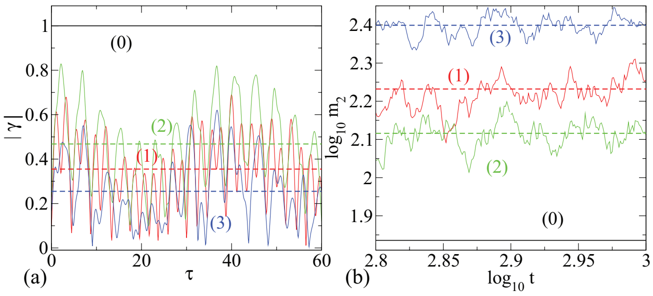

FIG. 2. (Color online) Comparison of (a) $|\gamma|$ and (b) m_2 in the Anderson model after complete random dephasing of a NM. Dephasing is switched off after a number k of kicks: $(k) = (0)$ black line, (1) red, (2) green, (3) blue. In (b), only times are shown when the wave packet is, on average, not spreading anymore. A direct correspondence between decoherence and delocalization is observed when changes in the averages of $|\gamma(\tau)|$ [dashed lines in (a)] and $m_2(t)$ [dashed lines in (b)] are compared.

be larger when the initially radiated norm (and, therefore, delocalization) has been larger, approximately confirming a correspondence between decoherence and delocalization.

A common result for both linear and nonlinear models is that coherence on average decreases for a large growing number of phase kicks. This situation is close to an experimental one where phase fluctuations exist persistently. We observe that decoherence by persistent dephasing leads to complete delocalization and the generic onset of a diffusive spreading regime, $m_2 \sim t$ and $P \sim \sqrt{t}$ (Figs. 4 and 5), as can be expected for similar dynamically disordered tight-binding Hamiltonians [27]. Note that the loss of norm from a discrete breather is properly characterized by the participation number. Here, the diffusive regime is preceded by jump-like increases of P, which correspond to radiation of substantial parts of norm as small amplitude waves. It is expected that discrete breathers remain robust upon radiation of small amplitude waves [40], which can be treated as linear background of the high amplitude breather [41]. Consequently, when dephasing is only performed very rarely, new self-trapping may occur leading to a series of stepwise increases of P [Fig. 5(a)].

In general, the transition times before the asymptotic diffusive regime is assumed can vary. Especially for the

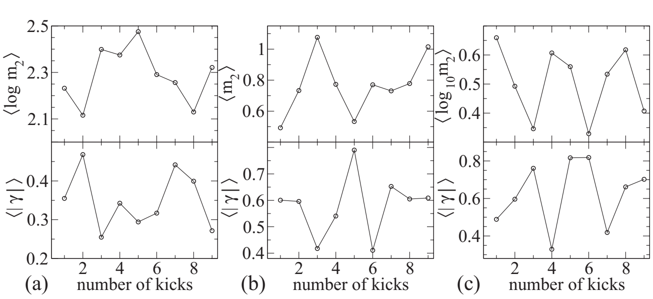

FIG. 3. Time averages of $|\gamma|$ (lower panel), m_2 (upper panel) in the (a) Anderson model, (b) Wannier-Stark ladder, (c) Aubry-André model when dephasing is switched off after a small number k of kicks. The averages for $k = 1$, 2 and 3 kicks in (a) correspond to the dashed lines in Fig. 2. Up to a number of 8 kicks [in (a), (c), 9 kicks in (b)] an increase (decrease) in γ corresponds to a decrease (increase) in m_2.

