Supplemental Material

(Dated: June 16, 2023)

I. LINEAR MAP

In this section we provide equations of motion for the short range network. For convenience we use a notation which slightly differs from the main text. We express the state vector $\vec{\Psi}$ as consisting of $N/2$ unit cells with sites A and B

$$\vec{\Psi}(t) = \{\psi_n^A(t), \psi_n^B(t)\}_{n=1}^{N/2} \ . \tag{1}$$

The linear time evolution of the system is governed by a discrete unitary map consisting of several transformations of vector $\vec{\Psi}$:

$$\hat{U}^{(0)} = \sum_n \hat{C}_n^{B,A} \sum_n \hat{C}_n^{A,B}, \tag{2}$$

where maps $\hat{C}_n^{A,B}$ and $\hat{C}_n^{B,A}$ are given by unitary matrices acting on the neighboring sites $(\psi_n^A, \psi_n^B)^T$:

$$\begin{aligned}\sum_n \hat{C}_n^{A,B}|\Psi(t)\rangle &= \sum_n \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_n^A(t) \\ \psi_n^B(t) \end{pmatrix}, \\ \sum_n \hat{C}_n^{B,A}|\Psi(t)\rangle &= \sum_n \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_n^B(t) \\ \psi_{n+1}^A(t) \end{pmatrix} \ .\end{aligned} \tag{3}$$

The resulting equations of motion of the linear evolution are as follows:

$$\begin{aligned}&\psi_n^A(t+1) = \cos^2\theta\psi_n^A(t) - \cos\theta\sin\theta\psi_{n-1}^B(t) \\ &+ \sin^2\theta\psi_{n-1}^A(t) + \cos\theta\sin\theta\psi_n^B(t) \\ \\ &\psi_n^B(t+1) = \sin^2\theta\psi_{n+1}^B(t) - \cos\theta\sin\theta\psi_n^A(t) \\ &+ \cos^2\theta\psi_n^B(t) + \cos\theta\sin\theta\psi_{n+1}^A(t) \ .\end{aligned} \tag{4}$$

The solution can be determined exactly from the standard ansatz $\left(\psi_n^A(t), \psi_n^B(t)\right)^T = e^{-i(\omega_k t - kn)} \left(\psi_k^A, \psi_k^B\right)^T$ with eigenfrequencies ω_k and wave numbers k. The dispersion relation $\omega(k)$ is given by:

$$\omega(k) = \pm \arccos\left(\cos^2\theta + \sin^2\theta\cos k\right), \tag{5}$$

with two dispersive bands ω_k^r $(r = 1, 2)$, see Fig. 1, and corresponding normal modes which form a complete set:

$$\vec{\Psi}_k^r = \sum_n e^{ikn}\psi_k^{r,p}, \quad p = A, B \ . \tag{6}$$

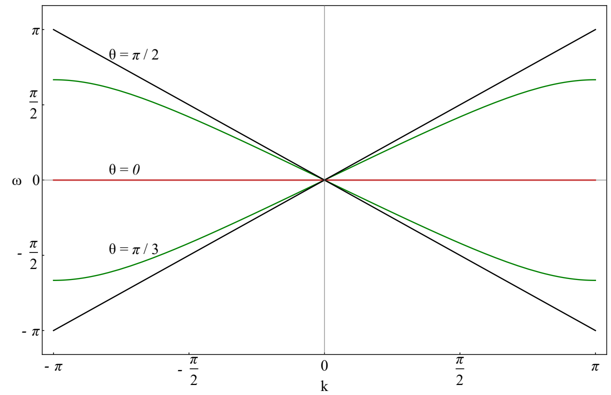

FIG. 1: The dispersion relation corresponding to Unitary Circuits (see eq. (5)). The parameter θ is varied to showcase a dispersionless flat band (red), a case of constant group velocity case (black) and the generic case (green).

