Enhancement of chaotic subdiffusion in disordered ladders with synthetic gauge fields

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We study spreading wave packets in a disordered nonlinear ladder with broken time-reversal symmetry induced by synthetic gauge fields. The model describes the dynamics of interacting bosons in a disordered and driven optical ladder within a mean-field approximation. The second moment of the wave packet $m_2 = g t^\alpha$ grows subdiffusively with the universal exponent $\alpha \simeq 1/3$ similar to the time-reversal case. However, the prefactor $g$ is strongly modified by the field strength and shows a nonmonotonic dependence. For a weak field, the prefactor increases since time-reversal enhanced backscattering is suppressed. For strong fields the spectrum of the linear wave equation reduces the localization length through the formation of gaps and narrow bands. Consequently the prefactor for the subdiffusive spreading law is suppressed.

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I. INTRODUCTION

The spreading of wave packets in disordered nonlinear lattices has regained a lot of interest recently. Many experiments were performed with ultracold atomic condensates in random optical potentials [1]. Due to disorder, linear waves which correspond to noninteracting condensates will eventually stop spreading and exponentially localize in low dimensions [2]. Mean-field treated two-body interactions lead to cubic nonlinear terms in the wave equations. They induce overlaps between the normal modes of the linear wave equation and ultimately lead to chaotic dynamics. The subsequent decoherence of phases of the normal modes breaks localization through incoherent spreading [3,4]. Numerical studies on wave packet spreading in several one-dimensional nonlinear disordered lattice models show that the wave packet exhibits a subdiffusive behavior for weak nonlinearity; namely, the second moment $m_2$ of a wave packet grows as $m_2 = g t^\alpha$ with the universal exponent $\alpha \simeq 1/3$ [4]. Several issues remain under debate, especially the asymptotic behavior of the second moment in the long time limit [5–9].

Previous studies focused only on time-reversal symmetry (TRS) cases. Here we consider the situation of broken TRS. For charged particles, TRS can be broken by turning on magnetic fields. For neutral particles, like bosons, we can introduce synthetic gauge fields [10]. The advantage of synthetic gauge fields on optical lattices is that strong fields, where the flux per lattice cell can take any value between zero and $2\pi$, can be realized with lasers [10]. This has to be contrasted to the case of electrons in metals [11]. Therefore synthetic gauge fields offer an opportunity for experimental studies on many lattice models under strong magnetic fields [12].

In the presence of magnetic fields, one may expect that wave spreading slows down due to the Lorentz force, which tends to localize an excitation. On the other hand, the magnetic field breaks TRS and therefore enhances the localization length in a disordered system. Moreover the magnetic field can also change the band structure of the corresponding linear and ordered system.

In this work, we study the influence of synthetic gauge fields on the spreading of nonlinear waves. It turns out that each of the effects mentioned above exhibits its dominant role in different parameter and time regimes.

II. MODEL

We consider the discrete nonlinear Schrödinger model on a two-leg ladder model with complex hopping terms:

$$
H = -t \sum_{l} (e^{-i2\pi q} \psi^*_{1,l} \psi_{1,l+1} + \psi^*_{2,l} \psi_{2,l+1} + \text{H.c.})
$$

$$
- t \sum_{l} (\psi^*_{1,l} \psi_{2,l} + \text{H.c.})
$$

$$
+ \sum_{\nu=1,2} \sum_{l} |\psi_{\nu,l}|^2 (\epsilon_{\nu,l} + \beta |\psi_{\nu,l}|^2).
$$

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The eigenvalue problem of the linear part of Hamiltonian (1) is

\[ E_\mu \Phi_{\mu,l} = -J \Phi_{\mu,l+1} - J^{-1} \Phi_{\mu,l-1} + D_l \Phi_{\mu,l}, \]  

(5)

where

\[ \Phi_{\mu,l} = \binom{\phi_{\mu,l}}{\phi_{\mu,l}} \]  