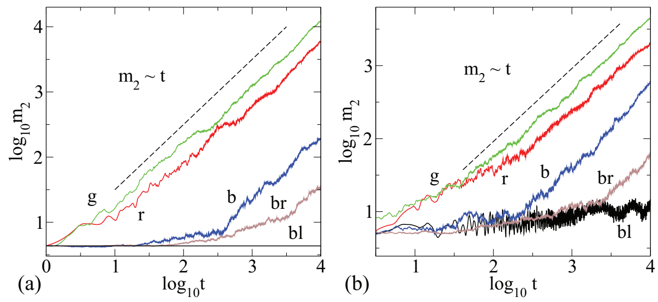

FIG. 4. (Color online) Diffusive spreading in linear chains (Anderson model) for persistent dephasing of initially one normal mode. (a) Complete random dephasing with a time $\triangle$ between phase kicks of $\triangle = 10$ red (r), 1 green (g), 0.01 blue (b), and 0.001 brown (br). The black (bl) line shows the evolution without dephasing for reference. (b) Quasiperiodic dephasing with strength $b = 0.0001$ black (bl), 0.0006 red (r), 0.01 green (g), 0.06 blue (b), 1.1 brown (br).

quasiperiodic dephasing, it may be expected that even for very small b normal diffusion commences after sufficiently long waiting time (see Ref. [12] for the Anderson model). This shows the highly sensitive dependence of localization on phase coherence and minimal phase fluctuations. Moreover, the results do not depend on the specific range $[0, \max \mu_l]$ of dephasing frequencies μ_l. Consequently, in experiments localization is at best an intermediate regime, since arbitrarily small fluctuations can destroy localization.

Another interesting effect of random dephasing is that increasing the rate or strength of dephasing does not necessarily lead to a decrease in localization and to faster spreading. It is rather observed that very strong and frequent dephasing even may suppress the onset of normal diffusion for an increasingly long time. As a result, the wave packet extent at the final time of integration becomes maximal for a certain optimal rate and strength of dephasing.

When the maximal extent corresponds to a diffusive spreading regime, it can be attributed a maximal diffusion constant D. This constant then also defines the optimal dephasing

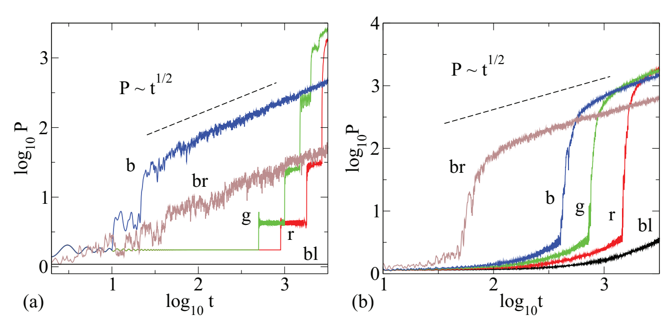

FIG. 5. (Color online) Diffusive spreading for persistent dephasing of a discrete breather. The crossover from a ballistic to diffusive spreading regime is seen as a jump-like increase of P. (a) Complete random dephasing with time $\triangle$ between phase kicks of $\triangle = 900$ red (r), 500 green (g), 10 blue (b), 0.1 brown (br). The black (bl) line shows the evolution without dephasing for reference. (b) Quasiperiodic dephasing with strength $b = 0.0001$ black (bl), 0.000 15 red (r), 0.0002 green (g), 0.000 25 blue (b), 0.0006 brown (br).

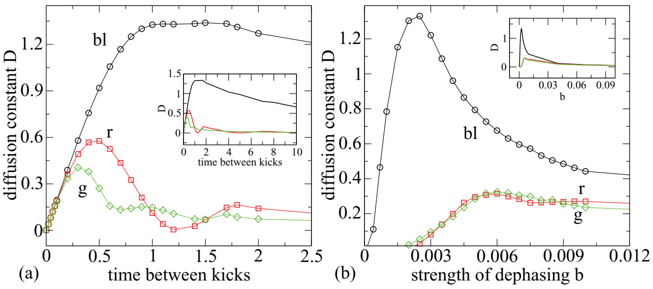

FIG. 6. (Color online) Dependence of the diffusion constant D on (a) rate and (b) strength of dephasing in the Anderson model [black circles (bl)], Wannier-Stark ladder [red squares (r)], and Aubry-André model [green diamonds (g)] with the asymptotic dependence in the insets. The occurrence of a global maximum in each model is observed. Each data point of D results from linear regression of an average of $m_2(t)$ over 100 realisations of sequences of random dephasing. The connecting lines guide the eye.

parameters for maximal delocalization at later times. Note that in the nonlinear model (Fig. 5) this is not the case. A discrete breather is destroyed best when the onset of normal diffusion can be avoided for a long time. Therefore, optimal rates and strengths are different for different final times considered and cannot be identified with a macroscopic constant.