II. NONLINEAR MAP

We rewrite the linear part of the evolution eq. (4) as follows:

$$\begin{aligned}&\alpha_n^A(t) \equiv \cos^2\theta\psi_n^A(t) - \cos\theta\sin\theta\psi_{n-1}^B(t) \\ &+ \sin^2\theta\psi_{n-1}^A(t) + \cos\theta\sin\theta\psi_n^B(t) \\ \\ &\alpha_n^B(t) \equiv \sin^2\theta\psi_{n+1}^B(t) - \cos\theta\sin\theta\psi_n^A(t) \\ &+ \cos^2\theta\psi_n^B(t) + \cos\theta\sin\theta\psi_{n+1}^A(t) \ .\end{aligned} \tag{7}$$

The nonlinearity inducing map $\hat{G}$ is applied after the linear part of the evolution:

$$\hat{U}_{\text{nonlin}} = \sum_n \hat{G}_n \sum_n \hat{C}_n^{B,A} \sum_n \hat{C}_n^{A,B}, \tag{8}$$

The nonlinearity is induced through an additional norm-dependent phase rotation as the result of the local linear evolution:

$$\hat{G}_n\psi_n^{A,B}(t) = e^{ig|\alpha^{A,B}(t)|^2}\psi_n^{A,B}(t) \tag{9}$$

The final equations of motion are:

$$\begin{aligned}&\psi_n^A(t+1) = e^{ig|\alpha_n^A|^2}\left[\cos^2\theta\psi_n^A(t) - \cos\theta\sin\theta\psi_{n-1}^B(t)\right. \\ &\left.+ \sin^2\theta\psi_{n-1}^A(t) + \cos\theta\sin\theta\psi_n^B(t)\right] \\ \\ &\psi_n^B(t+1) = e^{ig|\alpha_n^B|^2}\left[\sin^2\theta\psi_{n+1}^B(t) - \cos\theta\sin\theta\psi_n^A(t)\right. \\ &\left.+ \cos^2\theta\psi_n^B(t) + \cos\theta\sin\theta\psi_{n+1}^A(t)\right]\end{aligned} \tag{10}$$

III. SHORT RANGE NETWORK

The integrable limit is reached for $\theta = 0$. The system turns integrable and the equations of motion preserve the local norm (action) $|\psi_n^{A,B}|^2$:

$$\begin{aligned}\psi_n^A(t+1) &= e^{ig|\psi_n^A|^2}\psi_n^A(t)\\ \psi_n^B(t+1) &= e^{ig|\psi_n^B|^2}\psi_n^B(t)\end{aligned} \tag{11}$$

For small values of the parameter θ eq. (11) yields

$$\begin{aligned}\psi_n^A(t+1) &= e^{ig|\alpha_n^A|^2}\left[\psi_n^A(t) - \theta(\psi_{n-1}^B(t) - \psi_n^B(t))\right]\\ \psi_n^B(t+1) &= e^{ig|\alpha_n^B|^2}\left[\psi_n^B(t) + \theta(\psi_{n+1}^A(t) - \psi_n^A(t))\right].\end{aligned} \tag{12}$$

These equations of motion couple the actions through nearest neighbor terms and fall under the definition of a short range network.

IV. LONG RANGE NETWORK

The integrable limit is reached for $g = 0$ (see section I). The long range network of observables is obtained in the normal mode space of the model. The state vector can be represented as a sum of normal modes:

$$\vec{\Psi}(t) = \sum_k c_k^r(t)\vec{\Psi}_k^r. \tag{13}$$

where the index $r = 1, 2$ corresponds to one of the two bands and $\vec{\Psi}_k^r$ are the corresponding normal modes (see eq. (6)). In the linear setup the absolute values of the normal mode coefficients are conserved in time $|c_k^r(t)| = const$ and as such are integrals of motion. In the reciprocal space the network consists of disconnected nodes with c_k^r associated to each node. Let us expand the nonlinear evolution map eq.(8) for small values of the parameter g:

$$\hat{U}_{\text{nonlin}} = \hat{U}^{(0)} + ig\sum_{n,p}|\alpha_n^p|^2\hat{U}^{(0)}. \tag{14}$$

