# Supplemental Material for "Flat Bands Under Correlated Perturbations"

This Supplemental Material presents derivations of the localization length Eq.(6), the density of states Eqs.(7,8), and the profiles of the low energy eigenstates appearing in the main text.

*Localization Length* — For $\epsilon_n^+ = 0$, $f_n$ can be eliminated from the eigenmode equations Eq.(3), leaving

$$\left(\frac{(\epsilon_n^-)^2}{\overline{E}} - \overline{E} - 2t\right) p_n = 2(p_{n-1} + p_{n+1}). \tag{S1}$$

When $\overline{E} \ll W^2/4$ is small, the first term on the left hand side is resonantly enhanced and dominates. The ratio $R_n = p_{n+1}/p_n$ is approximated by

$$R_n \approx \frac{(\epsilon_n^-)^2}{2\overline{E}} - \frac{1}{R_{n-1}}, \tag{S2}$$

The decaying solution for small $\overline{E}$ is $R_{n-1}(\epsilon_n^-) \approx 2\overline{E}/(\epsilon_n^-)^2$, thus applying Eq.(5) we obtain

$$\begin{aligned}\xi^{-1} &= \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} \ln \left| \frac{2\overline{E}}{(\epsilon_n^-)^2} \right|, \\ &= \langle \ln \left| \frac{2\overline{E}}{(\epsilon_n^-)^2} \right| \rangle. \end{aligned} \tag{S3}$$

$\epsilon_n^-$ are uncorrelated random variables with a uniform probability distribution function (PDF),

$$f_\epsilon(x) = \begin{cases} \frac{1}{W}, & \text{if } |x| \leq \frac{W}{2} \\ 0, & \text{otherwise} \end{cases} \tag{S4}$$

thus the disorder average is

$$\begin{aligned}\xi^{-1} &= \frac{1}{W} \int_{-W/2}^{W/2} \ln \left| \frac{2\overline{E}}{x^2} \right| dx \\ &= 2 + \ln \left| \frac{8\overline{E}}{W^2} \right|, \end{aligned} \tag{S5}$$

which reproduces Eq.(6) (noting that $\overline{E}/W^2 \ll 1$ and taking $\xi$ to be positive).

Curiously, Eq.(6) incorrectly predicts $\xi^{-1} = 0$ at $\overline{E}/W^2 = 1/(8e^2) \approx 0.02$, well within the validity of the approximation $\overline{E}/W^2 \ll 1/4$. To explain this anomaly, we note that the perturbative result Eq.(S2) is only valid when $E/(\epsilon_n^-)^2 \ll 1 \Rightarrow \epsilon_n^- \gg \sqrt{E}$. Thus, the integral in Eq.(S5) requires a finite cutoff $a \sim \sqrt{E}$

$$\xi^{-1} = \frac{2}{W} \left( \int_0^a \ln |R(x)|\, dx + \int_a^{W/2} \ln \left| \frac{2\overline{E}}{x^2} \right| dx \right), \tag{S6}$$

and we require $a \ll W/2$ for the first term to be negligible. Thus, Eq.(6) is only a good approximation under the stricter condition $\sqrt{E}/W \ll 1$, which excludes the divergence of the localization length, $\xi^{-1} = 0$, at $\sqrt{E}/W \approx 0.13$.

*Density of States* — To obtain the density of states Eq.(7), we evaluate the PDF of the random variable $z = \epsilon_0^- \epsilon_1^-$. The product distribution $f_z(x)$ is given by

$$\begin{aligned} f_z(x) &= \int f_\epsilon(y) f_\epsilon(x/y) \frac{1}{|y|} dy, \\ &= \frac{1}{W} \int_{-W/2}^{W/2} \frac{1}{|y|} f_\epsilon(x/y) dy, \\ &= \frac{2}{W^2} \int_{2|x|/W}^{W/2} \frac{dy}{y}, \\ &= \begin{cases} \frac{2}{W^2} \left[ \ln \frac{W}{2} - \ln \frac{2|x|}{W} \right], & \text{if } |x| \leq \frac{W^2}{4}, \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{S7}$$

Eq.(7) follows by making the change of variables $E = z/2$, with $\rho(E) = 2 f_z(2E)$.

Similarly, we obtain Eq.(8) from the PDF of $z = \epsilon_0^2 = g(\epsilon_0)$ via

$$f_z(x) = 2 |\partial_x g^{-1}(x)| f_\epsilon(g^{-1}(x)), \tag{S8}$$

where $g^{-1}(x) = \sqrt{x}$. This yields

$$f_z(x) = \begin{cases} \frac{1}{W\sqrt{x}}, & \text{if } 0 < x < \frac{W^2}{4}, \\ 0, & \text{otherwise} \end{cases} \tag{S9}$$

which gives Eq.(8) after the change of variables $E = z/(2t)$.

By the same arguments as above, the incorrectly predicted vanishing of $\rho(\overline{E})$ at $\overline{E} = W^2/4$ occurs due to realizations of the potential outside the range of validity of the perturbative expansion, and instead the stricter condition $\sqrt{E}/W \ll 1$ is again required.

*Low Energy Eigenstates* — The initial conditions $f_{0,1}$ uniquely determine the eigenmode amplitude along the rest of the lattice. The eigenmode equations for sites $p_{0,1}$ read

$$\left( \frac{\epsilon_0^-}{2} - \frac{\overline{E}^2}{2\epsilon_0^-} - \frac{t\overline{E}}{\epsilon_0^-} \right) f_0 = p_{-1} + \frac{\overline{E} f_1}{\epsilon_1^-}, \tag{S10}$$

$$\left( \frac{\epsilon_1^-}{2} - \frac{\overline{E}^2}{2\epsilon_1^-} - \frac{t\overline{E}}{\epsilon_1^-} \right) f_1 = p_2 + \frac{\overline{E} f_0}{\epsilon_0^-}. \tag{S11}$$

Without loss of generality, we can set $f_0 = 1$. When $\overline{E}$ is small, from the calculation of the localization length we have $p_{-1,2} \approx 2\overline{E} p_{0,1}/(\epsilon_{-1,2}^-)^2 \approx 0$. Under this approximation, the above equations are solved to leading order in $\overline{E}$ to obtain, for $t = 0$,

$$\overline{E} = \pm \epsilon_0^- \epsilon_1^- / 2, f_1 = \pm 1, \tag{S12}$$

and when $t \neq 0$

$$\overline{E} = \epsilon_0^2/(2t), f_1 = \epsilon_0^-/(t\epsilon_1^-), \tag{S13}$$

which yield the eigenmode profiles appearing in the main text. To verify this result we also obtained eigenstates numerically for various realizations of disorder. The small $\overline{E}$ eigenstates indeed display a single strong maximum, with energy determined by disorder potential at this maximum according to the above equations.