For the linear models, the diffusion constants are shown in Fig. 6. Apart from some local maxima, there clearly exists a global maximum (see also Refs. [42,43] for the Anderson model) in a wide range of rates and strengths of dephasing. Its nature can be revealed with the following estimate in the Anderson model. Consider that the localization length corresponds to a mean free path of quasiparticles [44]. An optimal rate would leave the wave packet spread ballistically over the new accessible localization length ξ before the next phase kick is applied. Since after one kick only those NMs can be excited considerably, whose center of norm lies inside the localization volume (determined by ξ) of the initial wave packet, at most one localization volume becomes additionally accessible. An upper estimate of the localization volume of a NM is of the order of $330/W^2$, corresponding to a localization length of $100/W^2$ [45]. For $W = 5$, this gives a mean free path for quasiparticles of ≈ 4. With a maximal particle velocity (group velocity in Fourier basis) of two sites per time unit, an optimal rate corresponds to one kick every two time units. This upper estimate is in good agreement with the numerically obtained value [Fig. 6(a)] of one kick per 1.6 time units [46]. As a result, one could actually expect a ballistic spreading when the kicks match the times for completing the ballistic spreading into a new localization volume. However, we always observe the onset of a diffusive regime since the new localization volumes variate while the rate of dephasing is kept constant. In consequence of the above, optimal parameters of dephasing have to balance, on the one hand, decoherence and delocalization of the initial wave packet and, on the other hand, should not suppress too much a possibly super-diffusive spreading by imposing normal diffusion. Therefore, it becomes also clear that the optimal dephasing of a discrete breather cannot correspond to a diffusive regime, since stepwise radiation of small amplitude waves allows for ballistic transport to arbitrary distances in this case.

V. CONCLUSIONS

In conclusion, we investigated the effects of decoherence on linear and nonlinear localized wave packets and showed that phase coherence is a necessary condition for localization. Therefore, localization, especially in the Aubry-André model as well, is essentially a wave phenomenon and dephasing can be identified as a general mechanism of wave packet spreading. As soon as nondecaying random phase fluctuations occur, localization is destroyed with the onset of normal diffusion. Random phase fluctuations can also occur due to deterministic chaos and nonintegrability of nonlinear wave equations. Then the additional dependence of the effective diffusion constant on the wave packet density results in a slower subdiffusion, as considered in Ref. [31]. To maximize the wave packet extent at a given time, there is an optimal rate and strength of dephasing, resulting from a competition between quickly decohering the initial wave packet and not too much slowing down norm radiation to the exterior chain.

ACKNOWLEDGMENTS

The authors thank T. V. Laptyeva and J. D. Bodyfelt for very helpful discussions.

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] G. H. Wannier, Phys. Rev. **117**, 432 (1960).
[3] H. Fukuyama, R. A. Bari, and H. C. Fogedby, Phys. Rev. B **8**, 5579 (1973).
[4] A. J. Sievers and S. Takeno, Phys. Rev. Lett. **61**, 970 (1988).
[5] S. Flach and A. V. Gorbach, Phys. Rep. **467**, 1 (2008).
[6] S. Aubry and G. André, Ann. Isr. Phys. Soc. **3**, 133 (1980).
[7] J. Bellissard, D. Bessis, and P. Moussa, Phys. Rev. Lett. **49**, 701 (1982).
[8] S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, and E. D. Siggia, Phys. Rev. Lett. **50**, 1873 (1983).
[9] J. Bellissard, R. Lima, and D. Testard, Commun. Math. Phys. **88**, 207 (1983).
[10] M. Albert and P. Leboeuf, Phys. Rev. A **81**, 013614 (2010).
[11] W. H. Zurek, Rev. Mod. Phys. **75**, 717 (2003).
[12] Y. Yin, D. E. Katsanos, and S. N. Evangelou, Phys. Rev. A **77**, 022302 (2008).
[13] M. Schlosshauer, Rev. Mod. Phys. **76**, 1267 (2004).
[14] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[15] S. A. Gurvitz, Phys. Rev. Lett. **85**, 812 (2000).
[16] S. Aubry and R. Schilling, Physica D **238**, 2045 (2009).
[17] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982).
[18] E. Ott, T. M. Antonsen, and J. D. Hanson, Phys. Rev. Lett. **53**, 2187 (1984).