Using the normal mode representation of the state vector (13) we obtain the evolution equations of normal mode coefficients:

$$c_k^r(t+1) = e^{i\omega_k}c_k^r(t) + \frac{ig}{N}\sum_{\substack{r_1,r_2,r_3\\k_1,k_2,k_3}} e^{i(\omega_{k_1}^{r_1}+\omega_{k_2}^{r_2}-\omega_{k_3}^{r_3})} I_{k,k_1,k_2,k_3}^{r,r_1,r_2,r_3} c_{k_1}^{r_1}(t)c_{k_2}^{r_2}(t)\left(c_{k_3}^{r_3}(t)\right)^* \tag{15}$$

$$I_{k,k_1,k_2,k_3}^{r,r_1,r_2,r_3} = \delta_{k_1+k_2-k_3-k,0}\sum_p \psi_{k_1}^{r_1,p}\psi_{k_2}^{r_2,p}(\psi_{k_3}^{r_3,p})^*(\psi_k^{r,p})^*. \tag{16}$$

The number of triplet terms induced by nonlinearity in equation (15) is proportional to N^3, the selection rules result in Kronecker $\delta_{k_1+k_2-k_3-k,0}$ term in eq. (16) reducing the number of couplings to N^2, which falls under the definition of a long range network.

V. DEVIATION VECTORS

To compute the set of Lyapunov exponents we follow the evolution of tangent vectors $\{\vec{w}_i\}$. Each vector corresponds to the direction of the exponential growth or shrinking of the deviation from the initial trajectory - in total $2N$ vectors. The evolution of tangent vectors is done using the corresponding equations of motion derived below. We measure the magnitude of growth $\gamma(t) = |\vec{w}(t)|$ of each tangent vector and compute transient Lyapunov exponents $X_i(t) = 1/t\sum_\tau^t \log\gamma(\tau)$ after which the tangent vectors are orthonormalized using a Gram-Schmidt procedure. The evolution of positive transient Lyapunov exponents $X(t)$ is shown in Fig. 2. After an initial decay the transient Lyapunov exponents saturate. The saturated values are taken as final values for Lyapunov exponents Λ. Due to the conservation of the norm two exponents are expected to attain zero value. In the figure we see one of them (bottom most purple line) tending to zero with increasing time and no saturation.

A. Equations of Motion

We start from the nonlinear EoM (12):

$$\begin{aligned}\psi_n^A(t+1) &= e^{ig|\alpha_n^A(\Psi(t))|^2}\alpha_n^A(\Psi(t))\\ \psi_n^B(t+1) &= e^{ig|\alpha_n^B(\Psi(t))|^2}\alpha_n^B(\Psi(t)),\end{aligned} \tag{17}$$

where $\alpha_n^{A,B}$ are linear functions of the local components of the state vector $\vec{\Psi}(t)$ according to equations (4). We