(6)

is the \( \mu \)-th eigenmode and \( E_\mu \) is the corresponding eigenvalue. Using the expansion \( \psi(t) = \sum_\mu c_\mu(t) \Phi_{\mu,l} \) in Eq. (3), we obtain the equations of motion for the normal mode amplitudes:

\[ i \dot{c}_\mu = E_\mu c_\mu + \beta \sum_{\mu_1,\mu_2,\mu_3} I_{\mu,\mu_1,\mu_2,\mu_3} c_{\mu_1}^* c_{\mu_2} c_{\mu_3} \]  

(7)

with

\[ I_{\mu,\mu_1,\mu_2,\mu_3} \equiv \sum_l \sum_{i=1,2} \Phi_{\mu,l}^i J_i \Phi_{\mu_1,l}^i \Phi_{\mu_2,l}^i J_i \Phi_{\mu_3,l}. \]  

(8)

The overlap integrals \( I_{\mu,\mu_1,\mu_2,\mu_3} \) are random variables and their distribution plays a crucial role in the nonlinear wave packet spreading.

### III. LINEAR EQUATION PROPERTIES

#### A. Clean system

We first study the linear (\( \beta = 0 \)) and clean (\( \epsilon_{i,j} = 0 \)) case of Eq. (1). Let the width of the ladder be \( L \). Under the gauge chosen and imposing periodic boundary conditions \( \psi_{\epsilon,1} = \psi_{\epsilon,L} \), the Hamiltonian (1) has lattice translation invariance along the horizontal direction and therefore can be easily diagonalized in the momentum space [13,14]. The eigenvalue spectrum reads

\[ E_\epsilon(k) = -\cos(k) - \cos(2\pi q + k) \pm \sqrt{[\cos(k) - \cos(2\pi q + k)]^2 + 1}, \]  

(9)

where the momentum \( k \) is varying in the first Brillouin zone \( k \in [-\pi, \pi] \). Figure 2 shows the band structure for various magnetic fluxes. Interestingly the two bands open a gap when the flux exceeds a certain critical value \( q_c \approx 0.34 \), which is determined by the conditions \( E_\epsilon(k) = 0 \) and \( \partial E_\epsilon(k) / \partial k = 0 \). This behavior can be expected in a multibands system, since a magnetic field tends to reduce the kinetic energy of translational motion and therefore the bandwidth of a single band. A magnetic field in a system with few bands will therefore flatten each band and eventually lead to the appearance of new gaps.

![Figure 1: Two-leg ladder lattice exposed to synthetic gauge fields. A uniform synthetic magnetic flux per plaquette is \( 2\pi q \). The gauge is chosen to reside only on chain 1. Solid lines correspond to hopping \( t = 1 \) and dashed lines correspond to complex hopping.](image)

![Figure 2: The band structure of the corresponding linear and clean model of Eq. (1) for various magnetic fluxes: \( q = 0, 0.125, 0.42 \), and 0.5. The two bands open a gap when \( q > q_c \approx 0.34 \). The gap is largest for \( q = 1/2 \). At the same time, the bands become narrower.](image)
The localization length is enhanced almost in the whole energy band when $0 < q < q_c$, while it is reduced for $q_c < q \leq 1/2$ with $q_c \simeq 0.34$.

For small magnetic fluxes ($0 < q < q_c$), the localization length is enhanced in almost the whole energy band. This is a well known phenomenon of the two-dimensional Anderson model in the presence of magnetic fields [17]. It is due to the fact that magnetic fields, which destroy constructive interference by breaking TRS, reduce the return probability and therefore enhance the localization length.