[19] F. L. Moore, J. C. Robinson, C. Bharucha, P. E. Williams, and M. G. Raizen, Phys. Rev. Lett. **73**, 2974 (1994).
[20] H. Ammann, R. Gray, I. Shvarchuck, and N. Christensen, Phys. Rev. Lett. **80**, 4111 (1998).
[21] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, Nature (London) **453**, 891 (2008).
[22] S. S. Kondov, W. R. McGehee, J. J. Zirbel, and B. DeMarco, Science **334**, 66 (2011).
[23] F. Jendrzejewski, A. Bernard, K. Müller, P. Cheinet, V. Josse, M. Piraud, L. Pezzé, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, Nature Physics **8**, 398 (2012).
[24] B. Shapiro, J. Phys. A **45**, 143001 (2012).
[25] Y. Lahini, A. Avidan, F. Pozzi, M. Sorel, R. Morandotti, D. N. Christodoulides, and Y. Silberberg, Phys. Rev. Lett. **100**, 013906 (2008).
[26] G. Roati, C. D'Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature (London) **453**, 895 (2008).
[27] E. Schwarzer and H. Haken, Phys. Lett. A **42**, 317 (1972).
[28] J. Heinrichs, Z. Phys. B **89**, 115 (1992).
[29] S. Ostlund and R. Pandit, Phys. Rev. B **29**, 1394 (1984).
[30] D. L. Shepelyansky, Phys. Rev. Lett. **70**, 1787 (1993).
[31] S. Flach, D. O. Krimer, and C. Skokos, Phys. Rev. Lett. **102**, 024101 (2009).
[32] S. Flach, Chem. Phys. **375**, 548 (2010).
[33] T. V. Laptyeva, J. D. Bodyfelt, D. O. Krimer, C. Skokos, and S. Flach, Europhys. Lett. **91**, 30001 (2010).
[34] J. D. Bodyfelt, T. V. Laptyeva, C. Skokos, D. O. Krimer, and S. Flach, Phys. Rev. E **84**, 016205 (2011).
[35] G. Kopidakis, S. Komineas, S. Flach, and S. Aubry, Phys. Rev. Lett. **100**, 084103 (2008).
[36] A. D. Mirlin, Phys. Rep. **326**, 259 (2000).
[37] D. O. Krimer and S. Flach, Phys. Rev. E **82**, 046221 (2010).
[38] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, New York, 1995).
[39] C. Skokos, D. O. Krimer, S. Komineas, and S. Flach, Phys. Rev. E **79**, 056211 (2009).
[40] S. Flach, V. Fleurov, and A. V. Gorbach, Phys. Rev. B **71**, 064302 (2005).
[41] B. Rumpf, Physica D **238**, 2067 (2009).
[42] H. Yamada and K. S. Ikeda, Phys. Rev. E **59**, 5214 (1999).
[43] H. Ezaki and F. Shibata, Physica A **187**, 267 (1992).
[44] D. J. Thouless, J. Phys. C **6**, L49 (1973).
[45] B. Kramer and A. MacKinnon, Rep. Prog. Phys. **56**, 1469 (1993).
[46] The optimal dephasing rate in the Aubry-André model, where the localization length of all NMs is given by $\xi = \frac{1}{\ln(\zeta/2)}$, can be estimated in a similar way. Note that, therefore, one has to similarly assume a correspondence of ξ to a mean free path here. In the case of the Wannier-Stark ladder, the optimal dephasing rate can be estimated from the time it takes for half a Bloch-oscillation to be completed without perturbation, i.e., half a Bloch-period $\frac{T_B}{2} = \frac{\pi}{E}$.