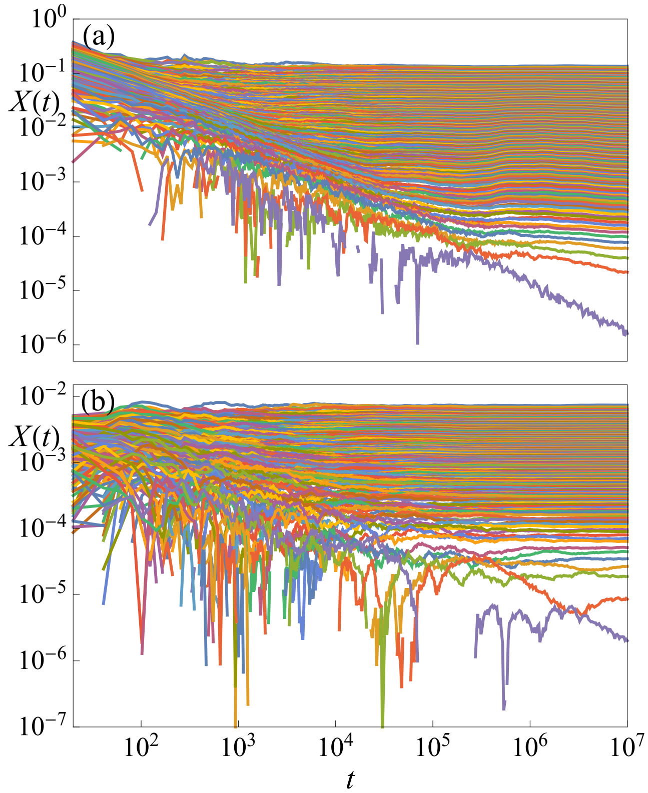

FIG. 2: The evolution of positive transient Lyapunov exponents. a) SRN case with angle $\theta = 0.1$ and nonlinearity $g = 1.0$, b) LRN case with angle $\theta = 0.33\pi$ and $g = 0.1$. For both cases system size $N = 200$.

consider a small deviation $\vec{\varepsilon}(t)$ from the initial trajectory $\vec{x}(t)$:

$$\vec{\psi} = \vec{x} + \vec{\varepsilon} \tag{18}$$

Substituting into (12):

$$\begin{aligned} \psi_n^A(t+1) &= e^{ig|\alpha_n^A[\vec{x}(t)+\vec{\varepsilon}(t)]|^2}\alpha_n^A\left[(\vec{x}(t)+\vec{\varepsilon}(t))\right] \\ \psi_n^B(t+1) &= e^{ig|\alpha_n^B[\vec{x}(t)+\vec{\varepsilon}(t))]|^2}\alpha_n^B\left[(\vec{x}(t)+\vec{\varepsilon}(t))\right]. \end{aligned} \tag{19}$$

Expanding the nonlinear term and keeping terms only in the 1st order of $\vec{\varepsilon}$ results in

$$\begin{aligned} &|\alpha_n^p[\vec{x}(t)+\vec{\varepsilon}(t)]|^2 = |\alpha_n^p[\vec{x}(t)] + \alpha_n^p[\vec{\varepsilon}(t)]|^2 = \\ &\alpha_n^p(\vec{x}(t))[\alpha_n^p(\vec{x}(t))]^* + \alpha_n^p(\vec{\varepsilon}(t))[\alpha_n^p(\vec{\varepsilon}(t))]^* + \\ &\alpha_n^p(\vec{\varepsilon}(t))[\alpha_n^p(\vec{x}(t))]^* + \alpha_n^p(\vec{x}(t))[\alpha_n^p(\vec{\varepsilon}(t))]^* \approx \\ &|\alpha_n^p(\vec{x}(t))|^2 + \Delta_n^p(t), \end{aligned} \tag{20}$$

where

$$\Delta_n^p(t) = \alpha_n^p(\vec{x}(t))[\alpha_n^p(\vec{\varepsilon}(t))]^* + c.c. \tag{21}$$

Thus we can rewrite the exponential term by expanding $e^{ig\Delta_n^{A,B}(t)}$:

$$e^{ig|\alpha_n^p[\vec{x}(t)+\vec{\varepsilon}(t))]|^2} = e^{ig|\alpha_n^p(\vec{x}(t))|^2}\left[1 + ig\Delta_n^p(t)\right] \tag{22}$$

With (19) and using the linearity of α_n^p we finally arrive at the following linear equations:

$$\varepsilon_n^p(t+1) = e^{ig|\alpha_n^p(\vec{x}(t))|^2}\Big\{\alpha_n^p[\vec{\varepsilon}(t)] + ig\Delta_n^p(t)\alpha_n^p[\vec{x}(t)]\Big\}. \tag{23}$$

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