Large magnetic fluxes ($q > q_c$) open an energy gap around $E = 0$ in the corresponding clean system (see Fig. 2). Therefore the density of states opens a pseudogap around $E = 0$ (see Fig. 4). States in that pseudogap are similar to states in the Lifshitz tails in the band edges and the localization length of these states is shortened. At the same time the two bands of the clean system have a smaller width. This reduction of kinetic energy leads to a reduction of the localization length at energies away from the pseudogap region, where the density of states is also enhanced. Indeed, in one-dimensional systems the localization length is proportional to the mean free path, and the enhanced density of states reduces the mean free path and therefore the localization length. At the largest value of the magnetic flux $q = 1/2$, TRS is restored. The restoring of this symmetry is another factor which leads to a suppression of the localization length in the whole energy range.

**IV. NONLINEAR WAVE PACKET SPREADING**

We launch a local excitation in the center of the ladder as an initial wave packet, namely, $\psi_{1,l}(\tau = 0) = \psi_{2,l}(\tau = 0) = \delta_{l,L/2}/\sqrt{2}$, and study the wave packet spreading. To characterize the wave packet spreading we calculate the second moment $m_2 \equiv \sum_{l} (l - \bar{l})^2 \langle |\psi_l(\tau)|^2 \rangle = \sum_{l} (l - \bar{l})^2 (|\psi_{1,l}(\tau)|^2 + |\psi_{2,l}(\tau)|^2)$ with $\bar{l} \equiv \sum_{l} |\langle l |\psi_{0}(0)\rangle|^2 = \sum_{l} \langle l |\psi_{1,l}(0)\rangle^2 + \langle l |\psi_{2,l}(0)\rangle^2$. We use the SBA3 symplectic integrator [18] to evaluate the wave function $\psi_l(\tau)$ and therefore $m_2$. The time evolution of $m_2$ for various magnetic fluxes is shown in Fig. 5 for $\beta = 1$ and disorder strength $W = 4$.

During a first short time regime ($\tau < 10$) the wave packet spreads ballistically, $m_2 \equiv g(q)\tau^\alpha$ with the exponent $\alpha \simeq 2$. In this regime, the prefactor $g(q)$ decreases with increasing the magnetic flux. This is due to the fact that the largest group velocity of waves in the corresponding linear and clean system decreases with increasing the magnetic flux. In one-dimensional disordered systems, the localization length is of the same order as the mean free path. We can therefore neglect the effect of the disorder in this short time regime. Nonlinearity does not affect so much the behavior of the spreading in this regime either. The strength of nonlinearity considered here is chosen such as to be in the weak chaos regime [19].

**FIG. 3.** (Color online) Localization length as a function of energy with disorder strength $W = 4$ for various magnetic fluxes: $q = 0$, $0.125$, $0.34$, $0.42$, and $0.5$. The system size $L = 10^3$. The localization length is enhanced almost in the whole energy band when $0 < q < q_c$, while it is reduced for $q_c < q \leq 1/2$ with $q_c \simeq 0.34$.

**FIG. 4.** (Color online) Density of states averaged over 20 disorder realizations with disorder strength $W = 4$ for various magnetic fluxes: $q = 0$, $0.125$, $0.34$, $0.42$, and $0.5$. The density of states opens a pseudogap around the center of the band when $q \sim q_c$ with $q_c \simeq 0.34$.

**FIG. 5.** (Color online) Disorder average over 100 disorder realizations of $\log_{10} m_2$ vs $\log_{10} \tau$ with disorder strength $W = 4$ and $\beta = 1$ for various magnetic fluxes: $q = 0$, $0.125$, $0.34$, $0.42$, and $0.5$. The two dashed lines guide the eye for $\tau^\alpha$ with $\alpha = 2$ and $1/3$, respectively.
implies that the interaction energy of the initial wave packet is small compared to the bandwidth of the linear wave equation. Therefore the ballistic wave packet spreading up to a distance of the order of the localization length is expected. Recent studies on the spreading dynamics of interacting bosons in homogeneous lattices show that in one-dimensional lattices, for weak interactions, the nonlinearity induced suppression of the expansion velocity is too weak to be observed [20], which is consistent with our observation. For substantially larger times (τ > 10^3) the wave packet exhibits a subdiffusive behavior. The second moment of the wave packet grows as m_2 = g(q)τ^α with the exponent α ≃ 1/3, which does not depend on the value of the flux. This subdiffusive spreading is caused by chaoticity of the wave packet dynamics which is due to resonances and nonintegrability. The exponent α = 1/3 has been shown to depend solely on the power of the nonlinear terms and on the dimensionality of the underlying lattice [4]. The presence of synthetic gauge fields does not affect these ingredients, underpinning the universality of the subdiffusive spreading exponent α = 1/3. However, the synthetic gauge field does affect the prefactor g(q). For small magnetic fluxes (0 < q < q_c) the prefactor is enhanced and for big magnetic fluxes (q_c < q ≤ 1/2) the prefactor is reduced. As we will show, this is due to the strong variation of the localization length with changing the magnetic flux.

In order to get more insight into the details of the dynamics of spreading wave packets, we plot in Figs. 6–8 the space-time dependence of the wave function density along the first ladder leg |ψ_{1,l}(τ)|^2 and the second ladder leg |ψ_{2,l}(τ)|^2 for different magnetic fields. We observe that the density quickly decays in the core of the wave packet in both legs, leading to a homogeneous spreading without any remnants of Anderson localization at the original excitation sites and irrespective of the value of the magnetic field.

Nonlinearity introduces resonances and breaks integrability. In the presence of nonlinearity, the amplitude of a localized normal mode is modified by a triplet of other excited modes \( \vec{\mu} \equiv (\mu_1, \mu_2, \mu_3) \) up to the first order in \( \beta \) as

\[
|c_1^{(1)}| = \beta \sqrt{n_{\mu_1} n_{\mu_2} n_{\mu_3}} R_{\mu, \vec{\mu}}^{-1} \sim \frac{E_{\mu, \vec{\mu}}}{I_{\mu, \mu_1, \mu_2, \mu_3}},
\]

where \( E_{\mu, \vec{\mu}} \) is a relevant mode that resides inside a finite volume of the order of the localization volume of the given mode \( \mu \). To measure the localization volume we use the quantity \( \sqrt{\sum_{\mu} n_{\mu}^2} \) [21], where \( n_{\mu}^2 \equiv \sum_l (l - \bar{l}_{\mu, l})^2 |\Phi_{\mu, l}|^2 \) and \( \bar{l}_{\mu} \equiv \sum_l l |\Phi_{\mu, l}|^2 \).

Perturbation theory breaks down with the onset of resonances when \( \sqrt{\sum_{\mu} n_{\mu}^2} < |c_1^{(1)}| \). For simplicity we assume that all modes inside the wave packet have the same norm \( n \). Then the resonance condition is \( \beta n < R_{\mu, \vec{\mu}} \). For a given normal mode \( \mu \), we define \( R_\mu = \min_{\vec{\mu}} R_{\mu, \vec{\mu}} \), where the minimum is taken inside the corresponding localization volume with \( \mu_1 \neq \mu_2 \neq \mu_3 \neq \mu \). Collecting \( R_\mu \) for many \( \mu \) and many disorder realizations, we obtain the probability density \( W(R_\mu) \) (Fig. 9). Following the argument of Ref. [19] the probability that a given mode has at least one triplet of other modes with which

FIG. 6. (Color online) Time evolution of the wave function density for \( q = 0 \). The upper left plot (a1) corresponds to the first leg of the ladder, and the upper right plot (a2) corresponds to the second leg. We plot \( \log_{10} |\psi_{1,l}(\tau)|^2 \) in a color code vs the ladder coordinate \( l \) and time \( \tau \). The three-dimensional plot (a3) at the bottom shows the density evolution \( |\psi_{1,l}(\tau)|^2 \) on a linear scale for the first leg. The second leg plot is very similar and therefore omitted.

FIG. 7. (Color online) Time evolution of the wave function density for \( q = 0.125 \). The upper left plot (b1) corresponds to the first leg of the ladder, and the upper right plot (b2) corresponds to the second leg. We plot \( \log_{10} |\psi_{1,l}(\tau)|^2 \) in a color code vs the ladder coordinate \( l \) and time \( \tau \). The three-dimensional plot (b3) at the bottom shows the density evolution \( |\psi_{1,l}(\tau)|^2 \) on a linear scale for the first leg. The second leg plot is very similar and therefore omitted. The color map follows Fig. 6.
it is resonant at a given value of $\beta$ is $P = \int_0^{\beta \alpha} W(x) dx$. We denote $C = W(R_\mu \rightarrow 0) \neq 0$. A bigger (smaller) $C$ indicates a stronger (weaker) resonance. We find that for stronger resonance the nonlinear spreading evolves faster (slower) (see Figs. 5 and 9).

We also study the distribution of $E_{\mu,\vec{\mu}}$ and $|I_{\mu,\mu_1,\mu_2,\mu_3}|^{-1}$ separately. We choose a localized eigenstate with energy $E_\mu$ and store $E_{\mu,\vec{\mu}}$ for all eigenstates $\mu_1 \neq \mu_2 \neq \mu_3 \neq \mu$ inside the localization volume associated with the state $\mu$. We repeat the same procedure for the next eigenstate. Collecting these data for many disorder realizations, we obtain the statistics of $|E_{\mu,\vec{\mu}}|$. The same method yields the statistics of $|I_{\mu,\mu_1,\mu_2,\mu_3}|^{-1}$. For further details we refer to Ref. [21]. In Figs. 10 and 11 we plot the distributions of $|E_{\mu,\vec{\mu}}|$ and $|I_{\mu,\mu_1,\mu_2,\mu_3}|^{-1}$ for various magnetic fluxes. The distribution of $|E_{\mu,\vec{\mu}}|$ is insensitive to the magnetic flux and is very close to a Gaussian distribution. The reason is that inside a localization volume there are many energy levels, and correlations between them are strong only when their distances are of the order of the mean level spacing. Since most contributions to $|E_{\mu,\vec{\mu}}|$ come from more distant levels, the distribution of $|E_{\mu,\vec{\mu}}|$ is approaching the distribution of sums over independent

![FIG. 8. (Color online) Time evolution of the wave function density for $q = 0.5$. The upper left plot (c1) corresponds to the first leg of the ladder, and the upper right plot (c2) corresponds to the second leg. We plot $\log_{10} |\psi_{1,(\tau)}|^2$ in a color code vs the ladder coordinate $l$ and time $\tau$. The three-dimensional plot (c3) at the bottom shows the density evolution $|\psi_{1,(\tau)}|^2$ on a linear scale for the first leg. The second leg plot is very similar and therefore omitted. The color map follows Fig. 6.](image)

![FIG. 9. (Color online) The probability distribution function (PDF) of $R_\mu$ inside a localization volume at disorder strength $W = 4$ for various magnetic fluxes: $q = 0, 0.125, 0.34, 0.42, and 0.5$. The plot is in log-linear scale.](image)

![FIG. 10. (Color online) The PDF of $|E_{\mu,\vec{\mu}}|$ inside a localization volume with disorder strength $W = 4$ for various magnetic fluxes: $q = 0, 0.125, 0.42, and 0.5$.](image)

![FIG. 11. (Color online) The PDF of $|I_{\mu,\mu_1,\mu_2,\mu_3}|^{-1}$ inside a localization volume with disorder strength $W = 4$ for various magnetic fluxes: $q = 0, 0.125, 0.42, and 0.5$. The plot is in log-log scale.](image)
where the normal modes in the wave packet is chaotic due to noise, the distribution of variables. The distribution of $|I_{\mu,\mu_1,\mu_2}|^{-1}$ however depends on the magnetic flux. For small magnetic fluxes the probability of large overlap integrals becomes smaller, while it is enhanced for larger magnetic fluxes.

In order to understand the variation of the prefactor $g(q)$ with the magnetic flux, we adopt the arguments of Refs. [4,19,21,22]. The main idea is to assume that a part of the normal modes in the wave packet is chaotic due to the nonlinearity. If the overlap $I_{\mu,\mu_1,\mu_2}$ does not fluctuate strongly, we can replace it by its average $\langle I \rangle$ and replace $\langle \rho \rangle$ by $n^{3/2}$ in Eq. (7), where $\langle I \rangle$ is the average overlap in the wave packet. Then we consider a Langevin type equation of motion of a mode $\hat{\beta}$ outside of the wave packet which however resides in a close neighborhood to the wave packet:

$$i\dot{c}_\beta \approx E_\beta c_\beta + \beta \langle I \rangle n^{3/2} \mathcal{P}(\beta n) f(\tau),$$

where $f(\tau)$ is generated by the chaotic dynamics of wave packet modes and is assumed to be an uncorrelated white noise, $\langle f(\tau)f(\tau') \rangle = \delta(\tau - \tau')$. Here $\xi \equiv \int \rho(E)\xi(E)dE$ is the average of the localization length over all the modes. Such chaotic dynamics was confirmed in recent quantitative studies [23]. From the numerical data we observe that $C/\xi$ is roughly independent of the flux, namely, $C \sim \xi$. In the weak chaos regime $C\beta n < 1$, $\mathcal{P} \sim C\beta n$, one obtains

$$m_2 \approx C' \beta^{4/3} \langle I \rangle^{2/3} \xi^{5/3} \tau^{1/3},$$

where $C'$ is a constant which does not depend on any physical parameter [24]. Figure 12 shows the comparison of the coefficient in Eq. (14) and the coefficient $g(q)$ extracted from the numerical data of Fig. 5 for various magnetic fluxes. The unknown coefficient $C'$ is fitted to be $|\log_{10} C'| = 0.91$. The prediction of Eq. (14) matches the numerical data reasonably well. In particular the observed increase of the prefactor for small magnetic fluxes and the subsequent decrease for larger magnetic fluxes are very well reproduced.

V. SUMMARY

We studied the subdiffusive spreading of nonlinear waves in a one-dimensional disordered lattice in the absence of time-reversal symmetry. Our results show that the much-debated weak chaos subdiffusion law with exponent $\alpha = 1/3$ keeps its universality even in the presence of synthetic gauge fields. Their main impact is to change the localization length. For small magnetic fluxes, the losing of time-reversal symmetry leads to a suppression of backscattering and an increase of the localization length. Consequently the prefactor $g$ of the subdiffusive spreading law increases. For large magnetic fluxes, the spectrum of the linear wave equations opens a gap filled with Lifshitz-tail-like localized states. In this regime, the localization length is reduced. It follows that the prefactor $g$ is decreasing. A theoretical estimate of the dependence of $g$ on the magnetic flux yields good agreement with numerical data.

It would be also interesting to extend this study to the regime of strong chaos ($\alpha = 1/2$) [25,26] and two-dimensional disordered lattices.

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[11] In most electronic lattice systems it would take a large magnetic field $B \sim 10^5$ T to generate a flux per lattice cell which is of the order of a flux quantum.


[13] One can also choose other gauges which may break the lattice translation symmetry of the Hamiltonian along the horizontal direction. However, the magnetic flux is transnational invariant, and therefore one can always define a magnetic translator which commutes with the Hamiltonian [14].


[24] Notice that here we calculate $m_2$ in the normal mode space and in the numerical simulation $m_3$ is evaluated in the real space. The two are closely related and show similar behaviors.
